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Statistical Approximation Properties of q-Balázs-Szabados-Stancu Operators

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1. Introduction

After Phillips [18], the approximation properties for q-analogue of operators were studied by several researchers .
We begin with some notations and definitions of q-calculus. For any non-negative integer r, the q-integer of the number r is defined as

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q} & \text{if} \quad q \neq 1 \\ r & \text{if} \quad q = 1 \end{cases},$$

where q is a positive real number.
The q-factorial is defined as

$$[r]_q! = \begin{cases} [1]_q\,[2]_q \ldots [r]_q & \text{if} \quad r = 1, 2, \ldots \\ r & \text{if} \quad r = 0. \end{cases}$$

For integers n, r with $0 \leq r \leq n$, the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q!\,[n-r]_q!}.$$

Details on q-integers can be found in [2, 4, 14].
Bernstein type rational functions were defined by Balázs [5]. Balázs and Szabados modified and studied approximation properties of these operators [6].
The q-analogue of the Balázs-Szabados operators were defined by Dogru [8] as follows

$$R_n(f; q, x) = \frac{1}{\prod_{s=0}^{n-1}(1+q^s a_n x)} \sum_{j=0}^{n} q^{j(j-1)/2} f\left(\frac{[j]_q}{b_n}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n x)^j, \tag{1}$$

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where $x \in [0,\infty)$, $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$ for all $n \in \mathbb{N}, q \in (0,1]$ and $0 < \beta \le \frac{2}{3}$.
Dogru also gave the following equalities

$$R_n(e_0; q, x) = 1, \tag{2}$$

$$R_n(e_1; q, x) = \frac{x}{1 + a_n x}, \tag{3}$$

$$R_n(e_2; q, x) = \frac{[n-1]_q}{[n]_q} \frac{q^2 x^2}{(1 + a_n x)(1 + a_n q x)} + \frac{x}{b_n(1 + a_n x)}, \tag{4}$$

where $e_k(x) = x^k$ for $k = 0, 1, 2$.
In (4), using the equality $[n]_q = q[n-1]_q + 1$, we get

$$R_n(e_2; q, x) = \frac{\left(1 - \frac{a_n}{b_n}\right) q x^2}{(1 + a_n x)(1 + a_n q x)} + \frac{x}{b_n(1 + a_n x)}. \tag{5}$$

We will use (5) instead of (4) throughout the paper.

The rational complex Balázs-Szabados operators were defined by Gal in [11]. He studied approximation properties of these operators on compact disks. In [13], the complex q-Balázs-Szabados operators were defined and the approximation properties of these operators were studied on compact disks.

$C[0,A]$ denotes the space of all continuous functions on $[0,A]$, $A > 0$ with the norm $\|f\| = \max_{x\in[0,A]} |f(x)|$ for all $f \in C[0,A]$.

We define the following q-Balázs-Szabados-Stancu operators

$$R_{n,q}^{(\alpha,\gamma)}(f; q, x) = \sum_{j=0}^{n} f\left(\frac{[j]_q + [\alpha]_q}{b_n + [\gamma]_q}\right) p_{n,j}(x; q),$$

where f is a real valued function defined on the all positive axis, $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$, $[\alpha]_q = \dfrac{1-q^{\alpha}}{1-q}$, $[\gamma]_q = \dfrac{1-q^{\gamma}}{1-q}$ for all $n \in \mathbb{N}$, $q \in (0,1]$, $0 < \beta \le \dfrac{2}{3}$ and $0 \le \alpha \le \gamma$,

$$p_{n,j}(x; q) = \frac{q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n x)^j}{\prod_{s=0}^{n-1} (1 + q^s a_n x)} \tag{6}$$

and

$$\prod_{s=0}^{n-1} (1 + q^s a_n x) = \sum_{j=0}^{n} q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n x)^j. \tag{7}$$

It is clear that $R_{n,q}^{(\alpha,\gamma)}$ are linear and positive operators.

We have the following lemma for the operators $R_{n,q}^{(\alpha,\gamma)}$.

