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A Fixed Point Theorem for a New Class of Set-Valued Mappings in $\mathbb{R}$-Complete (Not Necessarily Complete) Metric Spaces

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Abstract. In this paper, firstly, we introduce the notion of R-complete metric spaces. This notion let us to consider fixed point theorem in R-complete instead of complete metric spaces. Secondly, as motivated by the recent work of Amini-Harandi (Fixed Point Theory Appl. 2012, 2012:215), we explain a new generalized contractive condition for set-valued mappings and prove a fixed point theorem in R-complete metric spaces which extends some well-known results in the literature. Finally, some examples are given to support our main theorem and also we find the existence of solution for a first-order ordinary differential equation.

1. Introduction

In 1969, Nadler [5] extended the Banach contraction principle [2] to set-valued mappings as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that $H(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in T(z)$.*

Many fixed point theorems have been proved by various authors as generalizations to Nadler's theorem. One such generalization is due to Kaneko in [3] and Nicolae in [6]. Another generalization was proved by Mizoguchi and Takahashi [4] which is also well known as a positive response to a conjecture posed by Simeon Reich [7].

Nadler's theorem was generalized by Mizoguchi and Takahashi [4] in the following way.

Theorem 1.2. *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfying*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for all $x, y \in X$, where α is a mapping from $(0, \infty)$ into $[0, 1)$ such that $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0, \infty)$. Then T has a fixed point.

Recently, A. Amini-Harandi [1] introduced a new concept of set-valued contraction and proved a fixed point theorem which generalizes some well-known results in the literature, especially [10]. In this paper, we present an improvement and generalization of the main result of A. Amini-Harandi [1], M. Sgroi et al. [8], J. Tiammee et al. [9] and D. Wardowski [10].

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2. Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote, respectively, the sets of all natural numbers, rational numbers and real numbers. Also, for every nonempty set X denote $\mathcal{P}^*(X)$ the set of all nonempty subsets of X.
Let (X, d) be a metric space. We denote $CB(X)$ collections of all closed and bounded members of $\mathcal{P}^*(X)$.
For $A, B \in CB(X)$ and $x \in X$, define

$$D(x, A) := \inf\{d(x, a); a \in A\}$$

and

$$H(A, B) := \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}.$$

Notice that H is a metric on $CB(X)$, called the Hausdorff metric induced by d.
To set up our results in the next section, we introduce some definitions that play a major role in further sections.
Let X be a nonempty set, $A, B \subseteq X$ and R be an arbitrary binary relation over X. The binary relations R_1 and R_2 between A and B are defined as follows.
(1) $A \ \mathrm{R}_1\ B$ if $a \mathrm{R} b$, for all $a \in A$ and $b \in B$.
(2) $A \ \mathrm{R}_2\ B$ if for each $a \in A$ there exists $b \in B$ such that $a \mathrm{R} b$.
Next, we introduce two types of monotone set-valued mappings by using the relations R_1 and R_2.

Definition 2.1. *Let (X, d) be a metric space endowed a binary relation R over X and $T : X \to CB(X)$. Then T is said to be*
(i) monotone mapping of type (I) if

$$x, y \in X, x \ \mathrm{R} \ y \Rightarrow Tx \ \mathrm{R}_1 \ Ty;$$

(ii) monotone mapping of type (II) if

$$x, y \in X, x \ \mathrm{R} \ y \Rightarrow Tx \ \mathrm{R}_2 \ Ty;$$

Example 2.2. *Let $X = \{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots\} \cup \{0, 1\}$, $d(x, y) = |x - y|$, for all $x, y \in X$, and binary relation R over X defined by*

$$x \ \mathrm{R} \ y \iff \begin{cases} \frac{y}{x} \in \mathbb{N}, \\ or\ x = y = 0. \end{cases}$$

Let $T : X \to CB(X)$ defined by

$$Tx = \begin{cases} \{\frac{1}{2^n}, \frac{1}{2^{n+1}}\}, & if \quad x = \frac{1}{2^n}, n = 1, 2, \cdots, \\ \{0\}, & if \quad x = 0, \\ \{1, \frac{1}{2}, \frac{1}{4}\}, & if \quad x = 1. \end{cases}$$

It is easy to see that T is monotone of type (II) but not monotone of type (I). Since $\frac{1}{2} \mathrm{R}\, 1$ but $T(\frac{1}{2}) = \{\frac{1}{2}, \frac{1}{4}\} \not\mathrm{R}_1 \{1, \frac{1}{2}, \frac{1}{4}\} = T(1)$.

