Filomat 28:1 (2014), 119–129
DOI 10.2298/FIL1401119F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Degree Hypergroupoids Associated with Hypergraphs

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Abstract. In this paper, we present some connections between graph theory and hyperstructure theory. In this regard, we construct a hypergroupoid by defining a hyperoperation on the set of degrees of vertices of a hypergraph and we call it a degree hypergroupoid. We will see that the constructed hypergroupoid is always an H_v-group. We will investigate some conditions on a degree hypergroupoid to have a hypergroup. Further, we study the degree hypergroupoid associated with Cartesian product of hypergraphs. Finally, the fundamental relation and complete parts of a degree hypergroupoid are studied.

1. Introduction and preliminaries

The notion of hypergraph has been introduced around 1960 as a generalization of graph and one of the initial concerns was to extend some classical results of graph theory. In [2], there is a very good presentation of graph and hypergraph theory.

A hypergraph is a generalization of a graph in which an edge can connect any number of vertices. Formally, a *hypergraph* is a pair $\Gamma = (X, E)$, where X is a finite set of vertices and $E = \{E_1, \ldots, E_m\}$ is a set of hyperedges which are non-empty subsets of X. Figure 1 is an example of a hypergraph with 7 vertices and 4 hyperedges.

A hypergraph $\Gamma' = (X', E')$ is a subhypergraph of $\Gamma = (X, E)$ if $X' \subseteq X$ and $E' \subseteq E$. We note that every graph can be considered as a hypergraph. We denote the set of vertices of a graph G by $V(G)$. A *simple graph* is an undirected graph that has no loops (edges connected at both ends to the same vertex) and no more than one edge between any two different vertices. A *complete graph* is a simple graph with n vertices and an edge between every two vertices. We use the symbol K_n for a complete graph with n vertices. A *star graph* with n edges is a graph $S_n = (X, E)$ in which $X = \{x\} \cup \{x_1, \ldots, x_n\}$ and $E = \{x_i x \mid 1 \leq i \leq n\}$. x is called the center vertex of S_n.

Let $\Gamma = (X, E)$ be a hypergraph and $x, y \in X$. A hyperedge sequence $(E_1, \ldots, E_k)$ is called a *path of length k* from x to y if the following conditions are satisfied:

(1) $x \in E_1$ and $y \in E_k$,

(2) $E_i \neq E_j$ for $i \neq j$,

(3) $E_i \cap E_{i+1} \neq \emptyset$ for $1 \leq i \leq k-1$.

2010 *Mathematics Subject Classification*. Primary 20N20; Secondary 05C65
Keywords. Hypergraph, Hypergroupoid, Hypergroup, Fundamental relation
Received: 02 September 2013; Accepted: 11 November 2013
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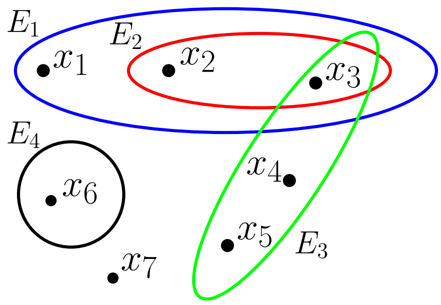

Figure 1: A hypergraph with 7 vertices and 4 hyperedges

We contract out there is a path of length zero between x and x.

In a hypergraph Γ, two vertices x and y are called *connected* if Γ contains a path from x to y. If two vertices are connected by a path of length 1, i.e. by a single hyperedge, the vertices are called *adjacent*. We use the notation $x — y$ to denote the adjacency of vertices x and y. A hypergraph is said to be *connected* if every pair of vertices in the hypergraph is connected. A *connected component* of a hypergraph is any maximal set of vertices which are pairwise connected by a path.

The length of shortest path between vertices x and y is denoted by $\mathrm{dist}(x, y)$ and the *diameter* of Γ is defined as follows:

$$\mathrm{diam}(\Gamma) = \begin{cases} max\{\mathrm{dist}(x, y) \mid x, y \in X\} & \text{if } \Gamma \text{ is connected,} \\ \infty & \text{otherwise.} \end{cases}$$

The hyperstructure theory was born in 1934, when Marty introduced the notion of a hypergroup [18]. Since then, many papers and several books have been written on this topic (see for instance [5–9, 20]). Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a map $* : H \times H \longrightarrow P^*(H)$ and the structure $(H, *)$ is called a *hypergroupoid*. A hypergroupoid $(H, *)$ is called *commutative* if for all $x, y \in H$ we have $x * y = y * x$. A hypergroupoid $(H, *)$ is called a *quasihypergroup* if for all x in H we have $x * H = H * x = H$, which means that $\bigcup\limits_{u \in H} x * u = \bigcup\limits_{v \in H} v * x = H$. A quasihypergroup $(H, *)$ is called

(1) a *hypergroup* if $*$ is associative, i.e., for all x, y, z of H we have $(x * y) * z = x * (y * z)$,

(2) an *H_v-group* if for all x, y, z of H we have $(x * y) * z \cap x * (y * z) \neq \emptyset$.

A hypergroup $(H, *)$ is called a *total hypergroup* if $x * y = H$, for all x, y of H. A non-empty subset K of a hypergroup $(H, *)$ is called a *subhypergroup* if for all x of K we have $x * K = K * x = K$.

Let $(H, *)$ and $(K, \diamond)$ be two hypergroupoids. A map $\varphi : H \longrightarrow K$ is called

(1) an *inclusion homomorphism* if for all $x, y \in H$ we have $\varphi(x * y) \subseteq \varphi(x) \diamond \varphi(y)$,

(2) a *homomorphism* if for all $x, y \in H$ we have $\varphi(x * y) = \varphi(x) \diamond \varphi(y)$.

If there exists a one to one (inclusion) homomorphism of H onto K, then we say that H is (inclusion) isomorphic to K and we write $(H \overset{i}{\cong} K)$ $H \cong K$.

