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# Multi-Valued Ćirić Contractions on Metric Spaces with Applications

**N. Hussain[a], N. Yasmin[b], N. Shafqat[b]**

[a] *Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[b] *Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, 60800, Pakistan*

**Abstract.** In this paper, using the concept of $w$-distance, we obtain fixed point results for multi-valued generalized $w$-contractive maps not involving the extended Hausdorff metric. Presented theorems are generalizations of recent fixed point theorems due to L.B.Ćirić [Nonlinear Analysis 71 (2009), 2716-2723], D. Klim and D. Wardowski [ J. Math. Anal. Appl. 334 (2007), 132-139], Y. Feng and S. Liu [J. Math. Anal. Appl. 317 (2006), 103-112], Latif and Abdou [Nonlinear Analysis 74 (2011), 1436-1444 and Fixed Point Theory and Appl., Volume 2009 (2009) Article ID 487161, 8 pp.] and several others. As an application of our results we establish common fixed point results for newly defined class of Banach operator pairs.

## 1. Introduction and Preliminaries

Let $(X, d)$ be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of $X$. Let $d(x, A)$ denotes the distance from $x$ to $A \subset X$ and $H$ the Hausdorff metric induced by $d$, that is,

$$H(A, B) = \max\{\sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A)\}.$$

By $Cl(X)$ and $Comp(X)$ we will denote the collection of all nonempty closed and all nonempty compact subsets of $X$, respectively.

In 1996, Kada, Suzuki and Takahashi [17] introduced the concept of $w$-distance on a metric space as follows:

A function $w : X \times X \to [0; \infty)$ is called $w$-distance on $X$ if it satisfies the following for any $x,\ y,\ z \in X$:

$(w1)$ $w(x, z) \le w(x, y) + w(y, z)$;
$(w2)$ a map $w(x, .) : X \to [0, \infty)$ is lower semicontinuous;
$(w3)$ for any $\epsilon > 0$, there exists $\delta > 0$ such that $w(z, x) \le \delta$ and $w(z, y) \le \delta$ imply $d(x, y) \le \epsilon$

Using the concept of $w$-distance, Kada, Suzuki and Takahashi improved Caristi's fixed point theorem, Ekeland's variational principle(EVP), the minimization theorem and the Kirk-Caristi fixed point theorem

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*Email addresses:* nhusain@kau.edu.sa (N. Hussain), nusyasmin@yahoo.com (N. Yasmin), naeem781625@yahoo.com (N. Shafqat)

for a $w$-distance. Further, Lin and Du [28] introduced the concept of a $\tau$-function which is an extension of a $w$-distance. They established a generalized EVP for lower semicontinuous from above functions and with a $\tau$-function. They also derived the minimization theorem, nonconvex equilibrium theorem, and common fixed point theorem for a family of multivalued maps and the flower petal theorem.

Let us give some examples of $w$-distance [2, 17].

(a) The metric $d$ is a $w$-distance on $X$.

(b) Let $X$ be a normed space with norm $\|.\|$. Then the functions $w_1$ and $w_2$ defined by $w_1(x, y) = \|x\| + \|y\|$ and $w_2(x, y) = \|y\|$ for every $x,\ y \in X$, are $w$-distance.

(c) Let $(X, d)$ be a metric space and let $g : X \to X$ a continuous operator. Then the function $w : X \times X \to [0, \infty)$ defined by $w(x, y) = \max\{d(g(x), y), d(g(x), g(y))\}$ for every $x, y \in X$ is a $w$-distance.

The following lemmas concerning $w$-distance are crucial for the proofs of our results.