Example 2.3. *Let $X = [0, 1)$ and let the metric on X be the Euclidean metric. Define binary relation R over X by x R y if $xy \in \{x, y\}$ for all $x, y \in X$. Let $T : X \to CB(X)$ be a mapping defined by*

$$T(x) = \begin{cases} \{\frac{1}{2}x^2, x\}, & x \in \mathbb{Q} \cap X, \\ \{0\}, & x \in \mathbb{Q}^c \cap X. \end{cases}$$

It is easy to see that T is monotone of type (I) and (II).

Definition 2.4. *Let Λ denote the class of those functions $\phi(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}_+$ which satisfy the following conditions*
(Λ_1) *ϕ is increasing in t_2, t_3, t_4 and t_5;*
(Λ_2) *$t_{n+1} < \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0)$ implies that $t_{n+1} < t_n$, for each positive sequence $\{t_n\}$;*
(Λ_3) *If $t_n, s_n \to 0$ and $u_n \to \gamma > 0$, as $n \to \infty$, then we have $\limsup_{n\to\infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1}) < \gamma$;*
(Λ_4) *$\phi(u, u, u, 2u, 0) \le u$, for each $u \in \mathbb{R}^+ = [0, +\infty)$.*

Example 2.5. *Let $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ defined by*

$$\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + L t_5$$

where $\alpha, \beta, \gamma, \delta, L \ge 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \ne 1$. We claim that $\phi \in \Lambda$. Indeed (Λ_1) obviously holds. To show (Λ_2), let $\{t_n\}$ be a positive sequence such that

$$\begin{aligned} t_{n+1} < \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0) &= \alpha t_n + \beta t_n + \gamma t_{n+1} + \delta(t_n + t_{n+1}) \\ &= (\alpha + \beta + \delta) t_n + (\gamma + \delta) t_{n+1}. \end{aligned}$$

Since $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \ne 1$, then we can conclude that $1 - (\gamma + \delta) > 0$ and hence

$$t_{n+1} < \frac{(\alpha + \beta + \delta)}{1 - (\gamma + \delta)} t_n = t_n.$$

It is obvious to see that the properties (Λ_3) and (Λ_4) hold for this function.

Definition 2.6. *[1] Let $F : (0, +\infty) \to \mathbb{R}$ and $\theta : (0, +\infty) \to (0, +\infty)$ be two mappings. Throughout the paper, let Δ be the set of all pairs (θ, F) satisfying the following:*
(δ_1) *$\theta(t_n) \not\to 0$ for each strictly decreasing sequence $\{t_n\}$;*
(δ_2) *F is a strictly increasing function;*
(δ_3) *For each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*
(δ_4) *If $t_n \downarrow 0$ and $\theta(t_n) \le F(t_n) - F(t_{n+1})$ for each $n \in \mathbb{N}$, then we have $\sum_{n=1}^{\infty} t_n < \infty$.*

Example 2.7. *[1] Let $F(t) = \ln(t)$ and $\theta(t) = -\ln(\alpha(t))$ for each $t \in (0, +\infty)$, where $\alpha : (0, \infty) \to (0, 1)$ satisfying $\limsup_{s\to t^+} \alpha(s) < 1$, for all $t \in [0, \infty)$. Then $(\theta, F) \in \Delta$.*

Definition 2.8. *Let $X \ne \emptyset$ and $\mathrm{R} \subseteq X \times X$ be a binary relation. A sequence $\{x_n\}$ is called a* R*-sequence if*

$$(\forall n \in \mathbb{N}, \quad x_n \,\mathrm{R}\, x_{n+1}).$$

Definition 2.9. *Let (X, d) be a metric space and* R *be a binary relation over X. Then X is said to be* R*-regular if for each sequence $\{x_n\}$ such that $x_n \,\mathrm{R}\, x_{n+1}$, for all $n \in \mathbb{N}$, and $x_n \to x$, for some $x \in X$, then $x_n \,\mathrm{R}\, x$, for all $n \in \mathbb{N}$ (briefly, (X, d, R) is called* R*-regular metric space).*

Definition 2.10. *Let (X, d) be a metric space and* R *be a binary relation over X. Then X is said to be* R*-complete if every Cauchy* R*-sequence is convergent (briefly, (X, d, R) is called* R*-complete metric space).*