The connections between hyperstructure theory and graph theory have been analyzed by many researchers (see for instance [1, 3, 4, 10, 11, 14, 16, 17, 19]). In [4], Corsini considered a hypergraph $\Gamma = (H, \{E_i\}_i)$ and constructed a hypergroupoid $H_\Gamma = (H, \circ)$ in which the hyperoperation $\circ$ on H has defined as follows:

$$\forall x, y \in H^2,\ x \circ y = E(x) \cup E(y),$$

where $E(x) = \bigcup\{E_i \mid x \in E_i\}$. Corsini proved that:

Theorem 1.1. *The hypergroupoid H_Γ satisfies for each $(x, y) \in H^2$ the following conditions:*

(1) $x \circ y = x \circ x \cup y \circ y$,

(2) $x \in x \circ x$,

(3) $y \in x \circ x \Longleftrightarrow x \in y \circ y$.

Also, he proved that:

Theorem 1.2. *Let $(H, \circ)$ be a hypergroupoid satisfying* (1), (2) *and* (3) *of Theorem 1.1. Then, $(H, \circ)$ is a hypergroup if and only if the following condition is valid:*

$$\forall (a, c) \in H^2, \quad c \circ c \circ c - c \circ c \subseteq a \circ a \circ a.$$

2. Degree hypergroupoids

Let $\Gamma = (X, \{E_i\}_i)$ be a hypergraph. For each $x \in X$ we define the degree $\deg(x)$ of x to be the number of hyperedges containing x. A hypergraph in which all vertices have the same degree is said to be *regular*. We define the degree neighborhood of x as follows:

$$D(x) = \bigcup\{E_i \mid \exists\ y \in E_i \text{ such that } \deg(y) = \deg(x)\}.$$

It is easy to check that $D(x) = D(y)$ if $\deg(x) = \deg(y)$. The set of all degrees of vertices of Γ will be denoted by D_Γ. For each $d \in D_\Gamma$ we define $\deg^{-1}(d) = \{x \in X \mid \deg(x) = d\}$. The hypergroupoid $(D_\Gamma, \circ_\Gamma)$ where the hyperoperation $\circ_\Gamma$ is defined by

$$\forall d, d' \in D_\Gamma, \quad d \circ_\Gamma d' = \{\deg(z) \mid z \in D(x) \text{ for some } x \in \deg^{-1}(d) \cup \deg^{-1}(d')\},$$

is called a *degree hypergroupoid*.

We note that if Γ is a hypergraph and Γ' is a connected component of Γ, then for each $d, d' \in D_{\Gamma'}$ we have $d \circ_{\Gamma'} d' \subseteq d \circ_\Gamma d'$.

Theorem 2.1. *The degree hypergroupoid $(D_\Gamma, \circ_\Gamma)$ has the following properties for each $d, d' \in D_\Gamma$:*

(1) $d \circ_\Gamma d' = d \circ_\Gamma d \cup d' \circ_\Gamma d'$ *(whence $d \circ_\Gamma d' = d' \circ_\Gamma d$),*

(2) $d \in d \circ_\Gamma d$,

(3) $d \in d' \circ_\Gamma d' \Longleftrightarrow d' \in d \circ_\Gamma d$.

By the above theorem we conclude that $\{d, d'\} \subseteq d \circ_\Gamma d'$ for all $d, d' \in D_\Gamma$ and so for all $d, d', d'' \in D_\Gamma$ we have $\{d, d', d''\} \subseteq (d \circ_\Gamma d') \circ_\Gamma d'' \cap d \circ_\Gamma (d' \circ_\Gamma d'')$. On the other hand, for all $d \in D_\Gamma$ we have $d \circ_\Gamma D_\Gamma = D_\Gamma$. These ones imply that every degree hypergroupoid is an H_v-group. Also, by using Theorem 1b of [4], for all $d \in D_\Gamma$ we have $d \circ_\Gamma d \circ_\Gamma d = \bigcup\limits_{d' \in d \circ_\Gamma d} d' \circ_\Gamma d'$.

Corollary 2.2. *Let Γ be a hypergraph and $d_j \in D_\Gamma$. If $d_i \circ_\Gamma d_i = D_\Gamma$, for every $d_i \in D_\Gamma - \{d_j\}$, then $d_j \circ_\Gamma d_j = D_\Gamma$.*

Proposition 2.3. *If Γ is a hypergraph with regular connected components, then for every $d, d' \in D_\Gamma$ we have $d \circ_\Gamma d' = \{d, d'\}$ and so $(D_\Gamma, \circ_\Gamma)$ is a hypergroup.*

Corollary 2.4. *Γ is a regular hypergraph if and only if D_Γ is a singleton set.*

Corollary 2.5. *If Γ is a hypergraph and $D_\Gamma = \{d_1, d_2\}$ and $d_1 \circ_\Gamma d_1 \neq \{d_1\}$, then $(D_\Gamma, \circ_\Gamma)$ is a total hypergroup.*

Proposition 2.6. *If Γ is a hypergraph and $|D_\Gamma| = 3$, then $(D_\Gamma, \circ_\Gamma)$ is a hypergroup.*

Proof. It suffices to show that $\circ_\Gamma$ is associative. Let $d_1, d_2, d_3 \in D_\Gamma$ be arbitrary elements. If $d_1 = d_2 = d_3$, then commutativity of $\circ_\Gamma$ implies that $(d_1 \circ_\Gamma d_2) \circ_\Gamma d_3 = d_1 \circ_\Gamma (d_2 \circ_\Gamma d_3)$. We show that $(d_1 \circ_\Gamma d_1) \circ_\Gamma d_2 = d_1 \circ_\Gamma (d_1 \circ_\Gamma d_2)$ where $d_1 \neq d_2$. If $d_1 \circ_\Gamma d_2 = D_\Gamma$, then the result is obvious. Suppose that $d_1 \circ_\Gamma d_2 = \{d_1, d_2\}$. Surely $d_3 \notin d_1 \circ_\Gamma d_1$. We have the following two cases.