**Lemma 1.1. ([17])** Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0. Then, for the $w$-distance $w$ on $X$ the following hold for every $x,\ y,\ z \in X$:

(a) if $w(x_n, y) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$, in particular, if $w(x, y) = 0$ and $w(x, z) = 0$, then $y = z$;

(b) if $w(x_n, y_n) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to $z$;

(c) if $w(x_n, x_m) \le \alpha_n$ for any $n,\ m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;

(d) if $w(y, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

**Lemma 1.2. ([28])** Let $K$ be a closed subset of $X$ and $w$ be a $w$-distance on $X$. Suppose that there exists $u \in X$ such that $w(u, u) = 0$. Then $w(u, K) = 0$ if and only if $u \in K$ (where $w(u, K) = inf_{y \in K} w(u, y)$)

The theory of nonlinear analysis has come out as one of the significant mathematical disciplines during the last 50 years. The fixed point theorem, generally known as the Banach Contraction Mapping Principle, appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis(see [1]-[35]). In 1969, the Banach Contraction Mapping Principle was extended nicely to multi-valued mappings by Nadler.

**Theorem 1.3.** *(Nadler [30]). Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $CB(X)$. Assume that there exists $\lambda \in [0, 1)$ such that*

$$H(T(x), T(y)) \le \lambda d(x, y) \text{ for all } x, y \in X.$$

*Then there exists $z \in X$ such that $z \in Tz$.*

The Banach's Contraction Mapping Principle and the Nadler's fixed point theorem have been extended in many directions (c.f.[1]-[35]).

Recently Feng and Liu proved the following theorem.

**Theorem 1.4.** *(Feng - Liu [9], Theorem 3.1]). Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$. If there exist constants $b, c \in (0, 1)$, $c < b$, such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$bd(x, y) \le d(x, T(x))$$

*and*

$$d(y, T(y)) \leq cd(x, y),$$

*then there exists $z \in X$ such that $z \in T(z)$ provided the function $f(x) = d(x, T(x))$ is lower semi-continuous.*

Very recently Klim and Wardowski generalized Theorem 1.4 of Feng and Liu.

**Theorem 1.5.** *(Klim - Wardowski [22], Theorem 2.1). Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$. Assume that the following conditions hold:*
*(i) the map $f : X \to \mathbb{R}$, defined by $f(x) = d(x, T(x))$, $x \in X$, is lower semi-continuous;*
*(ii) there exist a constant $b \in (0, 1)$ and a function $\varphi : [0, \infty) \to [0, b)$ satisfying*

$$\lim_{r \to t+} \sup \varphi(r) < b \text{ for each } t \in [0, \infty) \tag{1}$$

*and for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$b\, d(x, y) \leq d(x, T(x)) \tag{2}$$

*and*

$$d(y, T(y)) \leq \varphi(d(x, y))d(x, y). \tag{3}$$

*Then there exists $z \in X$ such that $z \in Tz$.*

**Theorem 1.6.** *(Klim - Wardowski [22], Theorem 2.2). Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$. Assume that the following conditions hold:*
*(i) the map $f : X \to \mathbb{R}$, defined by $f(x) = d(x, T(x))$, $x \in X$, is lower semi-continuous;*
*(ii) there exists $\varphi : [0, \infty) \to [0, 1)$ satisfying the condition*

$$\lim_{r \to t+} \sup \varphi(r) < 1 \text{ for each } t \in [0, \infty) \tag{4}$$

*and such that for any $x \in X$ there is $y \in T(x)$ satisfying the conditions*

$$d(x, y) = d(x, T(x)) \tag{5}$$

*and*

$$d(y, T(y)) \leq \varphi(d(x, y))d(x, y). \tag{6}$$

*Then there exists $z \in X$ such that $z \in Tz$.*

Very recently Ćirić generalized Theorems 1.5 and 1.6 of Klim and Wardowski. He proved the following two theorems.