Lemma 1.1. *The following equalities are satisfied for the operators* $R_{n,q}^{(\alpha,\gamma)}$

$$R_{n,q}^{(\alpha,\gamma)}(e_0;x) = 1, \tag{8}$$

$$R_{n,q}^{(\alpha,\gamma)}(e_1;x) = \frac{b_n x}{\left(b_n + [\gamma]_q\right)(1+a_n x)} + \frac{[\alpha]_q}{b_n + [\gamma]_q}, \tag{9}$$

$$R_{n,q}^{(\alpha,\gamma)}(e_2;x) = \frac{b_n^2\left(1-\frac{a_n}{b_n}\right)qx^2}{\left(b_n+[\gamma]_q\right)^2(1+a_n x)(1+a_n q x)} + \frac{b_n\left(2[\alpha]_q+1\right)x}{\left(b_n+[\gamma]_q\right)^2(1+a_n x)} + \frac{[\alpha]_q^2}{\left(b_n+[\gamma]_q\right)^2}, \tag{10}$$

where $e_k(x) = x^k$ *for* $k = 0,1,2$.

Proof. From (7), it is clear that

$$R_{n,q}^{(\alpha,\gamma)}(e_0;x) = 1.$$

With direct computation, we get

$$R_{n,q}^{(\alpha,\gamma)}(e_1;x) = \frac{b_n}{b_n+[\gamma]_q}R_n(e_1;q,x) + \frac{[\alpha]_q}{b_n+[\gamma]_q}R_n(e_0;q,x).$$

Using (2) and (3), we obtain desired result.
Similarly, with direct computation, we get

$$R_{n,q}^{(\alpha,\gamma)}(e_2;x) = \frac{b_n^2}{\left(b_n+[\gamma]_q\right)^2}R_n(e_2;q,x) + \frac{2[\alpha]_q b_n}{\left(b_n+[\gamma]_q\right)^2}R_n(e_1;q,x) + \frac{[\alpha]_q^2}{\left(b_n+[\gamma]_q\right)^2}R_n(e_0;q,x).$$

Using (2) ,(3) and (5), we obtain desired result. □

Lemma 1.2. *It holds the following equalities for the operators* $R_{n,q}^{(\alpha,\gamma)}$

$$R_{n,q}^{(\alpha,\gamma)}((e_1-x);x) = -\frac{[\gamma]_q x}{\left(b_n+[\gamma]_q\right)(1+a_n x)} - \frac{a_n x^2}{1+a_n x} + \frac{[\alpha]_q}{b_n+[\gamma]_q} \tag{11}$$

and

$$\begin{aligned} R_{n,q}^{(\alpha,\gamma)}\left((e_1-x)^2;x\right) &= \frac{a_n^2 q x^4 + a_n(q_n+1)x^3}{(1+a_n x)(1+a_n q_n x)} - \frac{2b_n a_n q x^3}{\left(b_n+[\gamma]_q\right)(1+a_n x)(1+a_n q x)} \\ &+ \frac{b_n^2\left(q-1-q\frac{a_n}{b_n}+\frac{[\gamma]_q^2}{b_n^2}\right)x^2}{\left(b_n+[\gamma]_q\right)^2(1+a_n x)(1+a_n q x)} + \frac{b_n\left(2[\alpha]_q+1\right)x}{\left(b_n+[\gamma]_q\right)^2(1+a_n x)} \\ &+ \frac{[\alpha]_q^2}{\left(b_n+[\gamma]_q\right)^2} - \frac{2[\alpha]_q x}{b_n+[\gamma]_q}. \end{aligned} \tag{12}$$

Proof. From Lemma 1.1, the proof can be obtained easily, so we omit the proof. □

2. Statistical Convergence of the Operators

The concept of the statistical convergence was introduced by Fast[9].
In this section, we will give a Bohman-Korovkin type statistical approximation theorem.
Firstly, we recall some definitions about the statistical convergence. The density of a set $K \subset \mathbb{N}$ is defined by

$$\delta\left\{k \leq n : k \in K\right\},$$

The natural density, δ, of a set $K \subset \mathbb{N}$ is defined by

$$\lim_{n\to\infty} \frac{1}{n}\left|K_n\right|,$$

provided the limits exist [16].
A sequence $x = (x_k)$ is called statistically convergent to a number L if, for every $\varepsilon > 0$

$$\delta\left\{k : \left|x_k - L\right| \geq \varepsilon\right\} = 0,$$

and it is denoted as $st - \lim_k x_k = L$.
Any convergent sequence is statistically convergent but not conversely. For example, the sequence

$$x_k = \begin{cases} L_{1,} & \text{if } k = m^2 \\ L_2, & \text{if } k \neq m^2 \end{cases}, \text{ for } m = 1, 2, ...$$

is statistically convergent to L_2 but not convergent in the ordinary sense when $L_1 \neq L_2$.
Now, we consider a sequence $q = (q_n)$ satisfying

$$st - \lim_n q_n = 1 \text{ and } st - \lim_n q_n^n = c, 0 \leq c < 1. \tag{13}$$