Case 1: Let $d_1 \circ_\Gamma d_1 = \{d_1\}$. Then,

$$d_1 \circ_\Gamma (d_1 \circ_\Gamma d_2) = d_1 \circ_\Gamma \{d_1, d_2\} = d_1 \circ_\Gamma d_1 \cup d_2 \circ_\Gamma d_2 = d_1 \circ_\Gamma d_2 = (d_1 \circ_\Gamma d_1) \circ_\Gamma d_2.$$

Case 2: Let $d_1 \circ_\Gamma d_1 = \{d_1, d_2\}$. Then

$$d_1 \circ_\Gamma (d_1 \circ_\Gamma d_2) = d_1 \circ_\Gamma \{d_1, d_2\} = d_1 \circ_\Gamma d_1 \cup d_2 \circ_\Gamma d_2 = d_1 \circ_\Gamma d_2 = \{d_1, d_2\} \circ_\Gamma d_2 = (d_1 \circ_\Gamma d_1) \circ_\Gamma d_2.$$

If d_1, d_2, d_3 are distinct elements of D_Γ, then $(d_1 \circ_\Gamma d_2) \circ_\Gamma d_3 = d_1 \circ_\Gamma (d_2 \circ_\Gamma d_3) = D_\Gamma$ which completes the proof. □

Lemma 2.7. *Let $\Gamma = (X, \{E_i\}_i)$ be a hypergraph such that $D(x) \cap D(y) \neq \emptyset$ for all $x, y \in X$. Then, for all $d \in D_\Gamma$ we have $d \circ_\Gamma d \circ_\Gamma d = D_\Gamma$.*

Proof. Let d be an arbitrary element of D_Γ. We show that $r \in d \circ_\Gamma d \circ_\Gamma d$ for all $r \in D_\Gamma$. Let $r = \deg(x)$ and $d = \deg(y)$. By assumption there exists $w \in D(x) \cap D(y)$. Since $w \in D(x)$, there exists an edge E_i containing a vertex w' of degree r such that $w \in E_i$. Similarly, there exists an edge E_j containing a vertex w'' of degree d such that $w \in E_j$. If $\deg(w) = k$, then by definition of $\circ_\Gamma$ we have $k \in d \circ_\Gamma d$ and $r \in k \circ_\Gamma k$. Since $k \circ_\Gamma k \subseteq k \circ_\Gamma d$ we have $r \in k \circ_\Gamma d \subseteq d \circ_\Gamma d \circ_\Gamma d$. □

Theorem 2.8. *Let $\Gamma = (X, \{E_i\}_i)$ be a hypergraph such that $D(x) \cap D(y) \neq \emptyset$ for all $x, y \in X$. Then, $(D_\Gamma, \circ_\Gamma)$ is a hypergroup.*

Proof. Use Lemma 2.7 and Theorem 1.2. □

Corollary 2.9. *If Γ is a hypergraph with $diam(\Gamma) \leq 2$, then $(D_\Gamma, \circ_\Gamma)$ is a hypergroup.*

Lemma 2.10. *Let Γ be a connected hypergraph and $D_\Gamma = \{d_1, d_2, d_3\}$. Then,*

(1) there exists $i \in \{1, 2, 3\}$ such that $d_i \circ_\Gamma d_i = D_\Gamma$,

(2) for every $i, j \in \{1, 2, 3\}$ with $i \neq j$ we have $d_i \circ_\Gamma d_j = D_\Gamma$.

Proof. (1) Since Γ is connected, we have $d_1 \circ_\Gamma d_1 - \{d_1\} \neq \emptyset$. Without loss of generality, assume $d_2 \in d_1 \circ_\Gamma d_1$. If y is a vertex of degree d_3, then connectivity of Γ implies that there exists a vertex x of degree d_1 or d_2 such that $y \in D(x)$. If $\deg(x) = d_1$ then we have $d_1 \circ_\Gamma d_1 = D_\Gamma$ otherwise we have $d_2 \circ_\Gamma d_2 = D_\Gamma$.

Proof of (2) is straightforward. □

Corollary 2.11. *Let Γ be a connected hypergraph and $D_\Gamma = \{d_1, d_2, d_3\}$. Then, $(D_\Gamma, \circ_\Gamma)$ is a total hypergroup or a hypergroup with the following table:*

$\circ_\Gamma$	d_1	d_2	d_3
d_1	$D_\Gamma - \{d_2\}$	D_Γ	D_Γ
d_2	D_Γ	$D_\Gamma - \{d_1\}$	D_Γ
d_3	D_Γ	D_Γ	D_Γ

Proposition 2.12. *Let $\Gamma = (X, \{E_i\}_i)$ be a hypergraph and $D_\Gamma = \{d_1, d_2\}$. Then, the following assertions are equivalent:*

(1) $d_1 \circ_\Gamma d_1 = \{d_1\}$,

(2) $d_2 \circ_\Gamma d_2 = \{d_2\}$,

(3) Γ is not connected,

(4) $(D_\Gamma, \circ_\Gamma)$ is a hypergroup with the following table:

$\circ_\Gamma$	d_1	d_2
d_1	$\{d_1\}$	$\{d_1, d_2\}$
d_2	$\{d_1, d_2\}$	$\{d_2\}$

Remark 2.13. *By Proposition 2.6, the degree hypergroupoid $(D_\Gamma, \circ_\Gamma)$ associated with the graph Γ of Figure 2 is a hypergroup and $D(x) \cap D(y) = \emptyset$. This shows that the converse of Theorem 2.8 is not true.*

