Filomat 28:9 (2014), 1935–1941
DOI 10.2298/FIL1409935Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Differences of Composition Operators from Weighted Bergman Spaces to Bloch Spaces

Xiangling Zhu[a], Weifeng Yang[b]

[a] *Department of Mathematics, JiaYing University, 514015, Meizhou, GuangDong, China*
[b] *Department of Mathematics and Physics, Hunan Institute of Engineering, 411104, Xiangtan, Hunan, China*

Abstract. The boundedness and compactness of the differences of two composition operators from weighted Bergman spaces to Bloch spaces in the unit disk are investigated in this paper.

1. Introduction

Let D denote the open unit disk in the complex plane $\mathbb{C}$ and $H(D)$ the space of all analytic functions in D. For $a \in D$, let σ_a be the Möbius transformation of D exchanging 0 for a, namely $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in D$. Let $\rho(z,a)$ denote the pseudo-hyperbolic distance between z and a, i.e.,

$$\rho(z,a) = |\sigma_a(z)| = \left|\frac{a-z}{1-\bar{a}z}\right|.$$

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space, denoted by A^p_α, is the set of all functions $f \in H(D)$ satisfying

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_D |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in D such that $A(D) = 1$.

The Bloch space, denoted by $\mathscr{B} = \mathscr{B}(D)$, is the set of all $f \in H(D)$ such that

$$\beta(f) = \sup_{z\in D}(1-|z|^2)|f'(z)| < \infty.$$

Under the norm $\|f\|_{\mathscr{B}} = |f(0)| + \beta(f)$, the Bloch space is a Banach space.

Throughout the paper, $S(D)$ denotes the set of all analytic self-maps of D. Associated with $\varphi \in S(D)$ is the composition operator C_φ defined by

$$(C_\varphi f)(z) = f(\varphi(z)),$$

for $f \in H(D)$. For a general reference on composition operator see the book [3]. For some results on composition are related operators from or into Bergman spaces and Bloch-type spaces see, for example, [1, 10–16, 22–24, 28, 31, 33] and the related references therein.

2010 *Mathematics Subject Classification*. Primary 47B33, Secondary 30H05.
Keywords. Difference, composition operator, Bloch space, weighted Bergman space.
Received: 22 August 2013; Accepted: 18 November 2013
Communicated by Dragana Cvetković Ilić
Email addresses: xiangling-zhu@163.com (Xiangling Zhu), yangweifeng09@163.com (Weifeng Yang)

To understand the topological structure of the set of composition operators on some function spaces, many researchers recently studied the differences of two composition operators, i.e.,

$$T = C_\varphi - C_\psi,$$

where $\varphi, \psi \in S(D)$. For the study of differences of composition operators, see, for example, [2, 4–9, 17–21, 25–27, 29, 30] and the references therein.

Motivated by [9], here we give some necessary and sufficient conditions for the boundedness and compactness of the differences of two composition operators from weighted Bergman spaces into Bloch spaces.

Constants are denoted by C in this paper, they are positive and not necessary the same in each occurrence.

2. Main Results and Proofs

In this section we give our main results. In order to prove the main results of this paper, the following auxiliary lemmas are needed. The first lemma can be found, for example, in [11].

Lemma 1. *Let $0 < p < \infty$ and $\alpha > -1$. If $f \in A^p_\alpha$, then*

$$|f(z)| \le C\frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}} \quad \text{and} \quad |f'(z)| \le C\frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha+p}{p}}}. \tag{1}$$

Lemma 2. *Let $0 < p < \infty, \alpha > -1$. Then there exists a constant $C > 0$ such that*

$$\sup_{f \in B_{A^p_\alpha}} |(1-|z|^2)^{\frac{2+\alpha+p}{p}} f'(z) - (1-|w|^2)^{\frac{2+\alpha+p}{p}} f'(w)| \le C\rho(z,w)$$

for $z, w \in D$, where $B_{A^p_\alpha} = \{f \in A^p_\alpha : \|f\|_{A^p_\alpha} \le 1\}$.