Example 2.11. *Let $X = \mathbb{Q}$. Suppose that $x \,\mathrm{R}\, y$ if and only if $x = 0$ or $y = 0$. Clearly, $\mathbb{Q}$ with the Euclidean metric is not a complete metric space, but it is* R*-complete. In fact, if $\{x_k\}$ is an arbitrary Cauchy* R*-sequence in $\mathbb{Q}$, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0$ for all $n \ge 1$. It follows that $\{x_{k_n}\}$ converges to $0 \in X$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent. It is easy to see that (X, d, R) is also* R*-regular metric space.*

Example 2.12. *Let $X = [0, 1)$. Suppose that*

$$x \;\mathrm{R}\; y \iff \begin{cases} x \le y \le \frac{1}{4}, \\ \text{or } x = 0. \end{cases}$$

Clearly, X with the Euclidian metric is not a complete metric space, but it is R*-complete. In fact, if* $\{x_k\}$ *is an arbitrary Cauchy* R*-sequence in X, then there exists a subsequence* $\{x_{k_n}\}$ *of* $\{x_k\}$ *for which* $x_{k_n} = 0$ *for all* $n \geq 1$ *or there exists a monotone subsequence* $\{x_{k_n}\}$ *of* $\{x_k\}$ *for which* $x_{k_n} \leq \frac{1}{4}$ *for all* $n \geq 1$*. It follows that* $\{x_{k_n}\}$ *converges to a point* $x \in [0, \frac{1}{4}] \subseteq X$*. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that* $\{x_k\}$ *is convergent. It is easy to see that* (X, d, R) *is also* R*-regular metric space.*

Example 2.13. *Let* $X = \mathbb{R}$*. Suppose* $x \,\mathrm{R}\, y$ *if and only* $x = 0$ *or* $0 \neq y \in \mathbb{Q}$*. It is easy to see that* (X, d, R) *is a* R*-complete but not* R*-regular metric space.*

Example 2.14. *Let* $X = \mathbb{R}$*, suppose* $x \,\mathrm{R}\, y$ *if*

$$x, y \in \left(n + \frac{2}{4}, n + \frac{3}{4}\right)$$

for some $n \in \mathbb{Z}$ *or*

$$x = 0.$$

It is easy to see that (X, d, R) *is a* R*-complete but not* R*-regular metric space.*

3. Fixed Point Theory

We prove main theorem of this section by using the technique in [1].

Theorem 3.1. *Let* (X, d, R) *be a* R*-complete (not necessarily complete),* R*-regular metric space and* $T : X \to CB(X)$ *be a set-valued mapping. Assume that there exists* $(\frac{\theta}{2}, F) \in \Delta$ *such that*

$$\theta(d(x,y)) + F(H(Tx,Ty)) \leq F(\phi(d(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx))), \tag{1}$$

for each $x \,\mathrm{R}\, y$*, with* $Tx \neq Ty$*, where* $\phi \in \Lambda$*. If the following conditions are satisfied*
(i) T is monotone of type (I);
(ii) There exists $x_0 \in X$ *such that* $\{x_0\} \,\mathrm{R}_2\, Tx_0$*;*
(iii) T is compact valued or F is continuous from the right;
Then T has a fixed point.

Proof. By assumption (*ii*), there exists $x_1 \in Tx_0$ such that $x_0 \,\mathrm{R}\, x_1$. By assumption (*i*), since T is monotone of type (*I*), then $Tx_0 \,\mathrm{R}_1\, Tx_1$. If $x_1 \in Tx_1$, then x_1 is fixed point of T and the proof is complete. Assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Since either T is compact valued or F is continuous from right, $x_1 \in Tx_0$ and

$$F(D(x_1, Tx_1)) < F(H(Tx_0, Tx_1)) + \frac{\theta(d(x_0, x_1))}{2}$$

then there exists $x_2 \in Tx_1$ with $x_1 \,\mathrm{R}\, x_2$ such that

$$F(d(x_1, x_2)) \leq F(H(Tx_0, Tx_1)) + \frac{\theta(d(x_0, x_1))}{2}.$$

Repeating this process, we find that there exists a R-sequence $\{x_n\}$ with initial point x_0 such that $x_{n+1} \in Tx_n$, $Tx_n \neq Tx_{n+1}$ and