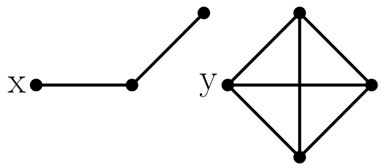

Figure 2: Γ

3. Degree graph of a hypergraph

For any hypergraph Γ, we construct its **degree graph**, denoted by G_Γ, as follows: the vertex set of G_Γ is the set of degrees of vertices of Γ, that is D_Γ, and vertices $d, d' \in D_\Gamma$ are adjacent if and only if there exists a hyperedge in Γ containing vertices x, y with $\deg(x) = d$ and $\deg(y) = d'$. It is clear that a degree graph is always a simple graph. For a vertex d in G_Γ, $\mathcal{N}_{G_\Gamma}(d)$ is the neighborhood of d that is the set of all vertices in G_Γ which are adjacent to d (each vertex is adjacent to itself). It is easy to check that for every $d \in D_\Gamma$ we have $d \circ_\Gamma d = \mathcal{N}_{G_\Gamma}(d)$ and $d \circ_\Gamma d \circ_\Gamma d = \bigcup_{u \in \mathcal{N}_{G_\Gamma}(d)} \mathcal{N}_{G_\Gamma}(u)$. Therefore, if γ is a connected component of G_Γ, then for every $d \in V(\gamma)$ we have $d \circ_\Gamma d \circ_\Gamma d \subseteq V(\gamma)$.

Lemma 3.1. *If Γ is a connected hypergraph, then G_Γ is a connected graph.*

As the following figure shows, the converse of Lemma 3.1 does not hold in general.

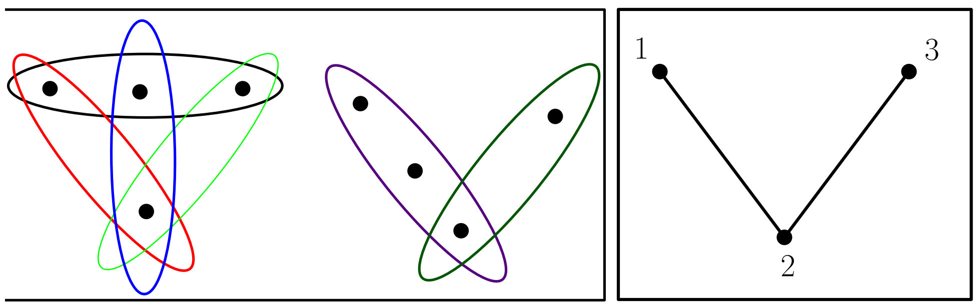

Figure 3: A disconnected hypergraph and it's degree graph

A question that comes to mind after defining degree graph is the following: Given a graph G with non-negative integer vertices, is there any hypergraph Γ such that G is the degree graph of Γ? The answer is **"yes"** as can be seen in Theorem 3.3.

Lemma 3.2. *Let d, d' be two distinct non-negative integers and let G be a simple graph on two vertices d and d'. Then, there exists a graph Γ such that $G_\Gamma = G$.*

Proof. We may assume without loss of generality that $d' < d$. If G has no edge, then Γ can be a graph containing two connected components K_d and $K_{d'}$. So, let the graph G be not null. We first prove the lemma in the case that d and d' are odd. Consider the star graph S_d with center vertex x and edges $x_1 x, \ldots, x_d x$. For $i = 1, \ldots, d$, we add the edges $x_i x_{\overline{i+1}}, x_i x_{\overline{i+2}}, \ldots, x_i x_{\overline{i+\frac{d'-1}{2}}}$ to the edges of S_d, where $\overline{i+r} \equiv i + r \pmod d$ and $1 \leq \overline{i+r} \leq d$, for each $1 \leq r \leq \frac{d'-1}{2}$. We denote the resulting graph by $G_{d',d}$. Obviously, degree graph of $G_{d',d}$ is G. Now, we prove the lemma in other cases. In the case that d' is odd and d is even, if $d' = d-1$, then it is sufficient to duplicate K_d and connect two vertices of degree $d-1$, otherwise we construct our desired graph by duplicating the graph $G_{d',d-1}$ and adding an edge between the vertices of degree $d-1$. In the case that d' is even and d is odd, we duplicate the graph $G_{d'-1,d}$ and for $i = 1, \ldots, d$, we add the edges $x_i x'_i$ where x'_i is duplicated vertex of x_i. Finally, whenever d' and d are even first we duplicate the graph $G_{d'-1,d-1}$ and then we connect the vertices of degree $d-1$ and for $i = 1, \ldots, d-1$, we add the edges $x_i x'_i$ where x'_i is duplicated vertex of x_i. □

Theorem 3.3. *If G is a simple graph such that its vertex set is a subset of non-negative integers, then there exists a graph Γ such that $G_\Gamma = G$.*

Proof. Suppose that the edge set of G is $\{E_1, \ldots, E_m\}$. By Lemma 3.2, for each E_i, $1 \leq i \leq m$, there exists a graph Γ_i such that G_{Γ_i} covers E_i. By putting Γ_i's together we will have a graph whose degree graph is G. □

Theorem 3.4. *Let $(H, *)$ be a finite hypergroupoid satisfying (1), (2), (3) of Theorem 1.1. Then, there exists a graph Γ such that $(D_\Gamma, \circ_\Gamma) \cong (H, *)$.*

Proof. Let $H = \{a_1, \ldots, a_n\}$ and let G be a graph with $V(G) = \{1, \ldots, n\}$ and $E(G) = \{ij \mid i \neq j \text{ and } a_i \in a_j * a_j\}$. By Theorem 3.3, there exists a graph Γ such that $G_\Gamma = G$. Clearly, $f : D_\Gamma \longrightarrow H$ defined by $f(i) = a_i$ is an isomorphism. □

Lemma 3.5. *If $diam(G_\Gamma) \leq 2$, then for every $d \in D_\Gamma$ we have $d \circ_\Gamma d \circ_\Gamma d = D_\Gamma$.*