Proof. From [20], we see that

$$|(1-|z|^2)^\beta f(z) - (1-|w|^2)^\beta f(w)| \le C\rho(z,w) \sup_{z\in D}(1-|z|^2)^\beta |f(z)|$$

for any $f \in H(D)$. Hence

$$|(1-|z|^2)^\beta f'(z) - (1-|w|^2)^\beta f'(w)| \le C\rho(z,w) \sup_{z\in D}(1-|z|^2)^\beta |f'(z)|$$

for any $f \in H(D)$. Then the result follows by Lemma 1 with $\beta = \frac{2+\alpha+p}{p}$.

Lemma 3. [11] *Let $0 < p < \infty, \alpha > -1$ and $\varphi \in S(D)$. Then $C_\varphi : A^p_\alpha \to \mathscr{B}$ is compact if and only if $C_\varphi : A^p_\alpha \to \mathscr{B}$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}} = 0. \tag{2}$$

The following lemma can be proved in a standard way (see, e.g., Proposition 3.11 in [3]).

Lemma 4. *Let $\varphi, \psi \in S(D)$ and $0 < p < \infty, \alpha > -1$. Then $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is compact if and only if $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathscr{B}$ which converges to zero uniformly on compact subsets of D, $\|(C_\varphi - C_\psi)f_k\|_{\mathscr{B}} \to 0$ as $k \to \infty$.*

We define

$$\mathscr{D}_\varphi(z) := \frac{(1-|z|^2)\varphi'(z)}{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}}, \qquad \mathscr{D}_\psi(z) := \frac{(1-|z|^2)\psi'(z)}{(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}}.$$

Now we are in a position to state and prove our main results in this paper.

Theorem 1. *Let $\varphi, \psi \in S(D)$ and $0 < p < \infty, \alpha > -1$. Then the following statements are equivalent.*

(i) $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ *is bounded;*

(ii)

$$\sup_{z\in D} |\mathscr{D}_\varphi(z)|\rho(\varphi(z), \psi(z)) < \infty \ \text{and} \ \sup_{z\in D} |\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| < \infty;$$

(iii)

$$\sup_{z\in D} |\mathscr{D}_\psi(z)|\rho(\varphi(z), \psi(z)) < \infty \ \text{and} \ \sup_{z\in D} |\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| < \infty.$$

Proof. $(i) \Rightarrow (ii)$. Assume that $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is bounded. For $a \in D$ with $a \neq 0$, set

$$f_a(z) = \frac{p(1-|a|^2)}{(2+\alpha+p)\bar{a}(1-\bar{a}z)^{\frac{2+\alpha+p}{p}}} \tag{3}$$

and

$$g_a(z) = \frac{p(a-z)(1-|a|^2)}{(2+\alpha+2p)\bar{a}(1-\bar{a}z)^{\frac{2+\alpha+2p}{p}}} + \frac{p^2(1-|a|^2)}{(2+\alpha+2p)(2+\alpha+p)\bar{a}^2(1-\bar{a}z)^{\frac{2+\alpha+p}{p}}}. \tag{4}$$

Then it is easy to check that $f_a, g_a \in A^p_\alpha$. Moreover $\sup_{a\in D} \|f_a\|_{A^p_\alpha}$ and $\sup_{a\in D} \|g_a\|_{A^p_\alpha}$ are bounded. Fix $w \in D$ with $\varphi(w) \neq 0$, we have