Under this conditions given in (13), it is clear that

$$st - \lim_n a_n = st - \lim_n \frac{1}{b_n} = st - \lim_n \frac{a_n}{b_n} = st - \lim_n \frac{1}{b_n + [\gamma]_q} = 0.$$

The useful connections of Korovkin type approximation theory were given by Altomare and Campiti in [1].
Recently, the statistical approximation of operators has also been investigated by several authors (see [7],[3],[17], [12], [19], [20], [22], [23], [21] and [24]).
Gadjiev and Orhan [10] proved the following Bohman-Korovkin type statistical approximation theorem for any sequence of positive linear operators.

Theorem 2.1. *([10]) If the sequence of positive linear operators $A_n : C[a,b] \to B[a,b]$ satisfies the conditions*

$$st - \lim_n \left\|A_n(e_\nu) - e_\nu\right\| = 0$$

with $e_\nu(t) = t^\nu$ for $\nu = 0, 1, 2$, then for any $f \in C[a,b]$, we have

$$st - \lim_n \left\|A_n(f) - f\right\| = 0.$$

Now, we can give the following main result for the operators $R_{n,q}^{(\alpha,\gamma)}$.

Theorem 2.2. *Let $q = (q_n)$ with $0 < q_n \leq 1$ be a sequence satisfying the conditions given in (13). If f is a continuous function on $[0, A]$ with $0 < A < \dfrac{1}{a_n}$ and bounded on the all positive axis, then it holds for the operators $R_{n,q}^{(\alpha,\gamma)}$*

$$st - \lim_n \left\| R_{n,q_n}^{(\alpha,\gamma)}(f;.) - f \right\| = 0.$$

Proof. From (8) in Lemma 1.1, it is clear that

$$st - \lim_n \left\| R_{n,q_n}^{(\alpha,\gamma)}(e_0;.) - e_0 \right\| = 0. \tag{14}$$

Using (11) in Lemma 1.2, we can write

$$\left| R_{n,q_n}^{(\alpha,\gamma)}(e_1;x) - e_1(x) \right| \leq \frac{[\gamma]_{q_n} |x|}{\left(b_n + [\gamma]_{q_n}\right) |1 - a_n |x||} + \frac{a_n |x|^2}{|1 - a_n |x||} + \frac{[\alpha]_{q_n}}{b_n + [\gamma]_{q_n}}. \tag{15}$$

Considering $0 < A < \frac{1}{a_n}$, taking maximum of both sides of (15) on $C[0, A]$, we get

$$\left\| R_{n,q_n}^{(\alpha,\gamma)}(e_1;.) - e_1 \right\| \leq \frac{[\gamma]_{q_n} A}{\left(b_n + [\gamma]_{q_n}\right)(1 - a_n A)} + \frac{a_n A^2}{1 - a_n A} + \frac{[\alpha]_{q_n}}{b_n + [\gamma]_{q_n}}. \tag{16}$$

For a given $\varepsilon > 0$, let us define the following sets:

$$\begin{aligned}
D &:= \left\{ k : \left\| R_{k,q_k}^{(\alpha,\gamma)}(e_1;.) - e_1 \right\| \geq \varepsilon \right\}, \\
D_1 &:= \left\{ k : \frac{[\gamma]_{q_k} A}{\left(b_k + [\gamma]_{q_k}\right)(1 - a_k A)} \geq \frac{\varepsilon}{3} \right\}, \\
D_2 &:= \left\{ k : \frac{a_k A^2}{1 - a_k A} \geq \frac{\varepsilon}{3} \right\}, \\
D_3 &:= \left\{ k : \frac{[\alpha]_{q_k}}{b_k + [\gamma]_{q_k}} \geq \frac{\varepsilon}{3} \right\}.
\end{aligned}$$