Proof. Clearly, for every $d \in D_\Gamma$ we have $d \circ_\Gamma d \circ_\Gamma d \subseteq D_\Gamma$. Let $d' \in D_\Gamma$ be an arbitrary element. If d, d' are adjacent vertices in G_Γ, then we have $d' \in d \circ_\Gamma d \subseteq d \circ_\Gamma d \circ_\Gamma d$. Otherwise, by assumption there exists a vertex $d'' \in V(G_\Gamma)$ which is adjacent to d' and d. Since $d \circ_\Gamma d \circ_\Gamma d = \bigcup\limits_{z \in d \circ_\Gamma d} z \circ_\Gamma z$ and $d'' \in d \circ_\Gamma d$, we have $d' \in d'' \circ_\Gamma d'' \subseteq d \circ_\Gamma d \circ_\Gamma d$. □

Lemma 3.6. *Let G_Γ be connected. If $3 \leq diam(G_\Gamma)$, then $(D_\Gamma, \circ_\Gamma)$ is not a hypergroup.*

Proof. By assumption, there exists a path $(\{d_1, d_2\}, \{d_2, d_3\}, \{d_3, d_4\})$ in G_Γ such that d_1, d_3 and d_2, d_4 and d_1, d_4 are not adjacent vertices. It is not difficult to see that $d_4 \notin (d_1 \circ_\Gamma d_1) \circ d_2$ whereas $d_4 \in d_1 \circ_\Gamma (d_1 \circ_\Gamma d_2)$. This shows that $\circ_\Gamma$ is not associative and so $(D_\Gamma, \circ_\Gamma)$ is not a hypergroup. □

Theorem 3.7. *If G_Γ is connected, then a necessary and sufficient condition for $(D_\Gamma, \circ_\Gamma)$ to be a hypergroup is $diam(G_\Gamma) \leq 2$.*

Proof. If $(D_\Gamma, \circ_\Gamma)$ is a hypergroup, then by Lemma 3.6 we have $\text{diam}(G_\Gamma) \leq 2$. Conversely, if $\text{diam}(G_\Gamma) \leq 2$, then by Lemma 3.5 and Theorem 1.2, $(D_\Gamma, \circ_\Gamma)$ is a hypergroup. □

Corollary 3.8. *If $(D_\Gamma, \circ_\Gamma)$ is a hypergroup, then the diameter of every connected component of G_Γ is less than or equal to 2.*

Corollary 3.9. *If $(D_\Gamma, \circ_\Gamma)$ is a hypergroup and γ is a connected component of G_Γ, then $(D_\gamma, \circ_\Gamma)$ is a subhypergroup of $(D_\Gamma, \circ_\Gamma)$.*

Theorem 3.10. *If $diam(G_\Gamma) = \infty$, then $(D_\Gamma, \circ_\Gamma)$ is a hypergroup if and only if the diameter of every connected component of G_Γ is less than or equal to 1.*

Proof. Let $(D_\Gamma, \circ_\Gamma)$ be a hypergroup. By Corollary 3.8, the diameter of every connected component of G_Γ is less than or equal to 2. Suppose that G_Γ has a connected component G_{Γ_1} of diameter 2. If d, d' are non-adjacent vertices in G_{Γ_1}, then $d' \in d \circ_\Gamma d \circ_\Gamma d - d \circ_\Gamma d$. Now, if $d'' \in V(G_\Gamma) - V(G_{\Gamma_1})$, then $d \circ_\Gamma d \circ_\Gamma d - d \circ_\Gamma d \not\subseteq d'' \circ_\Gamma d'' \circ_\Gamma d''$. This contradicts the Theorem 1.2. Conversely, assume that diameter of every connected component of G_Γ is less than or equal to 1. Then, for each $d \in V(G_\Gamma)$ we have $d \circ_\Gamma d \circ_\Gamma d = d \circ_\Gamma d$ and so by Theorem 1.2, $(D_\Gamma, \circ_\Gamma)$ is a hypergroup. □

4. Direct product of degree hypergroupoids

Let $\Gamma = (X, E)$ and $\Gamma' = (X', E')$ be two hypergraphs, where $E = \{E_1, \ldots, E_m\}$ and $E' = \{E'_1, \ldots, E'_n\}$. We define their product to be the hypergraph $\Gamma \times \Gamma'$ whose vertices set is $X \times X'$ and whose hyperedges are the sets $E_i \times E'_j$ with $1 \leq i \leq m$, $1 \leq j \leq n$. It is easy to see that for every $(x, y) \in X \times X'$ we have $\deg((x, y)) = \deg(x)\deg(y)$.

Lemma 4.1. *Vertices d and d' of $G_{\Gamma\times\Gamma'}$ are adjacent if and only if there exist $r, r' \in V(G_\Gamma)$ and $s, s' \in V(G_{\Gamma'})$ such that r—r', s—s', $d = rs$ and $d' = r's'$.*

Proof. Suppose that $d, d' \in V(G_{\Gamma\times\Gamma'})$ are adjacent. By definition, there exists a hyperedge $E_i \times E'_j$ containing vertices (x, y) and (u, v) with $d((x, y)) = d$ and $d((u, v)) = d'$. Now, it is sufficient to assume $r = \deg(x)$, $s = \deg(y)$, $r' = \deg(u)$ and $s' = \deg(v)$. Conversely, since r, r' are adjacent in G_Γ there exists a hyperedge E_i containing vertices x and y with $\deg(x) = r$ and $\deg(y) = r'$. Similarly, since s, s' are adjacent in $G_{\Gamma'}$ there exists a hyperedge E'_j containing vertices u and v with $\deg(u) = s$ and $\deg(v) = s'$. Therefore, by definition the vertices d and d' of $G_{\Gamma\times\Gamma'}$ are adjacent. □

Lemma 4.2. *If G_Γ and $G_{\Gamma'}$ are degree graphs of hypergraphs Γ and Γ' respectively, then the diameter of $G_{\Gamma\times\Gamma'}$ is less than or equal to $Max\{diam(G_\Gamma), diam(G_{\Gamma'})\}$.*