$$\begin{aligned}
\infty &> \|(C_\varphi - C_\psi)f_{\varphi(w)}\|_{\mathscr{B}} \geq \sup_{z\in D}(1-|z|^2)|((C_\varphi - C_\psi)f_{\varphi(w)})'(z)| \\
&\geq \left|\frac{(1-|w|^2)(1-|\varphi(w)|^2)\varphi'(w)}{(1-|\varphi(w)|^2)^{\frac{2+\alpha+2p}{p}}} - \frac{(1-|w|^2)(1-|\varphi(w)|^2)\psi'(w)}{(1-\overline{\varphi(w)}\psi(w))^{\frac{2+\alpha+2p}{p}}}\right| \\
&\geq |\mathscr{D}_\varphi(w)| - \left|\mathscr{D}_\psi(w)\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{\varphi(w)}\psi(w))^{\frac{2+\alpha+2p}{p}}}\right|
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\infty &> \|(C_\varphi - C_\psi)g_{\varphi(w)}\|_{\mathscr{B}} \geq \sup_{z\in D}(1-|z|^2)|((C_\varphi - C_\psi)g_{\varphi(w)})'(z)| \\
&\geq \frac{(1-|w|^2)(1-|\varphi(w)|^2)|\psi'(w)|}{|1-\overline{\varphi(w)}\psi(w)|^{\frac{2+\alpha+2p}{p}}}\left|\frac{\varphi(w)-\psi(w)}{1-\overline{\varphi(w)}\psi(w)}\right| \\
&= \left|\mathscr{D}_\psi(w)\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{\varphi(w)}\psi(w))^{\frac{2+\alpha+2p}{p}}}\right|\rho(\varphi(w), \psi(w)).
\end{aligned} \tag{6}$$

Set $D_1 = \{w \in D : \varphi(w) = 0\}$, $D_2 = \{w \in D : \psi(w) = 0\}$. Multiplying (5) by $\rho(\varphi(w), \psi(w))$ and using (6), we obtain

$$\sup_{w\in D\setminus D_1} |\mathscr{D}_\varphi(w)|\rho(\varphi(w), \psi(w)) < \infty. \tag{7}$$

Similarly we can obtain

$$\sup_{w\in D\setminus D_2} |\mathscr{D}_\psi(w)|\rho(\varphi(w), \psi(w)) < \infty. \tag{8}$$

From (5), we have

$$\begin{aligned}
\infty &> \|(C_\varphi - C_\psi)f_{\varphi(w)}\|_{\mathscr{B}} \\
&\geq \left|\frac{(1-|w|^2)(1-|\varphi(w)|^2)\varphi'(w)}{(1-|\varphi(w)|^2)^{\frac{2+\alpha+2p}{p}}} - \frac{(1-|w|^2)(1-|\varphi(w)|^2)\psi'(w)}{(1-\overline{\varphi(w)}\psi(w))^{\frac{2+\alpha+p}{p}}}\right| \\
&\geq |\mathscr{D}_\varphi(w) - \mathscr{D}_\psi(w)| - |\mathscr{D}_\psi(w)|\left|1 - \frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{\varphi(w)}\psi(w))^{\frac{2+\alpha+2p}{p}}}\right| \\
&\geq |\mathscr{D}_\varphi(w) - \mathscr{D}_\psi(w)| - C|\mathscr{D}_\psi(w)|\rho(\varphi(w),\psi(w)), \qquad (9)
\end{aligned}$$

which with (8) implies

$$\sup_{w\in D\setminus\{D_1\cup D_2\}} |\mathscr{D}_\varphi(w) - \mathscr{D}_\psi(w)| < \infty. \qquad (10)$$

If $\psi(w) = 0$ and $\varphi(w) \neq 0$, set

$$h_{\varphi(w)}(z) = \frac{\varphi(w) - z}{\overline{\varphi(w)}(1-\overline{\varphi(w)}z)^{\frac{2+\alpha+2p}{p}}} + \frac{p}{(2+\alpha+p)\overline{\varphi(w)}^2(1-\overline{\varphi(w)}z)^{\frac{2+\alpha+p}{p}}}.$$

We get

$$\begin{aligned}
\infty &> \|(C_\varphi - C_\psi)h_{\varphi(w)}\|_{\mathscr{B}} \geq \sup_{z\in D}(1-|z|^2)|((C_\varphi - C_\psi)h_{\varphi(w)})'(z)| \\
&\geq \frac{2+\alpha+2p}{p}(1-|w|^2)\left|\frac{(\varphi(w)-\psi(w))\psi'(w)}{(1-\overline{\varphi(w)}\psi(w))^{\frac{2+\alpha+3p}{p}}}\right| \\
&= \frac{2+\alpha+2p}{p}(1-|w|^2)|\psi'(w)\varphi(w)| = \frac{2+\alpha+2p}{p}|\mathscr{D}_\psi(w)|\rho(\varphi(w),\psi(w)),
\end{aligned}$$

which implies

$$\sup_{w\in D_2\setminus D_1} |\mathscr{D}_\psi(w)|\rho(\varphi(w),\psi(w)) < \infty. \qquad (11)$$