From (16), since $D \subseteq D_1 \cup D_2 \cup D_3$, we get

$$\begin{aligned}
\delta \left\{ k \leq n : \left\| R_{k,q_k}^{(\alpha,\gamma)}(e_1;.) - e_1 \right\| \geq \varepsilon \right\} \leq\ & \delta \left\{ k \leq n : \frac{[\gamma]_{q_k} A}{\left(b_k + [\gamma]_{q_k}\right)(1 - a_k A)} \geq \frac{\varepsilon}{3} \right\} \\
& + \delta \left\{ k \leq n : \frac{a_k A^2}{1 - a_k A} \geq \frac{\varepsilon}{3} \right\} + \delta \left\{ k \leq n : \frac{[\alpha]_{q_k}}{b_k + [\gamma]_{q_k}} \geq \frac{\varepsilon}{3} \right\}.
\end{aligned}$$

Under the condition given in (13), it is clear that

$$st - \lim_n \frac{[\gamma]_{q_n} A}{\left(b_n + [\gamma]_{q_n}\right)(1 - a_k A)} = st - \lim_n \frac{a_n A^2}{1 - a_n A} = st - \lim_n \frac{[\alpha]_{q_n}}{b_n + [\gamma]_{q_n}} = 0,$$

which implies

$$st-\lim_n \left\| R_{n,q_n}^{(\alpha,\gamma)}(e_1;.)-e_1 \right\| = 0. \tag{17}$$

Using (10) in Lemma 1.1, we can write

$$\begin{aligned} R_{n,q_n}^{(\alpha,\gamma)}(e_2;.)-e_2(x) &= -\frac{a_n^2 q_n x^4 + a_n(q_n+1)x^3}{(1+a_n x)(1+a_n q_n x)} + \frac{b_n^2\left(q_n - 1 - q_n\frac{a_n}{b_n} - \frac{2[\gamma]_{q_n}}{b_n} - \frac{[\gamma]_{q_n}^2}{b_n^2}\right)x^2}{\left(b_n+[\gamma]_{q_n}\right)^2(1+a_n x)(1+a_n q_n x)} \\ &\quad + \frac{b_n\left(2[\alpha]_{q_n}+1\right)x}{\left(b_n+[\gamma]_{q_n}\right)^2(1+a_n x)} + \frac{[\alpha]_{q_n}^2}{\left(b_n+[\gamma]_{q_n}\right)^2}. \end{aligned} \tag{18}$$

Considering $0 < A < \dfrac{1}{a_n}$, taking absolute value both sides of (18), and passing to norm on $C[0,A]$

$$\begin{aligned} \left\| R_{n,q_n}^{(\alpha,\gamma)}(e_2;.)-e_2 \right\| &\leq \frac{a_n^2 q_n A^4 + a_n(q_n+1)A^3}{(1-a_n A)(1-a_n q_n A)} + \frac{b_n^2\left(1 - q_n + q_n\frac{a_n}{b_n} + \frac{2[\gamma]_{q_n}}{b_n} + \frac{[\gamma]_{q_n}^2}{b_n^2}\right)A^2}{\left(b_n+[\gamma]_{q_n}\right)^2(1-a_n A)(1-a_n q_n A)} \\ &\quad + \frac{b_n\left(2[\alpha]_{q_n}+1\right)A}{\left(b_n+[\gamma]_{q_n}\right)^2(1-a_n A)} + \frac{[\alpha]_{q_n}^2}{\left(b_n+[\gamma]_{q_n}\right)^2}. \end{aligned} \tag{19}$$

If we choose

$$\begin{aligned} \lambda_n &= \frac{a_n^2 q_n A^4 + a_n(q_n+1)A^3}{(1-a_n A)(1-a_n q_n A)}, \\ \theta_n &= \frac{b_n^2\left(1 - q_n + q_n\frac{a_n}{b_n} + \frac{2[\gamma]_{q_n}}{b_n} + \frac{[\gamma]_{q_n}^2}{b_n^2}\right)A^2}{\left(b_n+[\gamma]_{q_n}\right)^2(1-a_n A)(1-a_n q_n A)}, \\ \eta_n &= \frac{b_n\left(2[\alpha]_{q_n}+1\right)A}{\left(b_n+[\gamma]_{q_n}\right)^2(1-a_n A)}, \\ \varphi_n &= \frac{[\alpha]_{q_n}^2}{\left(b_n+[\gamma]_{q_n}\right)^2} \end{aligned}$$

then, under the conditions given in (13), we have

$$st-\lim_n \lambda_n = st-\lim_n \theta_n = st-\lim_n \eta_n = st-\lim_n \varphi_n = 0. \tag{20}$$