$$F(d(x_n, x_{n+1})) \leq F(H(Tx_{n-1}, Tx_n)) + \frac{\theta(d(x_{n-1}, x_n))}{2}, \tag{2}$$

for all $n \in \mathbb{N}$. From (1), (2), (Λ_1) and (δ_2)we have

$$\begin{aligned}
&\theta(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1}))\\
&\leq \theta(d(x_{n-1}, x_n)) + F(H(Tx_{n-1}, Tx_n)) + \frac{\theta(d(x_{n-1}, x_n))}{2}\\
&\leq F(\phi(d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), D(x_{n-1}, Tx_n), D(x_n, Tx_{n-1})))\\
&+ \frac{\theta(d(x_{n-1}, x_n))}{2}\\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0)) + \frac{\theta(d(x_{n-1}, x_n))}{2}\\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0))\\
&+ \frac{\theta(d(x_{n-1}, x_n))}{2},
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{\theta(d(x_{n-1}, x_n))}{2} + F(d(x_n, x_{n+1}))\\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)),
\end{aligned} \tag{3}$$

for each $n \in \mathbb{N}$. This implies that

$$\begin{aligned}
&d(x_n, x_{n+1})\\
&< \phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0),
\end{aligned}$$

for each $n \in \mathbb{N}$. Then by (Λ_2), $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$. Since $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence, then by using (3), (Λ_1) and (Λ_4), we obtain that

$$\begin{aligned}
&\frac{\theta(d(x_{n-1}, x_n))}{2} + F(d(x_n, x_{n+1}))\\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0))\\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n) + d(x_{n-1}, x_n), 0))\\
&= F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), 2d(x_{n-1}, x_n), 0))\\
&\leq F(d(x_{n-1}, x_n)),
\end{aligned} \tag{4}$$

for each $n \in \mathbb{N}$.
Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$, for some $r \geq 0$. Now, we show that $r = 0$. On contrary, assume that $r > 0$. From (4) we get

$$\frac{1}{2}\sum_{i=1}^{n-1} \theta(d(x_i, x_{i+1})) \leq F(d(x_1, x_2)) - F(d(x_n, x_{n+1})) \tag{5}$$

for each $n \in \mathbb{N}$. Since $\{d(x_n, x_{n+1})\}$ is strictly decreasing, then from (δ_1) we obtain that $\theta(d(x_n, x_{n+1})) \nrightarrow 0$. Thus, $\sum_{i=1}^{\infty} \theta(d(x_i, x_{i+1})) = +\infty$, and then from (5) we have $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Then by (δ_3), $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$, that a contradiction. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

From (4), (6) and (δ_4), we have $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Then by triangle inequality $\{x_n\}$ is Cauchy R-sequence. Since X is R-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we prove that x is fixed point of T. If there exist a strictly increasing sequence $\{n_k\}$ such that $x_{n_k} \in Tx$ for all $k \in \mathbb{N}$, since Tx is closed and

$x_{n_k} \to x$, as $k \to \infty$, we get that $x \in Tx$ and proof is complete.
So, we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin Tx$, for each $n > n_0$. This implies that $Tx_n \neq Tx$, for each $n \geq n_0$. Now since X is a R-regular metric space by using (1) with $x = x_n$ and $y = x$, we obtain

$$\begin{aligned} &F(D(x_{n+1}, Tx)) < \theta(d(x_n, x)) + F(D(x_{n+1}, Tx)) \\ &\leq \theta(d(x_n, x)) + F(H(Tx_n, Tx)) \\ &\leq F(\phi(d(x_n, x), D(x_n, Tx_n), D(x, Tx), D(x_n, Tx), D(x, Tx_n))) \\ &\leq F(\phi(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x_n, Tx), d(x, x_{n+1}))), \end{aligned}$$

for each $n \geq n_0$.
Therefore

$$D(x_{n+1}, Tx) < \phi(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x_n, Tx), d(x, x_{n+1})), \tag{7}$$

for each $n \geq n_0$. Now if $x \in Tx$, then proof is complete. Let $x \notin Tx$ then by using (7) and (Λ_3) we have

$$\begin{aligned} &D(x, Tx) = \limsup_{n\to\infty} D(x_{n+1}, Tx) \\ &\leq \limsup_{n\to\infty} \phi(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x_n, Tx), d(x, x_{n+1})) \\ &< D(x, Tx), \end{aligned}$$

which is a contradiction. Hence $x \in Tx$ and proof is complete. □

Letting

$$\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + L t_5,$$

where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$, we get a generalization of Theorem 3.4 of [8].