From (9) and (11) we obtain

$$\sup_{w\in D_2\setminus D_1} |\mathscr{D}_\varphi(w) - \mathscr{D}_\psi(w)| < \infty. \qquad (12)$$

If $\psi(w) = 0$ and $\varphi(w) \neq 0$, similarly to the above proof we have

$$\sup_{w\in D_1\setminus D_2} |\mathscr{D}_\varphi(w) - \mathscr{D}_\psi(w)| < \infty, \ \sup_{w\in D_1\setminus D_2} |\mathscr{D}_\varphi(w)|\rho(\varphi(w),\psi(w)) < \infty. \qquad (13)$$

If $\varphi(w) = \psi(w) = 0$, taking $f_0 = z$ and using the boundedness of $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$, we obtain

$$\sup_{w\in D_1\cap D_2} |\mathscr{D}_\varphi(w) - \mathscr{D}_\psi(w)| = \sup_{w\in D_1\cap D_2}(1-|w|^2)|\varphi'(w) - \psi'(w)| \leq \|(C_\varphi - C_\psi)f_0\|_{\mathscr{B}} < \infty, \qquad (14)$$

$$\sup_{w\in D_1\cap D_2} |\mathscr{D}_\varphi(w)|\rho(\varphi(w),\psi(w)) = 0, \ \sup_{w\in D_1\cap D_2} |\mathscr{D}_\psi(w)|\rho(\varphi(w),\psi(w)) = 0. \qquad (15)$$

By (7),(13) and (15) we get

$$\sup_{z\in D} |\mathscr{D}_\varphi(z)|\rho(\varphi(z),\psi(z)) < \infty.$$

By (10), (12), (13) and (14) we get

$$\sup_{z\in D} |\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| < \infty.$$

(*ii*) ⇒ (*iii*). Assume the conditions in *(ii)* hold. Then

$$\sup_{z\in D}|\mathscr{D}_\psi(z)|\rho(\varphi(z),\psi(z)) \le \sup_{z\in D}|\mathscr{D}_\varphi(z)|\rho(\varphi(z),\psi(z)) + \sup_{z\in D}|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)|\rho(\varphi(z),\psi(z)) < \infty.$$

Therefore *(iii)* holds.

(*iii*) ⇒ (*i*). Let $f\in A^p_\alpha$ with $\|f\|_{A^p_\alpha}\le 1$. Using Lemmas 1 and 2 we have

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)|((C_\varphi - C_\psi)f)'(z)| = \sup_{z\in D}|(1-|z|^2)f'(\varphi(z))\varphi'(z) - (1-|z|^2)f'(\varphi(z))\psi'(z)|\\
=\ & \sup_{z\in D}\Big|\mathscr{D}_\varphi(z)(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}f'(\varphi(z)) - \mathscr{D}_\psi(z)(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}f'(\psi(z))\Big|\\
\le\ & \sup_{z\in D}\Big|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)\Big|(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}|f'(\varphi(z))|\\
&+\sup_{z\in D}\Big|\mathscr{D}_\psi(z)\Big|\Big|(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}f'(\varphi(z)) - (1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}f'(\psi(z))\Big|\\
\le\ & C\sup_{z\in D}\Big|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)\Big| + C\sup_{z\in D}\Big|\mathscr{D}_\psi(z)\Big|\rho(\varphi(z),\psi(z)) < \infty.
\end{aligned}$$

In addition, by Lemma 1 we have

$$|((C_\varphi - C_\psi)f)(0)| \le |f(\varphi(0))| + |f(\psi(0))| \le C\frac{\|f\|_{A^p_\alpha}}{(1-|\varphi(0)|^2)^{\frac{2+\alpha}{p}}} + C\frac{\|f\|_{A^p_\alpha}}{(1-|\psi(0)|^2)^{\frac{2+\alpha}{p}}} < \infty.$$