Again for a given $\varepsilon > 0$, let us define the following sets:

$$\begin{aligned} E &: = \left\{k : \left\|R_{k,q_k}^{(\alpha,\gamma)}(e_2; q_k, .) - e_2\right\| \geq \varepsilon\right\}, \\ E_1 &: = \left\{k : \lambda_k \geq \frac{\varepsilon}{4}\right\},\ E_2 := \left\{k : \theta_k \geq \frac{\varepsilon}{4}\right\}, \\ E_3 &: = \left\{k : \eta_k \geq \frac{\varepsilon}{4}\right\},\ E_4 := \left\{k : \varphi_k \geq \frac{\varepsilon}{4}\right\}. \end{aligned}$$

It is clear that $E \subseteq E_1 \cup E_2 \cup E_3 \cup E_4$, which implies

$$\begin{aligned} \delta\left\{k \leq n : \left\|R_{k,q_k}^{(\alpha,\gamma)}(e_2; .) - e_2\right\| \geq \varepsilon\right\} \leq\ & \delta\left\{k \leq n : \lambda_k \geq \frac{\varepsilon}{4}\right\} + \delta\left\{k \leq n : \theta_k \geq \frac{\varepsilon}{4}\right\} \\ & + \delta\left\{k \leq n : \eta_k \geq \frac{\varepsilon}{4}\right\} + \delta\left\{k \leq n : \varphi_k \geq \frac{\varepsilon}{4}\right\}. \end{aligned}$$

From (19), we obtain that

$$st - \lim_n \left\|R_{n,q_n}^{(\alpha,\gamma)}(e_2; .) - e_2\right\| = 0. \tag{21}$$

From (15), (17) and (21) and taking into account Theorem 2.1, the proof is finished. □

3. Rate of Statistical Convergence

In this part, we will give the order of statistical approximation of the operators $R_{n,q}^{(\alpha,\gamma)}$ by means of modulus of continuity and the elements of Lipschitz class functionals.

Let $f \in C[0, A]$. The modulus of continuity of f is defined by

$$\omega(f; \delta) = \sup_{\substack{|t-x|\leq\delta \\ x,t\in[0,A]}} \left|f(t) - f(x)\right|.$$

It is clear that $\lim\limits_{\delta\to 0^+} \omega(f; \delta) = 0$ for all $f \in C[0, A]$. Also, we have

$$\left|f(t) - f(x)\right| \leq \omega(f; \delta)\left(\frac{|t - x|}{\delta} + 1\right) \tag{22}$$

for any $\delta > 0$ and each $x, t \in [0, A]$.

A function $f \in C[0, A]$ belongs to $Lip_M(\theta)$ for $M > 0$ and $0 < \theta \leq 1$, provided that

$$\left|f(y) - f(x)\right| \leq \left|y - x\right|^\theta, \text{ for all } x, y \in [0, A]. \tag{23}$$

Theorem 3.1. *Let $q = (q_n)$ with $0 < q_n \leq 1$ be a sequence satisfying the conditions given in (13). If f is a continuous function on $[0, A]$ and bounded on the all positive axis, then it holds*

$$\left|R_{n,q_n}^{(\alpha,\gamma)}(f; x) - f(x)\right| \leq 2\omega(f; \delta_n(x)),$$

where

$$\delta_n(x) = \left(R_{n,q_n}^{(\alpha,\gamma)}\left((e_1 - x)^2; x\right)\right)^{1/2}. \tag{24}$$

Proof. From the linearity and positivity of the operators $R_{n,q_n}^{(\alpha,\gamma)}$ and using (22), we obtain

$$\begin{aligned} \left|R_{n,q_n}^{(\alpha,\gamma)}(f;x)-f(x)\right| &\leq R_{n,q_n}^{(\alpha,\gamma)}\left(\left|f(t)-f(x)\right|;x\right) \\ &\leq \omega(f;\delta(x))\left\{1+\frac{1}{\delta(x)}R_{n,q_n}^{(\alpha,\gamma)}(|e_1-x|;x)\right\}. \end{aligned} \tag{25}$$