Proof. Assume that $m = \mathrm{Max}\{\mathrm{diam}(G_\Gamma), \mathrm{diam}(G_{\Gamma'})\}$. Whenever $m = \infty$, there is nothing to prove. For the case $m < \infty$, let d and d' be arbitrary vertices of $G_{\Gamma\times\Gamma'}$. By Lemma 4.1, there exist $r, r' \in V(G_\Gamma)$ and $s, s' \in V(G_{\Gamma'})$ such that r—r', s—s', $d = rs$ and $d' = r's'$. Since $\mathrm{diam}(G_\Gamma) \leq m$, there exist vertices $r_1, \ldots, r_{m-1} \in V(G_\Gamma)$ (not necessarily distinct) such that r—r_1— … —r_{m-1}—r'. Reasoning in the same way, since $\mathrm{diam}(G_{\Gamma'}) \leq m$, there exist vertices $s_1, \ldots, s_{m-1} \in V(G_{\Gamma'})$ (not necessarily distinct) such that s—s_1— … —s_{m-1}—s'. Now, by using Lemma 4.1, we have rs —r_1s_1— … —$r_{m-1}s_{m-1}$—$r's'$. This means that there exists a path from d to d' of length less than or equal to m. Therefore, we have $\mathrm{diam}(G_{\Gamma\times\Gamma'}) \leq m$. □

If G_Γ and $G_{\Gamma'}$ are connected, then by Lemma 4.2, $G_{\Gamma\times\Gamma'}$ is connected. But the converse does not hold in general, as Example 4.4 shows.

Theorem 4.3. *If $(D_\Gamma, \circ_\Gamma)$ and $(D_{\Gamma'}, \circ_{\Gamma'})$ are hypergroups with connected degree graphs, then $(D_{\Gamma\times\Gamma'}, \circ_{\Gamma\times\Gamma'})$ is a hypergroup.*

Proof. By Theorem 3.7, we have $\mathrm{diam}(G_\Gamma) \leq 2$ and $\mathrm{diam}(G_{\Gamma'}) \leq 2$ and so by Lemma 4.2, we have $\mathrm{diam}(G_{\Gamma\times\Gamma'}) \leq 2$. Therefore, by Theorem 3.7, $(D_{\Gamma\times\Gamma'}, \circ_{\Gamma\times\Gamma'})$ is a hypergroup. □

The following example shows that the above result need not be true if G_Γ or $G_{\Gamma'}$ are not connected.

Example 4.4. *By Theorem 3.3, there exist graphs* Γ *and* Γ' *with the following degree graphs:*

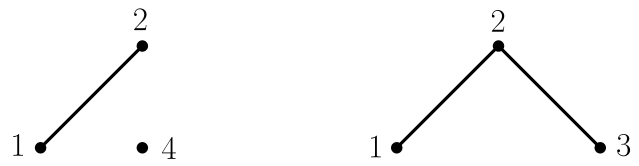

Figure 4: Degree graphs of G_Γ and $G_{\Gamma'}$.

By Proposition 2.6, $(D_\Gamma, \circ_\Gamma)$ *and* $(D_{\Gamma'}, \circ_{\Gamma'})$ *are hypergroups. By using Lemma 4.1, degree graph of* $\Gamma \times \Gamma'$ *has the following figure:*

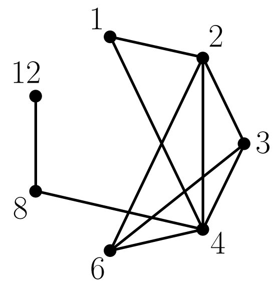

Figure 5: Degree graph of $G_{\Gamma\times\Gamma'}$.

Since diam$(G_{\Gamma\times\Gamma'}) = 3$*, by Theorem 3.7,* $(D_{\Gamma\times\Gamma'}, \circ_{\Gamma\times\Gamma'})$ *is not a hypergroup.*

The following example shows that the converse of Theorem 4.3 is not true.

Example 4.5. *Consider the following figure:*

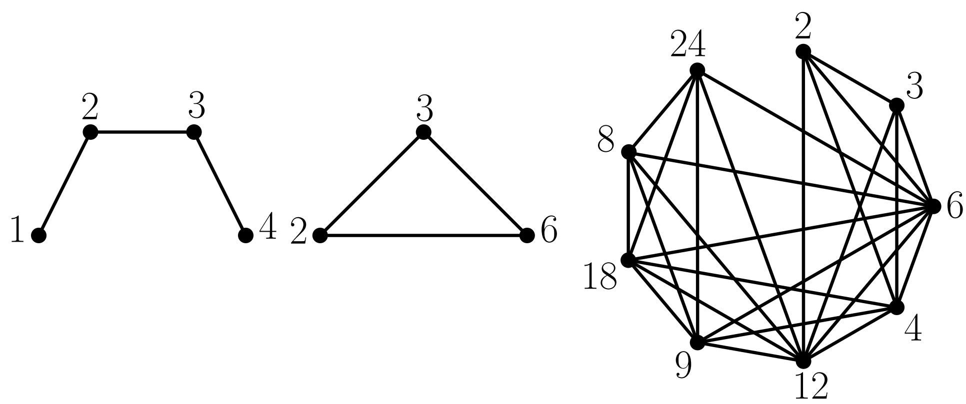

Figure 6: Degree graphs of G_Γ, $G_{\Gamma'}$ and $G_{\Gamma\times\Gamma'}$.

Since diam$(G_{\Gamma\times\Gamma'}) = 2$*, by Theorem 3.7,* $(D_{\Gamma\times\Gamma'}, \circ_{\Gamma\times\Gamma'})$ *is a hypergroup and since diam*$(G_\Gamma) = 3$*, by Lemma 3.6,* $(D_\Gamma, \circ_\Gamma)$ *is not a hypergroup.*

As can be seen by the Example 4.5, the inequality of Lemma 4.2 may be hold strictly. In the next lemma, we show that the equality can be hold under some conditions.