Hence $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is bounded. The proof of Theorem 1 is completed. □

To state the following theorem, we set

$$\Gamma(\varphi) = \{(z_n)\subset D : |\varphi(z_n)|\to 1\},\quad \Gamma(\psi) = \{(z_n)\subset D : |\psi(z_n)|\to 1\},$$

$$D(\varphi) = \{(z_n)\subset D : |\varphi(z_n)|\to 1, |\mathscr{D}_\varphi(z_n)|\nrightarrow 0\},\quad D(\psi) = \{(z_n)\subset D : |\psi(z_n)|\to 1, |\mathscr{D}_\psi(z_n)|\nrightarrow 0\}.$$

Theorem 2. *Let $\varphi,\psi\in S(D)$, $0<p<\infty$, $\alpha>-1$. Suppose that $C_\varphi, C_\psi : A^p_\alpha\to\mathscr{B}$ is bounded and $C_\varphi, C_\psi : A^p_\alpha\to\mathscr{B}$ neither of them is compact. Then $C_\varphi - C_\psi : A^p_\alpha\to\mathscr{B}$ is compact if and only if both (a) and (b) hold:*
(a) $D(\varphi) = D(\psi)\neq\emptyset$, $D(\varphi)\subset\Gamma(\psi)$.
(b) For $z_n\in\Gamma(\varphi)\cap\Gamma(\psi)$,

$$\lim_{n\to\infty}|\mathscr{D}_\varphi(z_n)|\rho(\varphi(z_n),\psi(z_n)) = \lim_{n\to\infty}|\mathscr{D}_\psi(z_n)|\rho(\varphi(z_n),\psi(z_n)) = \lim_{n\to\infty}|\mathscr{D}_\varphi(z_n)-\mathscr{D}_\psi(z_n)| = 0.$$

Proof. Necessity. First we assume that $C_\varphi - C_\psi : A^p_\alpha\to\mathscr{B}$ is compact. By the assumption that $C_\varphi : A^p_\alpha\to\mathscr{B}$ is not compact, from Lemma 3, there exists a sequence $(z_n)\subset D(\varphi)$ with $|\varphi(z_n)|\to 1$ such that $|\mathscr{D}_\varphi(z_n)|\nrightarrow 0$. For $a=\varphi(z_n)$, define f_a, g_a as in the proof of Theorem 1. We know that $f_a, g_a\in A^p_\alpha$ and converge to 0 uniformly on every compact subset of D as $|w|\to 1$. From Lemma 4, we have

$$\begin{aligned}
0 \ \leftarrow\ & \|(C_\varphi - C_\psi)f_{\varphi(z_n)}\|_{\mathscr{B}} \ge (1-|z_n|^2)|((C_\varphi - C_\psi)f_{\varphi(z_n)})'(z_n)|\\
\ge\ & |\mathscr{D}_\varphi(z_n)| - \Big|\mathscr{D}_\psi(z_n)\frac{(1-|\varphi(z_n)|^2)(1-|\psi(z_n)|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{\varphi(z_n)}\psi(z_n))^{\frac{2+\alpha+2p}{p}}}\Big|
\end{aligned}\tag{16}$$

and

$$\begin{aligned}
0 \ \leftarrow\ & \|(C_\varphi - C_\psi)g_{\varphi(z_n)}\|_{\mathscr{B}} \ge (1-|z_n|^2)|((C_\varphi - C_\psi)g_{\varphi(z_n)})'(z_n)|\\
\ge\ & \Big|\frac{\mathscr{D}_\psi(z_n)(1-|\varphi(z_n)|^2)(1-|\psi(z_n)|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{\varphi(z_n)}\psi(z_n))^{\frac{2+\alpha+2p}{p}}}\Big|\rho(\varphi(z_n),\psi(z_n)),
\end{aligned}\tag{17}$$

as $n \to \infty$. Multiplying (16) by $\rho(\varphi(z_n), \psi(z_n))$ and using (17), we get

$$\lim_{n\to\infty} |\mathscr{D}_\varphi(z_n)|\rho(\varphi(z_n), \psi(z_n)) = 0. \tag{18}$$