In (25), using Cauchy- Schwarz inequality, we get

$$\left|R_{n,q_n}^{(\alpha,\gamma)}(f;x)-f(x)\right| \leq \omega(f;\delta(x))\left\{1+\frac{1}{\delta(x)}\left(R_{n,q_n}^{(\alpha,\gamma)}\left((e_1-x)^2;x\right)\right)^{1/2}\right\}.$$

Finally, choosing $\delta(x)=\delta_n(x)$ as in (24), the proof is complete. □

Theorem 3.2. *Let $q=(q_n)$ with $0<q_n\leq 1$ be a sequence satisfying the conditions given in (13). If f is a continuous function on $[0,A]$ and bounded on the all positive axis then we have*

$$\left|R_{n,q_n}^{(\alpha,\gamma)}(f;x)-f(x)\right| \leq M\{\delta_n(x)\}^{\theta},$$

where $\delta_n(x)$ is given as in (24).

Proof. Using (23), we can write

$$\begin{aligned} \left|R_{n,q_n}^{(\alpha,\gamma)}(f;x)-f(x)\right| &\leq R_{n,q_n}^{(\alpha,\gamma)}\left(\left|f(t)-f(x)\right|;x\right) \\ &\leq MR_{n,q_n}^{(\alpha,\gamma)}\left(|t-x|^{\theta};x\right). \end{aligned}$$

Applying the Hölder inequality,we get

$$\left|R_{n,q_n}^{(\alpha,\gamma)}(f;x)-f(x)\right| \leq M\left(R_{n,q_n}^{(\alpha,\gamma)}\left((e_1-x)^2;x\right)\right)^{\theta/2},$$

and choosing $\delta_n(x)$ as given in (24), the proof is complete. □

4. An r-th Order Generalization of Operators $R_{n,q}^{(\alpha,\gamma)}$

By $C^{(r)}[0,A]$ we mean the space of all functions f for which their r-th derivative $f^{(r)}$ with $f^{(0)}(x)=f(x)$ are continuous on $[0,A]$ and bounded all positive axis for $A>0$ and $r=0,1,2....$.

Now, using the similar method by Kirov and Popova [15], we consider the following r-th order generalization

$$R_{n,q,r}^{(\alpha,\gamma)}(f;x)=\sum_{j=0}^{n}\sum_{i=0}^{r}p_{n,j}(x;q)\frac{f^{(i)}\left(\xi_{n,j}(q)\right)}{i!}\left(x-\xi_{n,j}(q)\right)^{i}, \tag{26}$$

where $n\in\mathbb{N}$, $\xi_{n,j}(q):=\frac{[j]_q+[\alpha]_q}{b_n+[\gamma]_q}$, $f\in C^{(r)}[0,A]$, $p_{n,j}(x;q)$ is as given in (6), $a_n=[n]_q^{\beta-1}$, $b_n=[n]_q^{\beta}$ with $0<\beta\leq\frac{2}{3}$ and $0\leq\alpha\leq\gamma$.

If we take $r=0$ in (26) then we get $R_{n,q,0}^{(\alpha,\gamma)}(f;x)=R_{n,q}^{(\alpha,\gamma)}(f;x)$.

We have the following approximation theorem for the operators $R_{n,q,r}^{(\alpha,\gamma)}$.

Theorem 4.1. *If $f \in C^{(r)}[0,A]$ such that $f^{(r)} \in Lip_M(\theta)$ then we have*

$$\left|R_{n,q,r}^{(\alpha,\gamma)}(f;x) - f(x)\right| \leq \frac{M\theta B(\theta,r)}{(r-1)!\,(\theta+r)} \left|R_{n,q}^{(\alpha,\gamma)}(\varphi;x)\right|,$$

where $\varphi(y) = \left|y-x\right|^{\theta+r}$ for each $x \in [0,A]$ and $B(\theta,r)$ denotes the Beta function.