Lemma 4.6. *If Γ and Γ' are hypergraphs with connected degree graphs such that $|D_{\Gamma\times\Gamma'}| = |D_\Gamma||D_{\Gamma'}|$, then diameter of $G_{\Gamma\times\Gamma'}$ is equal to $Max\{diam(G_\Gamma), diam(G_{\Gamma'})\}$.*

Proof. By Lemma 4.2, we have $\mathrm{diam}(G_{\Gamma\times\Gamma'}) \leq \mathrm{Max}\{\mathrm{diam}(G_\Gamma), \mathrm{diam}(G_{\Gamma'})\}$. We prove the reverse inequality. Let $r, r' \in V(G_\Gamma)$ and $s, s' \in V(G_{\Gamma'})$ be arbitrary elements. Put $d = rs$ and $d' = r's'$. If $\mathrm{diam}(G_{\Gamma\times\Gamma'}) = m$, then there exist vertices $d_1, \ldots, d_{m-1} \in V(G_{\Gamma\times\Gamma'})$ (not necessarily distinct) such that $d — d_1 — \ldots — d_{m-1} — d'$. Since $|D_{\Gamma\times\Gamma'}| = |D_\Gamma||D_{\Gamma'}|$, for each $1 \leq i \leq m-1$, there are unique elements $r_i \in V(G_\Gamma)$ and $s_i \in V(G_{\Gamma'})$, such that $d_i = r_i s_i$. Also, d and d' have no other decompositions. Now, by Lemma 4.1, we have $r — r_1 — \ldots — r_{m-1} — r'$ and $s — s_1 — \ldots — s_{m-1} — s'$. This shows that $d(r, r') \leq m$ and $d(s, s') \leq m$. □

The following corollary is an immediate consequence of Theorem 3.7 and Lemma 4.6.

Corollary 4.7. *Let Γ and Γ' be hypergraphs with connected degree graphs such that $|D_{\Gamma\times\Gamma'}| = |D_\Gamma||D_{\Gamma'}|$. Then, $(D_\Gamma, \circ_\Gamma)$ and $(D_{\Gamma'}, \circ_{\Gamma'})$ are hypergroups if and only if $(D_{\Gamma\times\Gamma'}, \circ_{\Gamma\times\Gamma'})$ is a hypergroup.*

Let $(H, *)$ and $(K, \diamond)$ be hypergroupoids. We define the hyperoperation $\otimes$ on the Cartesian product $H \times K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = \{(x, y) \mid x \in x_1 * x_2 \text{ and } y \in y_1 \diamond y_2\},$$

and so $(H \times K, \otimes)$ is a hypergroupoid.

Theorem 4.8. *Let Γ and Γ' be hypergraphs such that $|D_{\Gamma\times\Gamma'}| = |D_\Gamma||D_{\Gamma'}|$. Then, $D_{\Gamma\times\Gamma'} \overset{i}{\cong} D_\Gamma \times D_{\Gamma'}$.*

Proof. By assumption, for each $d \in D_{\Gamma\times\Gamma'}$, there are unique elements $r \in D_\Gamma$ and $s \in D_{\Gamma'}$, such that $d = rs$. Consider the map $\varphi : D_{\Gamma\times\Gamma'} \longrightarrow D_\Gamma \times D_{\Gamma'}$ defined by $\varphi(rs) = (r, s)$. It is clear that φ is well-defined, one to one and onto. We show that φ is an inclusion homomorphism. Let $r_1s_1, r_2s_2 \in D_{\Gamma\times\Gamma'}$ be arbitrary elements. Then,

$$\begin{aligned}
\varphi(r_1s_1 \circ_{\Gamma\times\Gamma'} r_2s_2) &= \varphi\big(\{rs \mid rs — r_1s_1 \text{ or } rs — r_2s_2\}\big) \\
&= \{(r, s) \mid (r — r_1 \text{ and } s — s_1) \text{ or } (r — r_2 \text{ and } s — s_2)\} \\
&\subseteq \{(r, s) \mid (r — r_1 \text{ or } r — r_2) \text{ and } (s — s_1 \text{ or } s — s_2)\} \\
&= \{(r, s) \mid r \in r_1 \circ_\Gamma r_2 \text{ and } s \in s_1 \circ_{\Gamma'} s_2\} \\
&= (r_1, s_1) \otimes (r_2, s_2) \\
&= \varphi(r_1s_1) \otimes \varphi(r_2s_2).
\end{aligned}$$

□

Corollary 4.9. *Let Γ and Γ' be hypergraphs such that $|D_{\Gamma\times\Gamma'}| = |D_\Gamma||D_{\Gamma'}|$. Then, $D_{\Gamma\times\Gamma'} \cong D_\Gamma \times D_{\Gamma'}$ if G_Γ or $G_{\Gamma'}$ is a complete graph.*

It is easy to verify that if Γ and Γ' are hypergraphs such that $|D_{\Gamma\times\Gamma'}| = |D_\Gamma||D_{\Gamma'}|$, then $G_{\Gamma\times\Gamma'}$ is a complete graph if and only if G_Γ and $G_{\Gamma'}$ are complete graphs. Hence, the following corollary is an immediate consequence of Corollary 4.9.