Similarly to the above proof we have

$$\lim_{n\to\infty} |\mathscr{D}_\psi(z_n)|\rho(\varphi(z_n), \psi(z_n)) = 0. \tag{19}$$

Since $|\mathscr{D}_\varphi(z_n)| \nrightarrow 0$, (18) implies that $\lim_{n\to\infty} \rho(\varphi(z_n), \psi(z_n)) = 0$. Hence, for any $z_n \in D(\varphi)$, $\lim_{n\to\infty} |\varphi(z_n) - \psi(z_n)| = 0$. Therefore

$$D(\varphi) \subset \Gamma(\psi). \tag{20}$$

In addition, we have

$$|\mathscr{D}_\varphi(z_n) - \mathscr{D}_\psi(z_n)| - C|\mathscr{D}_\psi(z_n)|\rho(\varphi(z_n), \psi(z_n)) \le \|(C_\varphi - C_\psi) f_{\varphi(z_n)}\|_{\mathscr{B}} \to 0$$

as $n \to \infty$. Hence by (19), we get

$$\lim_{n\to\infty} |\mathscr{D}_\varphi(z_n) - \mathscr{D}_\psi(z_n)| = 0. \tag{21}$$

Hence, from (20) and (21), we have $D(\varphi) \subset D(\psi)$. Similarly to the above proof we can obtain that $D(\psi) \subset D(\varphi)$. Therefore $D(\varphi) = D(\psi)$.

For any sequence $\{z_n\}$ such that $|\varphi(z_n)| \to 1$, $|\psi(z_n)| \to 1$ and $|\mathscr{D}_\varphi(z_n)| \to 0$, we have

$$\lim_{n\to\infty} |\mathscr{D}_\varphi(z_n)|\rho(\varphi(z_n), \psi(z_n)) = 0. \tag{22}$$

In addition,

$$\begin{aligned} 0 \leftarrow \quad & \|(C_\varphi - C_\psi) f_{\psi(z_n)}\|_{\mathscr{B}} \ge (1 - |z_n|^2)|((C_\varphi - C_\psi) f_{\psi(z_n)})'(z_n)| \\ = \quad & |\mathscr{D}_\varphi(z_n) - \mathscr{D}_\psi(z_n)| - C|\mathscr{D}_\varphi(z_n)|\rho(\varphi(z_n), \psi(z_n)) \end{aligned} \tag{23}$$

as $n \to \infty$. We obtain $\lim_{n\to\infty} |\mathscr{D}_\varphi(z_n) - \mathscr{D}_\psi(z_n)| = 0$ and hence $\lim_{n\to\infty} |\mathscr{D}_\varphi(z_n)| = \lim_{n\to\infty} |\mathscr{D}_\psi(z_n)| = 0$. Therefore $\lim_{n\to\infty} |\mathscr{D}_\psi(z_n)|\rho(\varphi(z_n), \psi(z_n)) = 0$.

Sufficiency. Now we assume that (a) and (b) hold. From the assumption and Theorem 3.1 of [11], we have

$$\sup_{z\in D} |\mathscr{D}_\varphi(z)| < \infty, \quad \sup_{z\in D} |\mathscr{D}_\psi(z)| < \infty. \tag{24}$$