Proof. From (26), we can write

$$f(x) - R_{n,q,r}^{(\alpha,\gamma)}(f;x) = \sum_{j=0}^{n} p_{n,j}(x;q)\left\{f(x) - \sum_{i=0}^{r} \frac{f^{(i)}\left(\xi_{n,j}(q)\right)}{i!}\left(x-\xi_{n,j}(q)\right)^{i}\right\}. \tag{27}$$

Using the well-known Taylor's formula, we get

$$f(x) - \sum_{i=0}^{r} \frac{f^{(i)}\left(\xi_{n,j}(q)\right)}{i!}\left(x-\xi_{n,j}(q)\right)^{i} = \tag{28}$$

$$\frac{\left(x-\xi_{n,j}(q)\right)^{r}}{(r-1)!}\int_{0}^{1}(1-t)^{r-1}\left[f^{(r)}\left(\xi_{n,j}(q)+t\left(x-\xi_{n,j}(q)\right)\right) - f^{(r)}\left(\xi_{n,j}(q)\right)\right]dt.$$

Since $f^{(r)} \in Lip_M(\theta)$, we see that

$$\left|f^{(r)}\left(\xi_{n,j}(q)+t\left(x-\xi_{n,j}(q)\right)\right) - f^{(r)}\left(\xi_{n,j}(q)\right)\right| \leq Mt^{\theta}\left|x-\xi_{n,j}(q)\right|^{\theta}. \tag{29}$$

Now, using (29) in (28) and considering the fact that

$$\int_{0}^{1}(1-t)^{r-1}t^{\theta}dt = \frac{\theta B(\theta,r)}{\theta+r},$$

we have

$$\left|f(x) - \sum_{i=0}^{r} \frac{f^{(i)}\left(\xi_{n,j}(q)\right)}{i!}\left(x-\xi_{n,j}(q)\right)^{i}\right| \leq \frac{M\theta B(\theta,r)}{(r-1)!\,(\theta+r)}\left|x-\xi_{n,j}(q)\right|^{r+\theta}. \tag{30}$$

Taking into account (30) in (27), we get the desired result. □

Remark 4.2. *The function φ in Theorem 4.1 belongs to $C[0,A]$ and $\varphi(x) = 0$. Also, for any $x, y \in [0,A]$, $r \in \mathbb{N}$, and $\theta \in [0,1)$, since*

$$\left|\varphi(y) - \varphi(x)\right| \leq \left|y-x\right|^{r}\left|y-x\right|^{\theta} \leq \left|y-x\right|^{\theta},$$

we get that $\varphi \in Lip_1(\theta)$.

Under the light of Remark 4.2, the following result is obtained from Theorem 3.1 and Theorem 3.2.

Corollary 4.3. *Let $q=(q_n)$ with $0<q_n\leq 1$ be a sequence satisfying the conditions given in (13). If $f\in C^{(r)}[0,A]$ such that $f^{(r)}\in Lip_M(\theta)$ then we have*

$$i)\ \left|R_{n,q_n,r}^{(\alpha,\gamma)}(f;x)-f(x)\right|\leq\frac{2M\theta B(\theta,r)}{(r-1)!\,(\theta+r)}\omega(\varphi;\delta_n(x)),$$

$$ii)\ \left|R_{n,q_n,r}^{(\alpha,\gamma)}(f;x)-f(x)\right|\leq\frac{M\theta B(\theta,r)}{(r-1)!\,(\theta+r)}\{\delta_n(x)\}^{\theta},$$

where $\delta_n(x)$ as given in (24).

Remark 4.4. *$\delta_n(x)$, given in (24), is defined on $[0,A]$ for sufficiently large natural numbers. Under the conditions given in (13), it is clear that $st-\lim_n\delta_n(x)$, which implies $st-\lim_n\omega(f;\delta_n(x))=0$.*

Consequently, Theorem 3.1 and Theorem 3.2 give us the rate of statistical convergence of the operators $R_{n,q_n}^{(\alpha,\gamma)}(f;x)$ to $f(x)$ on $[0,A]$.

Remark 4.5. *Under the hypothesis of Corollary 4.3, we see that $st-\lim_n\omega(\varphi;\delta_n(x))=0$ since $st-\lim_n\delta_n(x)$. Considering Theorem 4.1, (i) and (ii) in Corollary 4.3 give us the rate of statistical convergence of the operators $R_{n,q_n,r}^{(\alpha,\gamma)}(f;x)$ to $f(x)$ on $[0,A]$ provided that $f\in C^{(r)}[0,A]$ such that $f^{(r)}\in Lip_M(\theta)$ for $r\in\mathbb{N}$.*

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