Corollary 3.2. *Let (X, d, R) be a* R*-complete (not necessarily complete),* R*-regular metric space and $T : X \to CB(X)$ be a set-valued mapping. Assume that there exists $(\frac{\theta}{2}, F) \in \Delta$ such that*

$$\begin{aligned} &\theta(d(x, y)) + F(H(Tx, Ty)) \\ &\leq F(\alpha d(x, y) + \beta D(x, Tx) + \gamma D(y, Ty) + \delta D(x, Ty) + L D(y, Tx)), \end{aligned}$$

for each $x \mathrm{R} y$, with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. If the following conditions are satisfied
(i) T is monotone of type (I);
(ii) There exists $x_0 \in X$ such that $\{x_0\} \mathrm{R}_2 Tx_0$;
(iii) T is compact valued or F is continuous from the right;
Then T has a fixed point.

Proof. By using Example 2.1 of [1], we can easily show that this corollary is a generalization of Theorem 3.4 of [8]. □

Letting

$$\phi(t_1, t_2, t_3, t_4, t_5) = t_1$$

we get a generalization of Theorem 2.4 of [1].

Corollary 3.3. *Let (X, d, R) be a* R*-complete (not necessarily complete),* R*-regular metric space and $T : X \to CB(X)$ be a set-valued mapping. Assume that there exists $(\frac{\theta}{2}, F) \in \Delta$ such that*

$$\theta(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)),$$

for each $x \mathrm{R} y$, with $Tx \neq Ty$. If the following conditions are satisfied
(i) T is monotone of type (I);
(ii) There exists $x_0 \in X$ such that $\{x_0\} \mathrm{R}_2 Tx_0$;
(iii) T is compact valued or F is continuous from the right;
Then T has a fixed point.

In below we explain a generalization of Theorem 3.2 of [9].

Corollary 3.4. *Let* (X, d, R) *be a* R*-complete* (*not necessarily complete*), R*-regular metric space and* $T : X \to CB(X)$ *be a set-valued mapping. Assume that*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y), \tag{8}$$

for each $x \,\mathrm{R}\, y$*, with* $Tx \neq Ty$ *where* α *is a function from* $(0, \infty)$ *into* $(0, 1)$ *such that* $\limsup_{s\to t^+} \alpha(s) < 1$ *for all* $t \in [0, \infty)$*. If the following conditions are satisfied*
(*i*) *T is monotone of type* (*I*)*;*
(*ii*) *There exists* $x_0 \in X$ *such that* $\{x_0\} \,\mathrm{R}_2\, Tx_0$*;*
Then T has a fixed point.

Proof. Let $F(t) = \ln(t)$, $\theta(t) = -\ln(\alpha(t))$ for each $t \in (0, \infty)$, and $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ defined by $\phi(t_1, t_2, t_3, t_4, t_5) = t_1$ then $(\theta, F) \in \Delta$ and $\phi \in \Lambda$. Hence by using Theorem 3.1, T has a fixed point. □

Now we illustrate our main results by the following examples.

Example 3.5. *Let* (X, d) *be a metric space, where* $X = \{1, 2, 3, 4\}$, $d(1, 2) = d(1, 3) = 1$, $d(1, 4) = \frac{7}{4}$ *and* $d(2, 3) = d(2, 4) = d(3, 4) = 2$*. Let* $T : X \to CB(X)$ *be given by* $T1 = T4 = \{1, 4\}$, $T2 = T3 = \{4\}$ *and* $\mathrm{R} = \{(1, 1), (1, 2), (1, 3), (1, 4), (4, 1), (4, 4)\}$ *be a binary relation over X. Since X is finite set then every Cauchy sequence in* (X, d) *is equivalent constant and so convergent. Then* (X, d) *is a* R*-complete metric space. It is easy to see that:*
(1) *T is monotone of type* (*I*)*;*
(2) *There exists* $x_0 \in X$ *such that* $\{x_0\} \,\mathrm{R}_2\, Tx_0$*;*
(3) *X is a* R*-regular metric space;*
(4) *Inequality*

$$1 + \ln(H(Tx, Ty)) \leq \ln(\alpha.d(x, y) + L.D(y, Tx)),$$

holds for each $x \,\mathrm{R}\, y$*, with* $Tx \neq Ty$*, where* $\alpha = 1$ *and* $L = 4$*. Then by Corollary 3.2, T has a fixed point.*

