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New Results for Srivastava's λ-Generalized Hurwitz-Lerch Zeta Function

Min-Jie Luo[a], R.K. Raina[b]

[a]*Department of Mathematics, East China Normal University, Shanghai 200241, People's Republic of China*
[b]*M.P. University of Agriculture and Technology,*
Present address: *10/11, Ganpati Vihar, Opposite Sector 5, Udaipur-313002, Rajasthan, India*

Abstract. In view of the relationship with the Krätzel function, we derive a new series representation for the λ-generalized Hurwitz-Lerch Zeta function introduced by H.M. Srivastava [Appl. Math. Inf. Sci. 8 (2014) 1485–1500] and determine the monotonicity of its coefficients. An integral representation of the Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$-series is rederived by applying the Abel's summation formula (which provides a slight modification of the result given by Pogány [Integral Transforms Spec. Funct. 16 (8) (2005) 685–689]) and this modified form of the result is then used to obtain a new integral representation for Srivastava's λ-generalized Hurwitz-Lerch Zeta function. Finally, by making use of the various results presented in this paper, we establish two sets of two-sided inequalities for Srivastava's λ-generalized Hurwitz-Lerch Zeta function.

1. Introduction

Throughout our present investigation, we use the following notations:

$$\mathbb{N} := \{1, 2, \cdots\}, \ \ \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \ \ \mathbb{Z}^- := \{-1, -2, \cdots\} \ \text{ and } \ \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\},$$

where $\mathbb{Z}$ denotes the set of integers. The symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the set of real, positive real, and complex numbers, respectively. The normalized Fox-Wright function ${}_p\Psi_q^*[\cdot; z]$ is defined by (see [18, p. 516, Eq. (1)] and [20, p. 493, Eq. (2.1)])

$${}_p\Psi_q^*\left[\begin{matrix}(\lambda_p, \rho_p)\\(\mu_p, \sigma_p)\end{matrix}; z\right] \equiv {}_p\Psi_q^*\left[\begin{matrix}(\lambda_1, \rho_1), \cdots, (\lambda_p, \rho_p)\\(\mu_1, \sigma_1), \cdots, (\mu_p, \sigma_p)\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{([\lambda_p])_{\rho_p n}}{([\mu_q])_{\sigma_q n}} \frac{z^n}{n!} \tag{1}$$

$$\left(\lambda_j, \mu_k \in \mathbb{C} \ \text{ and } \ \rho_j, \sigma_k \in \mathbb{R}_+ \ \ (j = 1, \cdots, p; k = 1, \cdots, q)\right),$$

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Email addresses: mathwinnie@live.com (Min-Jie Luo), rkraina_7@hotmail.com (R.K. Raina)

where $([\lambda_p])_{\rho_p n} := [\lambda_1]_{\rho_1 n} \cdots [\lambda_p]_{\rho_p n}$ and $[\lambda]_\rho$ $(\lambda, \rho \in \mathbb{C})$ is defined in terms of gamma functions by

$$[\lambda]_\rho := \frac{\Gamma(\lambda+\rho)}{\Gamma(\lambda)} = \begin{cases} 1 & (\rho = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\rho = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases}$$

As usual, let

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \quad \text{and} \quad \nabla := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right)\left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right).$$

The series given by (1) converges in the whole complex z-plane when $\Delta > -1$; and when $\Delta = 0$, the series (1) converges for $|z| < \nabla$.

The λ-generalized Hurwitz-Lerch Zeta function which is introduced and investigated by Srivastava [14] is here defined by

$$\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a;b,\lambda) \equiv \Phi^{\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}}\, {}_p\Psi_q^* \left[\begin{matrix} (\lambda_p, \rho_p), \\ (\mu_p, \sigma_p) \end{matrix}; ze^{-t} \right] \mathrm{d}t \tag{2}$$

$$(\min\{\Re(a), \Re(s)\} > 0;\ \Re(b) \geq 0;\ \lambda \geq 0).$$

The series representation of $\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a;b,\lambda)$ involving Fox's H-function is given by (see [14, p. 1489, Theorem 1]; see also [16, p. 260, Eq. (1.2)], [17, p. 1020, Eq. (1.9)] and [9, p. 106, Theorem 2.1])

$$\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a;b,\lambda) = \frac{1}{\lambda\Gamma(s)} \sum_{n=0}^\infty \frac{([\lambda_p])_{\rho_p n}}{([\mu_q])_{\sigma_q n} n!} H^{2,0}_{0,2}\left[(a+n)\, b^{\frac{1}{\lambda}} \,\middle|\, \begin{matrix} \overline{\qquad\qquad} \\ (s,1), \left(0, \frac{1}{\lambda}\right) \end{matrix} \right] \frac{z^n}{(a+n)^s} \tag{3}$$

$$(\min\{\Re(a), \Re(s)\} > 0;\ \Re(b) > 0;\ \lambda > 0),$$

where $\lambda_j \in \mathbb{C}$ $(j = 1, \cdots, p)$ and $\mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \cdots, q)$; $\rho_j > 0$ $(j = 1, \cdots, p)$; $\sigma_j > 0$ $(j = 1, \cdots, q)$; $1 + \Delta \geq 0$, and the equality in the convergence condition holds true for $|z| < \nabla$.