**Theorem 1.7.** *(Ćirić [6], Theorem 2.1). Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$. Suppose that the function $f : X \to \mathbb{R}$, defined by $f(x) = d(x, T(x))$, $x \in X$, is lower semi-continuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\lim_{p \to t+} \sup \varphi(p) < 1 \text{ for each } t \in [0, \infty) \tag{7}$$

*such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$\sqrt{\varphi(f(x))}d(x, y) \leq f(x) \tag{8}$$

*and*

$$f(y) \leq \varphi(f(x))d(x, y). \tag{9}$$

*Then there exists $z \in X$ such that $z \in Tz$.*

**Theorem 1.8.** *(Ćirić [6], Theorem 2.2). Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$* into a collection of all non-empty proximinal subsets of $X$. *Suppose that the function $f : X \to \mathbb{R}$, defined by $f(x) = d(x, T(x)), x \in X$, is lower semi-continuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\lim_{p \to t+} \sup \varphi(p) < 1 \text{ for each } t \in [0, \infty) \tag{10}$$

*such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$\sqrt{\varphi(d(x, y))}d(x, y) \leq d(x, T(x)) \tag{11}$$

*and*

$$d(y, T(y)) \leq \varphi(d(x, y))d(x, y). \tag{12}$$

*Then there exists $z \in X$ such that $z \in Tz$.*

Using $w$-distance, Latif and Abdou [26] have established following generalization of Theorem 1.4 of Feng and Liu and Theorem 1.5 of Klim and Wardowski.

**Theorem 1.9.***(Latif - Abdou [26], Theorem 2.2). Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$. Assume that the following conditions hold:*
*(i) the map $f : X \to \mathbb{R}$, defined by $f(x) = w(x, T(x)), x \in X$, is lower semi-continuous;*
*(ii) there exist a constant $b \in (0, 1)$ and a function $\varphi : [0, \infty) \to [0, b)$ satisfying*

$$\lim_{r \to t+} \sup \varphi(r) < b \text{ for each } t \in [0, \infty)$$

*and for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$b\, w(x, y) \leq w(x, T(x)) \tag{13}$$

*and*

$$w(y, T(y)) \leq \varphi(w(x, y))w(x, y). \tag{14}$$

*Then there exists $z \in X$ such that $f(z) = 0$. Further, if $w(z, z) = 0$, then $z \in Tz$.*

The aim of this paper is to obtain further generalizations of very recent fixed point Theorems 1.7 and 1.8 due to Ćirić [6], which in turn extend and improve Theorem 1.4 of Feng and Liu, Theorem 1.5 of Klim and Wardowski and recent results in [10, 27] and many others. As an application of our results we establish common fixed point results for newly defined class of Banach operator pairs.

## 2. Main Results

Recall that if $X$ is a topological space and $f : X \to \mathbb{R}$ is a function, then $f$ is called lower semi-continuous, if for any $\{x_n\} \subseteq X$ and $x \in X$,

$$x_n \to x \text{ implies } f(x) \leq \lim_{n \to \infty} \inf f(x_n).$$

In the sequel, otherwise specified, we shall assume that $X$ is a metric space with metric $d$ and $w$ is a $w$-distance on $X$.

**Theorem 2.1.** *Let $(X,d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$. Suppose that the function $f : X \to \mathbb{R}$, defined by $f(x) = w(x, T(x)), x \in X$, is lower semi-continuous and that there exists a function $\varphi : [0,\infty) \to [a, 1), 0 < a < 1$, satisfying*

$$\lim_{p\to t+} \sup \varphi(p) < 1 \text{ for each } t \in [0,\infty) \tag{15}$$

*such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$[\varphi(f(x))]^r w(x,y) \le f(x), \text{ where } 0 < r < 1 \tag{16}$$

*and*

$$f(y) \le \varphi(f(x))w(x,y). \tag{17}$$

*Then there exists $z \in X$ such that $f(z) = 0$. Further, if $w(z,z) = 0$, then $z \in Tz$.*

*Proof.* Since by definition of $\varphi$ we have $\varphi(f(x)) < 1$ for each $x \in X$, it follows that for any $x \in X$ there exists $y \in T(x)$ such that (16) holds.