Example 3.6. *Let* $X = \{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots\} \cup \{1\}$, $d(x, y) = |x - y|$*, for all* $x, y \in X$*, and binary relation* R *defined over X by*

$$x \;\mathrm{R}\; y \iff \frac{y}{x} \in \mathbb{N}.$$

Let $T : X \to CB(X)$ *defined by*

$$Tx = \begin{cases} \{\frac{1}{2^n}\}, & \text{if} \quad x = \frac{1}{2^n}, n = 1, 2, \cdots, \\ \{1, \frac{1}{2}\}, & \text{if} \quad x = 1. \end{cases}$$

Now we can easily show that
(1) *X is a* R*-complete (not complete metric space) and* R*-regular metric space. Furthermore, every* R*-sequence is convergence;*
(2) *T is monotone of type* (*I*)*;*
(3) *There exists* $x_0 \in X$ *such that* $\{x_0\} \,\mathrm{R}_2\, Tx_0$*;*
(4) *Inequality*

$$1 + \ln(H(Tx, Ty)) \leq \ln(\alpha.d(x, y) + L.D(y, Tx)),$$

holds for for each $x \,\mathrm{R}\, y$*, with* $Tx \neq Ty$*, where* $\alpha = 1$ *and* $L = 2$*. Then by Corollary 3.2, T has a fixed point.*

Example 3.7. *Consider the sequence* $\{S_n\}$ *as follows:*

$$S_1 = 1 \times 2,$$
$$S_2 = 1 \times 2 + 2 \times 3,$$
$$S_3 = 1 \times 2 + 2 \times 3 + 3 \times 4,$$
$$\cdots$$
$$S_n = 1 \times 2 + 2 \times 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}, n \in \mathbb{N}.$$

Let $X = \{S_n : n \in \mathbb{N}\}$ *and* $d(x, y) = |x - y|$, $x, y \in X$. *For all* $S_n, S_m \in X$ *define* $S_n \ \mathrm{R} \ S_m$ *if and only if* $(1 = n \leq m)$. *Hence* (X, d, R) *is a* R*-complete and* R*-regular metric space. Define set-valued mapping* $T : X \to CB(X)$ *by the formulae:*

$$Tx = \begin{cases} \{S_{n-1}, S_{n+1}\}, & \text{if} \quad x = S_n, n = 3, 4, \cdots, \\ \{S_1\}, & \text{if} \quad x = S_1, S_2. \end{cases}$$

It is easy to see that T *is monotone mapping of type* (I) *and* $\{S_1\} \ \mathrm{R}_2 \ TS_1$.
Now since,

$$\lim_{n\to\infty} \frac{H(T(S_n), T(S_1))}{d(S_n, S_1)} = 1,$$

then T *is not* R*-contraction.*
First, observe that

$$S_n \ \mathrm{R} \ S_m \ , \ T(S_n) \neq T(S_m) \Longleftrightarrow (1 = n, m > 2).$$

On the other hand, for every $m \in \mathbb{N}, m > 2$ *we have*

$$1 + \ln(H(TS_1, TS_m)) \leq \ln(\alpha.d(S_1, S_m) + L.D(S_m, TS_1)),$$

where $\alpha = 1$ *and* $L = 9$. *Then by Corollary 3.2,* T *has a fixed point.*

4. Applications to Ordinary Differential Equations

Our purpose here is to apply Corollary 3.4 to prove the existence of a solution for the following differential equation:

$$\begin{cases} u'(t) = f(t, u(t)), & a.e. \ t \in I = [0, T], \\ u(0) = a, \quad a \geq 1, \end{cases} \tag{9}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:
(c1) $f(s, x) \geq 0$ for all $x \geq 0$ and $s \in I$,
(c2) there exists $\alpha \in L^1(I)$ such that

$$|f(s, x) - f(s, y)| \leq \alpha(s)|x - y|$$

for all $t \in I$ and $x, y \geq 0$ with $xy \geq (x \vee y)$, where $x \vee y = x$ or y.