It is worth mentioning here that by using the fact that (see [9, p. 106, Remark 2.2])

$$\lim_{b\to 0} H^{2,0}_{0,2}\left[(a+n)\, b^{\frac{1}{\lambda}} \,\middle|\, \begin{matrix} \overline{\qquad\qquad} \\ (s,1), \left(0, \frac{1}{\lambda}\right) \end{matrix} \right] = \lambda\Gamma(s) \quad (\lambda > 0),$$

the series representation (3) reduces to the following form (see also [19]):

$$\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a;0,\lambda) = \Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a) = \sum_{n=0}^\infty \frac{([\lambda_p])_{\rho_p n}}{([\mu_q])_{\sigma_q n} n!} \frac{z^n}{(a+n)^s},$$

and this family of Hurwitz-Lerch Zeta functions is defined along with its conditions of convergence in [18, p. 517, Definition] and [20]. It now becomes apparent that the function (2) contains the Hurwitz (or generalized) Zeta function and the Riemann Zeta function as its special cases. For more information about the Hurwitz (or generalized) Zeta function and the Riemann Zeta function, we may refer to [13], [15] and [20].

Recently, the study of Srivastava's λ-generalized Hurwitz-Lerch Zeta function and its special cases has attracted remarkable interest and many papers have appeared subsequently on this subject. Two-sided inequalities for some special cases of this λ-generalized Hurwitz-Lerch Zeta function, especially, the line of approach adopted by using the Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$-series have been considered by Jankov *et al.* [4] and Srivastava *et al.* [18]. Motivated essentially by these works, we consider in the present paper establishing of similar two-sided inequalities for the Srivastava λ-generalized Hurwitz-Lerch Zeta function.

2. Some Prerequisite Results

In this section, we establish several important results which give some useful characteristics of the λ-generalized Hurwitz-Lerch Zeta function defined above by (2).

The Krätzel function is defined for $x > 0$, $\rho \in \mathbb{R}$ and $\nu \in \mathbb{C}$ being such that $\Re(\nu) < 0$ when $\rho \leq 0$, by the integral (see [7, p. 110, Eq. (1.1)]; see also [1] and [6])

$$Z_\rho^\nu(x) = \int_0^\infty t^{\nu-1} e^{-t^\rho - \frac{x}{t}} \mathrm{d}t.$$

Lemma 2.1. *Let $a, b, \lambda > 0$ and $s \in \mathbb{R}$. Then*

$$\int_0^\infty t^{s-1} e^{-at - \frac{b}{t^\lambda}} \mathrm{d}t = \frac{1}{\lambda a^s} Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right), \tag{4}$$

where the function $Z_{1/\lambda}^{s/\lambda}\left(u^\lambda b\right)$ is a decreasing function of u on $(0, \infty)$.

Proof. The formula (4) follows immediately by using the substitution $t = \frac{1}{a} v^{\frac{1}{\lambda}}$ in the integral on the left-hand side. To show that $Z_{1/\lambda}^{s/\lambda}\left(u^\lambda b\right)$ is a decreasing function of u on $(0, \infty)$, we rearrange (4) in the following form:

$$Z_{1/\lambda}^{s/\lambda}\left(u^\lambda b\right) = \int_0^\infty v^{\frac{s}{\lambda}-1} e^{-v^{\frac{1}{\lambda}} - \frac{u^\lambda b}{v}} \mathrm{d}v. \tag{5}$$

Differentiating (5) with respect to u, we have

$$\frac{\mathrm{d}}{\mathrm{d}u} Z_{1/\lambda}^{s/\lambda}\left(u^\lambda b\right) = -b\lambda u^{\lambda-1} Z_{1/\lambda}^{s/\lambda-1}\left(u^\lambda b\right) < 0,$$

and thus the function $Z_{1/\lambda}^{s/\lambda}\left(u^\lambda b\right)$ is decreasing on $(0, \infty)$. □

Remark 2.2. *The Krätzel function $Z_\rho^\nu(x)$ $(\nu, \rho \in \mathbb{R}; x > 0)$ is completely monotonic on $(0, \infty)$, implying thereby that*

$$(-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m Z_\rho^\nu(x) \geq 0$$

for all $x > 0$ and $m \in \mathbb{N}_0$ (see [1, p. 718, Theorem 1, (b)]). For $m = 1$, the above inequality clearly shows that $Z_\rho^\nu(x)$ is decreasing on $(0, \infty)$, but it fails to show that the function $Z_{1/\lambda}^{s/\lambda}\left(u^\lambda b\right)$ of u is also decreasing on $(0, \infty)$ by using only the complete monotonicity property of the Krätzel function. However, in view of the paper [8, p. 391, Corollary 1], we infer that the function $Z_\rho^\nu\left(u^\lambda b\right)$ $(0 \leq \lambda \leq 1, b > 0)$ is completely monotonic on $(0, \infty)$ and this fact shows that only for $0 < \lambda \leq 1$, the function $Z_{1/\lambda}^{s/\lambda}\left(u^\lambda b\right)$ of u is decreasing on $(0, \infty)$.