Let $\{f_n\}$ be a sequence in A^p_α such that $\|f_n\|_{A^p_\alpha} \le 1$ and converges to 0 uniformly on every compact subset of D. To prove that $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is compact, by Lemma 4, we need to prove $\|(C_\varphi - C_\psi) f_n\|_{\mathscr{B}} \to 0$ as $n \to \infty$. Suppose not, since $f_n(\varphi(0)), f_n(\varphi(0)) \to 0$ as $n \to \infty$, we may assume that for some $\varepsilon > 0$, $\|(C_\varphi - C_\psi) f_n\|_{\mathscr{B}} > \varepsilon$ for all n. Then there exists a sequence $z_n \in D$ such that

$$|\mathscr{D}_\varphi(z_n)(1 - |\varphi(z_n)|^2)^{\frac{2+\alpha+p}{p}} f_n'(\varphi(z_n)) - \mathscr{D}_\psi(z_n)(1 - |\psi(z_n)|^2)^{\frac{2+\alpha+p}{p}} f_n'(\psi(z_n))| > \varepsilon. \tag{25}$$

for every n. This implies that $\max\{|\varphi(z_n)|, |\psi(z_n)|\} \to 1$, as $n \to \infty$ by the facts (24) and $\{f_n'\}$ also converges to 0 uniformly on every compact subset of D. Assume that $|\varphi(z_n)| \to 1$ and $\psi(z_n) \to w$, for some complex number w. If $|w| < 1$, then $z_n \notin \Gamma(\varphi) \cap \Gamma(\psi)$. Since $D(\varphi) \subset \Gamma(\psi)$, we have $|\mathscr{D}_\varphi(z_n)| \to 0$. On the other hand, by the boundedness of $C_\psi : A^p_\alpha \to \mathscr{B}$ we get $\psi \in \mathscr{B}$, i.e., we have

$$|\mathscr{D}_\psi(z_n)|(1 - |\psi(z_n)|^2)^{\frac{2+\alpha+p}{p}} = (1 - |z_n|^2)|\psi'(z_n)| < \infty.$$

Moreover, $|w| < 1$ yields $f_n'(\psi(z_n)) \to 0$. This contradicts (25). We obtain $|w| = 1$. Therefore $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$. From the assumption we obtain that

$$\begin{aligned} & |\mathscr{D}_\varphi(z_n)(1 - |\varphi(z_n)|^2)^{\frac{2+\alpha+p}{p}} f_n'(\varphi(z_n)) - \mathscr{D}_\psi(z_n)(1 - |\psi(z_n)|^2)^{\frac{2+\alpha+p}{p}} f_n'(\psi(z_n))| \\ & \le |\mathscr{D}_\varphi(z_n) - \mathscr{D}_\psi(z_n)| + C|\mathscr{D}_\varphi(z_n)|\rho(\varphi(z_n), \psi(z_n)) \to 0, \end{aligned}$$

as $n \to \infty$. This also contradicts (25). The proof of this theorem is finished. $\square$

Acknowledgments. The first author is supported by project of Department of Education of Guangdong Province(No.2012KJCX0096) and Natural Science Foundation of Guangdong Province(No.S2013010011978). The second author is supported by Hunan Provincial Natural Science Foundation of China (No. 13JJ4099).