Let $x_0 \in X$ be any initial point. Then from (16) and (17) we can choose $x_1 \in T(x_0)$ such that

$$[\varphi(f(x_0))]^r\, w(x_0,x_1) \le f(x_0) \tag{18}$$

and

$$f(x_1) \le \varphi(f(x_0))w(x_0,x_1). \tag{19}$$

From (18) and (19) we get

$$\begin{aligned} f(x_1) &\le \varphi(f(x_0))w(x_0,x_1) \\ &= [\varphi(f(x_0))]^{1-r}([\varphi(f(x_0))]^r w(x_0,x_1)) \\ &\le [\varphi(f(x_0))]^{1-r} f(x_0). \end{aligned}$$

Thus

$$f(x_1) \le [\varphi(f(x_0))]^{1-r}\, f(x_0). \tag{20}$$

Now we choose $x_2 \in T(x_1)$ such that

$$[\varphi(f(x_1))]^r w(x_1,x_2) \le f(x_1)$$

and

$$f(x_2) \le \varphi(f(x_1))w(x_1,x_2).$$

Hence we get

$$f(x_2) \le [\varphi(f(x_1))]^{1-r} f(x_1).$$

Continuing this process we can choose an iterative sequence $\{x_n\}_{n=0}^{\infty}$ satisfying $x_{n+1} \in T(x_n)$ and such that

$$[\varphi(f(x_n))]^r w(x_n,x_{n+1}) \le f(x_n) \tag{21}$$

and

$$f(x_{n+1}) \leq [\varphi(f(x_n))]^s f(x_n) \tag{22}$$

for all $n \geq 0$, where $s = 1 - r < 1$.

Now we shall show that $f(x_n) \to 0$ as $n \to \infty$.

From (22) and $\varphi(t) < 1$ we conclude that $\{f(x_n)\}_{n=0}^{\infty}$ is a strictly decreasing sequence of nonnegative real numbers. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} f(x_n) = \delta + . \tag{23}$$

Now we shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then, taking the limit on both sides of (22) and have in mind the assumption (15), we have

$$\delta \leq \lim_{f(x_n)\to\delta+} [\varphi(f(x_n))]^s \, \delta < \delta,$$

a contradiction. Thus $\delta = 0$, that is, $\lim_{n\to\infty} f(x_n) = 0$.

Now we shall show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Let

$$\alpha = \lim_{f(x_n)\to 0} [\varphi(f(x_n))]^s.$$

Then by the assumption (15), $\alpha < 1$. Let $q$ be such that $\alpha < q < 1$. Then there is some $n_0 \in N$ such that $[\varphi(f(x_n))]^s < q$ for all $n \geq n_0$. Thus from (22),

$$f(x_{n+1}) \leq q f(x_n) \quad \text{for each} \quad n \geq n_0.$$

Hence, by induction,

$$f(x_{n+1}) \leq q^{n+1-n_0} f(x_{n_0}) \quad \text{for all} \quad n \geq n_0. \tag{24}$$

Since $\varphi(t) \geq a > 0$ for all $t \geq 0$, from (21) we get $w(x_n, x_{n+1}) \leq (1/(a)^r) f(x_n)$. Thus by (24),

$$w(x_n, x_{n+1}) \leq \frac{1}{(a)^r} q^{n-n_0} f(x_{n_0}) \quad \text{for all} \quad n \geq n_0. \tag{25}$$

Then, using the triangle inequality, for each $m > n \geq n_0$ we have

$$w(x_n, x_m) \leq \sum_{k=n}^{m-1} w(x_k, x_{k+1}) \leq \frac{1}{(a)^r} \sum_{k=n}^{m-1} q^{k-n_0} f(x_{n_0}) \leq \frac{1}{(a)^r} \frac{q^{n-n_0}}{1-q} f(x_{n_0}).$$

From Lemma 1.1 we conclude, as $q < 1$, that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since $X$ is complete, there is some $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{26}$$