Corollary 4.10. *Let Γ and Γ' be hypergraphs such that $|D_{\Gamma\times\Gamma'}| = |D_\Gamma||D_{\Gamma'}|$. Then, $D_{\Gamma\times\Gamma'} \cong D_\Gamma \times D_{\Gamma'}$ if $G_{\Gamma\times\Gamma'}$ is a complete graph.*

Theorem 4.11. *If $\varphi : D_{\Gamma\times\Gamma'} \longrightarrow D_\Gamma \times D_{\Gamma'}$ defined by $\varphi(rs) = (r, s)$ is an isomorphism, then G_Γ or $G_{\Gamma'}$ is a complete graph.*

Proof. Suppose that $G_{\Gamma'}$ is not complete and $s_1, s_2 \in V(G_{\Gamma'})$ are non-adjacent vertices. Let r_1, r_2 be arbitrary vertices of $V(G_\Gamma)$. It is sufficient to show that $r_1 — r_2$. As in the proof of Theorem 4.8, we have

$$\varphi(r_1s_1 \circ_{\Gamma\times\Gamma'} r_2s_2) = \{(r, s) \mid (r — r_1 \text{ and } s — s_1) \text{ or } (r — r_2 \text{ and } s — s_2)\},$$

$$\varphi(r_1s_1) \otimes \varphi(r_2s_2) = \{(r, s) \mid (r — r_1 \text{ or } r — r_2) \text{ and } (s — s_1 \text{ or } s — s_2)\}.$$

Since $\varphi(r_1s_1 \circ_{\Gamma\times\Gamma'} r_2s_2) = \varphi(r_1s_1) \otimes \varphi(r_2s_2)$ and $(r_1, s_2) \in \varphi(r_1s_1) \otimes \varphi(r_2s_2)$, we have $(r_1, s_2) \in \varphi(r_1s_1 \circ_{\Gamma\times\Gamma'} r_2s_2)$ and so $r_1 — r_2$. □

5. Some other properties of degree hypergroupoids

One of the main tools to study hyperstructures are fundamental relations. Fundamental relations have been introduced on hypergroups by Koskas [15] and then studied mainly by Corsini [5] and Freni [12, 13] concerning hypergroups, Vougiouklis [20] concerning H_v-groups and many others.

Let $(H, *)$ be an H_v-group, $n \geq 2$ a natural number and let $U_H(n)$ be the set of all hyperproducts of n elements in H. We define the relation β_n on H as follows:

$$x\beta_n y \text{ iff there exists } P \in U_H(n) \text{ such that } \{x, y\} \subseteq P.$$

Let $\beta = \bigcup_{i=1}^{n} \beta_i$, where $\beta_1 = \{(a, a) \mid a \in H\}$. It is easy to see that β is reflexive and symmetric. We denote by $\widehat{\beta}$ the transitive closure of β and define it as follows:

$$x\widehat{\beta}y \quad \text{if} \quad \begin{array}{l} \text{there exists a natural number } k \text{ and elements} \\ x = a_1, a_2, \ldots, a_{k-1}, a_k = y \text{ in } H \text{ such that} \\ a_1\beta a_2, a_2\beta a_3, \ldots, a_{k-1}\beta a_k. \end{array}$$

Obviously, $\widehat{\beta}$ is an equivalence relation and we have $\beta = \widehat{\beta}$ if β is transitive. Freni [12] proved that the relation β defined on a hypergroup is transitive.

If $(H, *)$ is an H_v-group and R is an equivalence relation on H, then the set of all equivalence classes will be denoted by H/R, i.e., $H/R = \{R(x) \mid x \in H\}$. We denote by β^* the *fundamental relation* on H. β^* is the smallest equivalence relation on H such that H/β^* is a group with respect to the following operation:

$$\beta^*(a) \odot \beta^*(b) = \beta^*(c), \quad \text{for all } c \in a * b.$$

Theorem 5.1. *Let $(H, *)$ be an H_v-group. Then the fundamental relation β^* is the transitive closure of the relation β.*

Proof. See Theorem 2.1 of [20]. □

Theorem 5.2. *If Γ is a hypergraph and $(D_\Gamma, \circ_\Gamma)$ is the H_v-group associated with Γ, then $|D_\Gamma/\beta^*| = 1$ and $\beta^* = \beta_2 = D_\Gamma^2$, where β^* is the fundamental relation on D_Γ.*

Proof. For every $d, d' \in D_\Gamma$ we have $\{d, d'\} \subseteq d \circ d'$ and so we have $D_\Gamma^2 \subseteq \beta_2 \subseteq \beta$. On the other hand we have $\beta \subseteq \widehat{\beta} \subseteq D_\Gamma^2$ and therefore we have $D_\Gamma^2 = \widehat{\beta}$. Now, by using Theorem 5.1 we have $|D_\Gamma/\beta^*| = 1$ and $\beta^* = \beta_2 = D_\Gamma^2$. □

By the above theorem, fundamental relation of every degree hypergroupoid D_Γ is the full relation D_Γ^2. But, the converse is not true as the following example shows.

Example 5.3. *Let $(H = \{a, b\}, *)$ be an H_v-group with the following table:*

$*$	a	b
a	a	b
b	b	$\{a, b\}$

One easily checks that $\beta_2 = H^2$ and the fundamental relation on $(H, *)$ is equal to β_2, but $(H, *)$ is not a degree hypergroupoid.

Let $(H, *)$ be an H_v-group and A be a non-empty subset of H. We say that A is a *complete part* of H if for any natural number n and for all hyperproducts $P \in H_H(n)$, the following implication holds:

$$A \cap P \neq \emptyset \Longrightarrow P \subseteq A.$$

Proposition 5.4. *Let Γ be a hypergraph and $(D_\Gamma, \circ_\Gamma)$ be the degree hypergroupoid associated with Γ. Then, complete part of $(D_\Gamma, \circ_\Gamma)$ is equal to D_Γ.*

Proof. Suppose that A is a complete part of D_Γ and $b \in D_\Gamma$ is an arbitrary element. Clearly, for every $a \in A$ we have $A \cap a \circ_\Gamma b \neq \emptyset$ which implies that $a \circ_\Gamma b \subseteq A$ and so $b \in A$. This completes the proof. □

Consider the H_v-group H of Example 5.3. It is easy to verify that H is the only complete part of H but H is not a degree hypergroupoid. This shows that the converse of Proposition 5.4 is not true.

Acknowledgments. The authors are highly grateful to the referees for their valuable comments and suggestions for improving the paper.

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