References

[1] R. F. Allen and F. Colonna, Weighted composition operators on the Bloch space of a bounded homogeneous domain, *Operator Theory: Advances and Applications* **202** (2010), 11-37
[2] J. Bonet, M. Lindström and E. Wolf, Differences of composition operators between weighted Banach spaces of holomorphic functions, *J. Austral. Math. Soc.* **84** (2008), 9-20.
[3] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Math., CRC Press, Boca Raton, 1995.
[4] Z. Fang and Z. Zhou, Differences of composition operators on the Bloch space in the polydisc, *Bull. Austral. Math. Soc.* **79** (2009), 465-471.
[5] T. Hosokawa, Differences of weighted composition operators on the Bloch spaces, *Complex Anal. Oper. Theory*, **3** (2009), 847-866.
[6] T. Hosokawa and S. Ohno, Topological structures of the sets of composition operators on the Bloch spaces, *J. Math. Anal. Appl.* **314** (2006), 736-748.
[7] T. Hosokawa and S. Ohno, Differences of composition operators on the Bloch spaces, *J. Operator Theory*, **57** (2007), 229-242.
[8] Z. Jiang and S. Stević, Compact differences of weighted composition operators from weighted Bergman spaces to weighted-type spaces, *Appl. Math. Comput.* **217** (2010), 3522-3530.
[9] S. Li, Differences of generalized composition operators on the Bloch space, *J. Math. Anal. Appl.* **394** (2012) 706-711.
[10] S. Li and S. Stević, Composition followed by differentiation between Bloch type spaces, *J. Comput. Anal. Appl.* **9** (2) (2007), 195-205.
[11] S. Li and S. Stević, Weighted composition operators from Bergman-type spaces into Bloch spaces, *Proc. Indian Acad. Sci. Math. Sci.* **117** (2007), 371-385.
[12] S. Li and S. Stević, Weighted composition operators from H^∞ to the Bloch space on the polydisc, *Abstr. Appl. Anal.* Vol. 2007, Article ID 48478, (2007), 12 pages.
[13] S. Li and S. Stević, Products of composition and integral type operators from H^∞ to the Bloch space, *Complex Variables Elliptic Equations* **53** (5) (2008), 463-474.
[14] S. Li and S. Stević, Riemann-Stieltjes operators between different weighted Bergman spaces, *Bull. Belg. Math. Soc. Simon Stevin* **15** (4) (2008), 677-686.
[15] S. Li and S. Stević, Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball, *Taiwanese J. Math.* **12** (2008), 1625-1639.
[16] Z. Lou, Composition operators on Bloch type spaces, *Analysis (Munich)*, **23** (2003), 81-95.
[17] B. MacCluer, S. Ohno and R. Zhao, Topological structure of the space of composition operators on H^∞, *Integral Equations Operator Theory*, **40** (2001), 481-494.
[18] J. Manhas, Compact diffrences of weighted composition operators on weighted Banach spaces of analytic functions, *Integral Equations Operator Theory*, **62** (2008), 419-428.
[19] J. Moorhouse, Compact differences of composition operators, *J. Funct. Anal.* **219** (2005), 70-92.
[20] P. Nieminen, Compact differences of composition operators on Bloch and Lipschitz spaces, *Comput. Methods Function Theory*, **7** (2007), 325-344.
[21] J. Shapiro and C. Sundberg, Isolation amongst the composition operators, *Pacific J. Math.* **145** (1990), 117-152.
[22] S. Stević, Essential norms of weighted composition operators from the α-Bloch space to a weighted-type space on the unit ball, *Abstr. Appl. Anal.* Vol. 2008, Article ID 279691, (2008), 11 pages.
[23] S. Stević, On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball, *Discrete Dyn. Nat. Soc.* Vol. 2008, Article ID 154263, (2008), 14 pages.
[24] S. Stević, Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball, *Appl. Math. Comput.* **212** (2009), 499-504.
[25] S. Stević, Essential norm of differences of weighted composition operators between weighted-type spaces on the unit ball, *Appl. Math. Comput.* **217** (2010), 1811-1824.
[26] S. Stević and Z. Jiang, Differences of weighted composition operators on the unit polydisk, *Siberian Math. J.* **52** (2) (2011), 358-371.
[27] S. Stević and E. Wolf, Differences of composition operators between weighted-type spaces of holomorphic functions on the unit ball of $\mathbb{C}^n$, *Appl. Math. Comput.* **215** (2009), 1752-1760.
[28] S. Ueki, Order bounded weighted composition operators mapping into the Bergman space, *Complex. Anal. Operat. Theory* **6** (3) (2012), 549-560.
[29] E. Wolf, Compact differences of composition operators, *Bull. Aust. Math. Soc.* **77** (2008), 161-165.
[30] K. Yang and Z. Zhou, Essential norm of the difference of composition operators on Bloch space, *Czechoslovak Math. J.* **60** (2010), 1139-1152.
[31] W. Yang, Generalized weighted composition operators from the $F(p,q,s)$ space to the Bloch-type space, *Appl. Math. Comput.* **218** (2012), 4967-4972.
[32] K. Zhu, *Operator Theory on Function Spaces*, Marcel Dekker, Inc. Pure and Applied Mathematics 139, New York and Basel, 1990.
[33] X. Zhu, Weighted composition operators from logarithmic Bloch spaces to a class of weighted-type spaces in the unit ball, *Ars Combin.* **91** (2009), 87-95.