We now show that $z$ is a fixed point of $T$. Since by assumption $f(x) = w(x, T(x))$ is lower semi-continuous, we have

$$0 \leq w(z, T(z)) = f(z) \leq \lim_{n\to\infty} \inf f(x_n) = 0.$$

Hence $f(z) = w(z, T(z)) = 0$. Since $w(z, z) = 0$ and $Tz$ is closed, so by Lemma 1.2, $z \in T(z)$. Thus we proved that $z$ is a fixed point of $T$. □

**Remark 2.2.** (a). If $r = \frac{1}{2}$ in Theorem 2.1, we obtain Theorem 2.1 [25].

(b). If $r = \frac{1}{2}$ and $w = d$ in Theorem 2.1, we obtain Theorem 1.7.

Using Theorem 2.1, we obtain following extension of Thoerem 2.2 in [25]

**Theorem 2.3.** Suppose that all the hypotheses of Theorem 2.1 except the function $f : X \to \mathbb{R}$, defined by $f(x) = w(x, T(x)), x \in X$, is lower semi-continuous hold. Assume that

$$\inf\{w(x, v) + w(x, T(x)) : x \in X\} > 0,$$

for every $v \in X$ with $v \notin T(v)$. Then $Fix(T) \neq \emptyset$.

Now we shall prove a fixed point theorem for multi-valued nonlinear $w$-contractions which is a generalization of results of Mizoguchi and Takahashi [29], Feng and Liu [9] and Klim and Wardowski [22].

**Theorem 2.4.** *Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$. Suppose that the function $f : X \to R$, defined by $f(x) = w(x, T(x)), x \in X$, is lower semi-continuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\lim_{p \to t+} \sup \varphi(p) < 1 \quad \text{for each} \quad t \in [0, \infty) \tag{27}$$

*such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$[\varphi(w(x, y))]^r w(x, y) \leq w(x, T(x)), \quad \text{where} \quad 0 < r < 1 \tag{28}$$

*and*

$$w(y, T(y)) \leq \varphi(w(x, y))w(x, y). \tag{29}$$

*Then there exists $z \in X$ such that $f(z) = 0$. Further, if $w(z, z) = 0$, then $z \in Tz$.*

*Proof.* Replacing $\varphi(f(x))$ with $\varphi(w(x, y))$ and following the lines in the proof of Theorem 2.1, one can construct an iterative sequence $\{x_n\}_{n=0}^{\infty}$ in $X$ such that $x_{n+1} \in T(x_n)$ and satisfying

$$[\varphi(w(x_n, x_{n+1}))]^r w(x_n, x_{n+1}) \leq w(x_n, T(x_n)) \tag{30}$$

and

$$w(x_{n+1}, T(x_{n+1})) \leq [\varphi(w(x_n, x_{n+1}))]^s \, w(x_n, T(x_n)) \tag{31}$$

for all $n \geq 0$ , where $s = 1 - r < 1$. From (31) we conclude that $\{w(x_n, T(x_n))\}_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n \to \infty} w(x_n, T(x_n)) = \delta + . \tag{32}$$

The equality (32) corresponds to the equality (23) in the proof of Theorem 2.1, but now (32) is not enough. Since in the assumptions of this theorem appears $\varphi(w(x, y))$, we now need to prove that there exists a subsequence $\{w(x_{n_k}, x_{n_k+1})\}_{k=0}^{\infty}$ of $\{w(x_n, x_{n+1})\}_{n=0}^{\infty}$ such that $\lim_{k \to \infty} w(x_{n_k}, x_{n_k+1}) = \eta+$ for some $\eta \geq 0$.

From (30), as $\varphi(t) \geq a$, we get

$$w(x_n, x_{n+1}) \leq \frac{1}{(a)^r} w(x_n, T(x_n)). \tag{33}$$

From (32) and (33) we conclude that the sequence $\{w(x_n, x_{n+1})\}_{n=0}^{\infty}$ is bounded. Therefore, there is some $\theta \geq 0$ such that

$$\lim_{n\to\infty} \inf w(x_n, x_{n+1}) = \theta. \tag{34}$$

Since $x_{n+1} \in T(x_n)$, it follows that $w(x_n, x_{n+1}) \geq w(x_n, T(x_n))$ for each $n \geq 0$. This implies that $\theta \geq \delta$.