Note that $f : I \times \mathbb{R} \to \mathbb{R}$ is not necessarily Lipschitz from the given condition (c2). For example, the function

$$f(s, x) = \begin{cases} sx, & x \leq \frac{1}{2}, \\ 0, & x > \frac{1}{2} \end{cases}$$

satisfies the conditions (c1) and (c2) while f is not continuous. Also, for $s \neq 0$,

$$\left| f(s, \frac{1}{2}) - f(s, \frac{2}{3}) \right| = s\frac{1}{2} > s\frac{1}{6} = s\left| \frac{1}{2} - \frac{2}{3} \right|.$$

Theorem 4.1. *Under above assumptions, the differential equation (9) has a positive solution.*

Proof. Let $X = \{u \in C(I, \mathbb{R}) : u(t) > 0, \ \forall t \in I\}$. We consider the following binary relation over X:

$$x \ \mathrm{R} \ y \iff x(t)y(t) \geq (x(t) \vee y(t))$$

for all $t \in I$. Let $A(t) = \int_0^t |\alpha(s)| ds$. Then $A'(t) = |\alpha(t)|$ for almost every $t \in I$. Define

$$\|x\|_A = \sup_{t \in I} e^{-A(t)} |x(t)|, \quad d(x, y) := \|x - y\|_A$$

for all $x, y \in X$. It is easy to see that (X, d) is a metric space.

Now, we show that X is R-complete (not necessarily complete). Take a Cauchy R-sequence $\{x_n\}$ in X. It is easy to show that $\{x_n\}$ is convergent to a point $x \in C(I, \mathbb{R})$. Observe that $C(I, \mathbb{R})$ is a Banach space with this norm since it is equivalent to the maximum norm. It is enough to show the $x \in X$. Fix $t \in I$. The definition of relation R implies that

$$x_n(t) \, x_{n+1}(t) \geq (x_n(t) \vee x_{n+1}(t))$$

for each $n \in \mathbb{N}$. Since $x_n(t) > 0$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ in $\{x_n\}$ for which $x_{n_k}(t) \geq 1$ for each $k \in \mathbb{N}$. The convergence of this sequence of real numbers to $x(t)$ implies that $x(t) \geq 1$. But since $t \in I$ is arbitrary, it follows that $x \geq 1$ and hence $x \in X$. By similar reason, we can prove that (X, d, R) is a R-regular metric space. Define a mapping $\mathcal{F} : X \to X$ by

$$\mathcal{F}u(t) = \int_0^t f(s, u(s)) ds + a.$$

Note that the fixed points of $\mathcal{F}$ are the solutions of (9). To complete the proof, we need the following steps:

Step 1: $\mathcal{F}$ is monotone of type (I).
In fact, for all $x, y \in X$ with $x \ \mathrm{R} \ y$ and $t \in I$,

$$\mathcal{F}x(t) = \int_0^t f(s, x(s)) ds + a \geq 1,$$

which implies that $\mathcal{F}x(t)\mathcal{F}y(t) \geq \mathcal{F}x(t)$ and so $\mathcal{F}x \ \mathrm{R} \ \mathcal{F}y$. Moreover, for each $x \in X$, $x \ \mathrm{R} \ \mathcal{F}x$.

Step 2: $\mathcal{F}$ satisfies in contractive condition (8).
In fact, for all $x, y \in X$ with $x \ \mathrm{R} \ y$ and $t \in I$, the condition (c2) implies that

$$\begin{aligned}
e^{-A(t)} |\mathcal{F}x(t) - \mathcal{F}y(t)| &\leq e^{-A(t)} \int_0^t |f(s, x(s)) - f(s, y(s))| ds \\
&\leq e^{-A(t)} \int_0^t |\alpha(s)| e^{A(s)} e^{-A(s)} \, |x(s) - y(s)| ds \\
&\leq e^{-A(t)} \Big(\int_0^t |\alpha(s)| e^{A(s)} ds \Big) \, \|x - y\|_A \\
&\leq e^{-A(t)} (e^{A(t)} - 1) \, \|x - y\|_A \\
&\leq (1 - e^{-\|\alpha\|_1}) \, \|x - y\|_A
\end{aligned}$$

and so

$$\|\mathcal{F}x - \mathcal{F}y\|_A \leq (1 - e^{-\|\alpha\|_1}) \, \|x - y\|_A.$$

Since $1 - e^{-\|\alpha\|_1} < 1$, $\mathcal{F}$ satisfies in contractive condition (8).

Thus, Corollary 3.4 applies that the operator $\mathcal{F}$ has a fixed point. □

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