Now we shall show that $\theta = \delta$. At first suppose that $\delta = 0$. Then from (32) and (33) we have

$$\lim_{n\to\infty} w(x_n, x_{n+1}) = 0.$$

Thus, if $\delta = 0$, then $\theta = \delta$.

Suppose now that $\delta > 0$ and suppose, to the contrary, that $\theta > \delta$. Then $\theta - \delta > 0$ and so from (32) and (34) there is a positive integer $n_0$ such that

$$w(x_n, T(x_n)) < \delta + \frac{\theta - \delta}{4} \quad \text{for all} \quad n \geq n_0 \tag{35}$$

and

$$\theta - \frac{\theta - \delta}{4} < w(x_n, x_{n+1}) \quad \text{for all} \quad n \geq n_0. \tag{36}$$

Then from (36), (30) and (35) we get

$$\begin{aligned} [\varphi(w(x_n, x_{n+1}))]^r(\theta - \frac{\theta - \delta}{4}) &< [\varphi(w(x_n, x_{n+1}))]^r w(x_n, x_{n+1}) \\ &\leq w(x_n, T(x_n)) < \delta + \frac{\theta - \delta}{4} \end{aligned}$$

for all $n \geq n_0$. Hence we get

$$[\varphi(w(x_n, x_{n+1}))]^r \leq \frac{\theta + 3\delta}{3\theta + \delta} \quad \text{for all} \quad n \geq n_0. \tag{37}$$

Set

$$h = \frac{\theta + 3\delta}{3\theta + \delta}, \quad \text{that is,} \quad h = 1 - \frac{2(\theta - \delta)}{3\theta + \delta}.$$

Since we suppose that $\theta > \delta$, then $h < 1$. Now from (31) and (37),

$$w(x_{n+1}, T(x_{n+1})) \leq h.w(x_n, T(x_n)) \quad \text{for all} \quad n \geq n_0.$$

Hence it is easy to show, by induction, that

$$w(x_{n_0+k}, T(x_{n_0+k})) \leq h^k.w(x_{n_0}, T(x_{n_0})) \text{ for any } k \geq 1. \tag{38}$$

Since we suppose that $\delta > 0$ and as $h < 1$, there is a positive integer $k_0$ such that

$$h^{k_0}.w(x_{n_0}, T(x_{n_0})) < \delta.$$

Then by (38), as $\delta \leq w(x_n, T(x_n))$ for each $n \geq 0$, we have

$$\delta \leq w(x_{n_0+k_0}, T(x_{n_0+k_0})) \leq h^{k_0}.w(x_{n_0}, T(x_{n_0})) < \delta,$$

a contradiction. Therefore, our assumption $\theta > \delta$ is wrong. Thus $\theta = \delta$.

Now we shall show that $\theta = 0$. Since $\theta = \delta \leq w(x_n, T(x_n)) \leq \mathrm{w}(\mathrm{x}_n, \mathrm{x}_{n+1})$, then (34) we can read as $\lim_{n\to\infty} \inf w(x_n, x_{n+1}) = \theta +$ . Thus, there exists a subsequence $\{w(x_{n_k}, x_{n_k+1})\}_{k=0}^{\infty}$ of $\{w(x_n, x_{n+1})\}_{n=0}^{\infty}$ such that

$$\lim_{k\to\infty} w(x_{n_k}, x_{n_k+1}) = \theta + .$$

Now by (15) we have

$$\lim_{w(x_{n_k}, x_{n_k+1})\to\theta+} \sup[\varphi(w(x_{n_k}, x_{n_k+1}))]^s < 1. \tag{39}$$

From (31),

$$w(x_{n_k+1}, T(x_{n_k+1})) \leq [\varphi(w(x_{n_k}, x_{n_k+1}))]^s w(x_{n_k}, T(x_{n_k})).$$

Taking the limit as $k \to \infty$ and using (32), we get

$$\begin{aligned}
\delta &= \lim_{k\to\infty} \sup w(x_{n_k+1}, T(x_{n_k+1})) \\
&\leq (\lim_{k\to+\infty} \sup[\varphi(w(x_{n_k}, x_{n_k+1}))]^s)(\lim_{k\to\infty} \sup w(x_{n_k}, T(x_{n_k}))) \\
&= \left(\lim_{w(x_{n_k}, x_{n_k+1})\to\theta+} \sup[\varphi(w(x_{n_k}, x_{n_k+1}))]^s\right)\delta.
\end{aligned}$$

If we suppose that $\delta > 0$, then from this inequality we have

$$1 \leq \lim_{w(x_{n_k}, x_{n_k+1})\to\theta+} \sup[\varphi(w(x_{n_k}, x_{n_k+1}))]^s,$$

a contradiction with (39). Thus $\delta = 0$. Then from (32) and (33) we get

$$\lim_{n\to\infty} w(x_n, x_{n+1}) = 0 + . \tag{40}$$

Now by (40) and (32) we have

$$\alpha = \lim_{w(x_n, x_{n+1})\to 0+} \sup[\varphi(w(x_n, x_{n+1}))]^s < 1.$$

Proceeding as in the proof of Theorem 2.1, one can prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and that

$$z = \lim_{n\to\infty} x_n$$

is a fixed point of $T$. □

**Remark 2.5.** (a) If $r = \frac{1}{2}$ in Theorem 2.3, we obtain Theorem 2.3 [25].

(b) If $r = \frac{1}{2}$ and $w = d$ in Theorem 2.3, we obtain Theorem 1.8.

Following the proof of Theorem 2.4, we get the following extension of Theorem 2.5 in [25].

**Theorem 2.6.** Suppose that all the hypotheses of Theorem 2.4 except the function $f : X \to \mathbb{R}$, defined by $f(x) = w(x, T(x)), x \in X$, is lower semi-continuous hold. Assume that

$$\inf\{w(x, v) + w(x, T(x)) : x \in X\} > 0,$$

for every $v \in X$ with $v \notin T(v)$. Then $Fix(T) \neq \emptyset$.

The ordered pair $(T, I)$ of two self mappings of a metric space $(X, d)$ is called a *Banach operator pair*, if the set $F(I)$ is $T$-invariant, namely $T(F(I)) \subseteq F(I)$. Obviously, commuting pair $(T, I)$ is a Banach operator pair but the converse does not hold, in general; see [4, 11] and examples below. If $(T, I)$ is a Banach operator pair, then $(I, T)$ need not be a Banach operator pair (cf. Example 1 [4]). Recently, Chen and Li [4] introduced the class of Banach operator pairs, as a new class of noncommuting maps and it has been further studied by Hussain [11], Hussain and Cho [12], Khan and Akbar [21] and Pathak and Hussain [31]. Espinola and Hussain [8] further extended this concept to the case when $T$ is multivalued as follows(see also [13, 14] and references therein).

Let $I : X \to X$ and $T : X \to 2^X$ with $T(x) \neq \emptyset$ for $x \in X$. The ordered pair $(T, I)$ is a *Banach operator pair* if $T(x) \subseteq F(I)$ for each $x \in F(I)$. Further $I$ and $T$ are said to be *commuting mappings* if $I(T(x)) = T(I(x))$ for all $x \in X$ and $I$ and $T$ are *weakly compatible mappings* if $I$ and $T$ commute on each $x$ in the set $C(I, T) := \{x \in X : Ix \in Tx\}$ of coincidence points of $I$ and $T$.

**Example 2.7.** Let $X = \mathbb{R}$ with usual norm and $C = [1, \infty)$. Let $T(x) = \{x^2\}$ and $I(x) = 2x - 1$, for all $x \in C$. Then $F(I) = \{1\}$. Note that $(T, I)$ is a Banach operator pair but $T$ and $I$ are not commuting.

**Example 2.8.** Let $X = [0, 1]$ with the usual metric $d$. Define a $w$-distance function $w : X \times X \to [0, \infty)$, by

$$w(x, y) = y \quad for \quad all \quad x,\ y \in X.$$

Let $T : X \to Cl(X)$ be defined by:

$$T(x) = \begin{cases} \{\frac{1}{2}x^2\}, & \text{for } x \in [0, \frac{15}{32}) \cup (\frac{15}{32}, 1], \\ \{\frac{17}{96}, \frac{1}{4}, 1\}, & \text{for } x = \frac{15}{32}. \end{cases}$$

Define $I : X \to X$ by:

$$I(x) = \begin{cases} 0, & \text{for } x \in [0, \frac{15}{32}) \cup (\frac{15}{32}, 1], \\ 1, & \text{for } x = \frac{15}{32}. \end{cases}$$

Then $F(I) = \{0\}$ and $T(0) = \{0\} \subseteq F(I)$ imply that $(T, I)$ is a Banach operator pair. Further, $I(\frac{15}{32}) = 1 \in T(\frac{15}{32})$ but $TI(\frac{15}{32}) = T(1) = \{\frac{1}{2}\}$ and $IT(\frac{15}{32}) = I(\{\frac{17}{96}, \frac{1}{4}, 1\}) = \{0\}$ are mutually disjoint. Thus $T$ and $I$ are not weak compatible.

As an application of Theorem 2.1, we obtain following common fixed point result for Banach operator pair.

**Theorem 2.9.** *Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$, $I$ be a mapping from $X$ into $X$ and $(T, I)$ be a Banach operator pair. Suppose that $F(I)$ is nonempty and closed subset of $(X, d)$, the function $f : X \to \mathbb{R}$, defined by $f(x) = w(Ix, T(x)), x \in X$, is lower semi-continuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\limsup_{p \to t+} \varphi(p) < 1 \quad \text{for each} \quad t \in [0, \infty) \tag{41}$$

*such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$[\varphi(f(x))]^r w(x, y) \leq f(x), \quad \text{where} \quad 0 < r < 1 \tag{42}$$

*and*

$$f(y) \leq \varphi(f(x)) w(x, y). \tag{43}$$

*Then there exists $z \in X$ such that $f(z) = 0$. Further, if $w(z, z) = 0$, then $z = Iz \in Tz$.*

*Proof.* Notice that $F(I)$ being closed subset of complete space $(X, d)$ is complete. Since the pair $(T, I)$ is a Banach operator pair, $T(x) \subset F(I)$ for each $x \in F(I)$, and therefore, $T(x) \cap F(I) \neq \emptyset$ for $x \in F(I)$. The mapping $T(.) \cap F(I) : F(I) \to Cl(F(I))$ being the restriction of $T$ on $F(I)$ satisfies all the conditions of Theorem 2.1. Now the result follows as in Theorem 2.1. $\square$

**Corollary 2.10.** *Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $Cl(X)$, $I$ be a mapping from $X$ into $X$ and $(T, I)$ be a Banach operator pair. Suppose that $F(I)$ is nonempty and closed subset of $(X, d)$, the function $f : X \to \mathbb{R}$, defined by $f(x) = d(Ix, T(x)), x \in X$, is lower semi-continuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\limsup_{p \to t+} \varphi(p) < 1 \text{ for each } t \in [0, \infty) \tag{44}$$

*such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

$$[\varphi(f(x))]^r d(x, y) \leq f(x), \text{ where } 0 < r < 1 \tag{45}$$

*and*

$$f(y) \leq \varphi(f(x))d(x, y). \tag{46}$$

*Then there exists $z \in X$ such that $z = Iz \in Tz$.*

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