Applying the above Lemma 2.1 to (2), we get the following series representation for the function $\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z, s, a; b, \lambda)$.

Theorem 2.3. *Let a, b, λ and s be positive real numbers. Then*

$$\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z, s, a; b, \lambda) = \frac{1}{\lambda \Gamma(s)} \sum_{n=0}^\infty \frac{([\lambda_p])_{n\rho_p}}{([\mu_q])_{n\sigma_q} n!} Z_{1/\lambda}^{s/\lambda}\left((a+n)^\lambda b\right) \frac{z^n}{(a+n)^s}. \tag{6}$$

$$\left(\lambda_j \in \mathbb{R}\, (j = 1, \cdots, p) \text{ and } \mu_j \in \mathbb{R}\backslash\mathbb{Z}_0^-\, (j = 1, \cdots, q);\ \rho_j > 0\, (j = 1, \cdots, p);\ \sigma_j > 0\, (j = 1, \cdots, q);\ 1 + \Delta \geq 0\right). \tag{7}$$

Proof. By making use of the series representation of the normalized Fox-Wright function (1) occurring in the integrand of (2) and integrating the series term by term, we get

$$\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a;b,\lambda) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{([\lambda_p])_{n\rho_p}}{([\mu_q])_{n\sigma_q}} \frac{z^n}{n!} \int_0^\infty t^{s-1} e^{-(a+n)t-\frac{b}{t^\lambda}} \mathrm{d}t, \tag{8}$$

which with the help of Lemma 2.1 leads to the series representation (6). □

Remark 2.4. *The Krätzel function $Z^{\nu}_{\rho}(x)$ of real variable x $(x > 0)$ can be extended to complex z by considering its H-function representation ([7]; see also [6]). For $\rho > 0$, $\nu \in \mathbb{C}$ and $z \in \mathbb{C}$ $(z \neq 0)$, one can express*

$$Z^{\nu}_{\rho}(z) = \frac{1}{\rho} H^{2,0}_{0,2}\left[z \,\middle|\, \begin{matrix} \overline{\phantom{(0,1),\left(\frac{\nu}{\rho},\frac{1}{\rho}\right)}} \\ (0,1), \left(\frac{\nu}{\rho}, \frac{1}{\rho}\right) \end{matrix} \right].$$

Thus, the integral in (8) *can be expressed in terms of the H-function and holds true for some suitable complex parameters. That is*

$$\int_0^\infty t^{s-1} e^{-(a+n)t-\frac{b}{t^\lambda}} \mathrm{d}t = \frac{1}{\lambda (a+n)^s} H^{2,0}_{0,2}\left[(a+n)\, b^{\frac{1}{\lambda}} \,\middle|\, \begin{matrix} \overline{\phantom{(s,1),\left(0,\frac{1}{\lambda}\right)}} \\ (s,1), \left(0, \frac{1}{\lambda}\right) \end{matrix} \right],$$

which evidently leads us to the series representation (3).

We now state the following useful inequalities which hold true rather independently. These inequalities provide simple extensions to the inequality given earlier by Pogány and Srivastava [11, p. 131, Theorem 3].

Lemma 2.5. *Let the real-valued function $f \geq 0$ be twice continuously differentiable at the origin. Suppose also that $f(0) = 1$ and $f'(0) > f''(0) > [f'(0)]^2$. Then*

$$e^{f'(0)x} \leq f(x) \leq [1 - f'(0)] + f'(0)\, e^x \quad (x > 0) \tag{9}$$

and

$$e^{-f'(0)x} \leq f(-x) \leq [1 - f'(0)] + f'(0)\, e^{-x} \quad (x > 0), \tag{10}$$

where the equality signs in (9) *and* (10) *hold true when $x = 0$.*

Proof. For the proofs of the inequalities (9) and (10), one may refer to [11, p. 131]. □

Thus, for the normalized Fox-Wright function defined by (1) and following the arguments and details verified in [11, pp. 131–132], we have from Lemma 2.5 that

$$e^{\Omega^* x} \leq {}_p\Psi^*_q\left[\begin{matrix} (\lambda_p, \rho_p) \\ (\mu_q, \sigma_q) \end{matrix}; x \right] \leq 1 - \Omega^* + \Omega^* e^x \quad (x > 0) \tag{11}$$

and

$$e^{-\Omega^* x} \leq {}_p\Psi^*_q\left[\begin{matrix} (\lambda_p, \rho_p) \\ (\mu_q, \sigma_q) \end{matrix}; -x \right] \leq 1 - \Omega^* + \Omega^* e^{-x} \quad (x > 0), \tag{12}$$

where

$$\Omega^* := \frac{([\lambda_p])_{\rho_p}}{([\mu_q])_{\sigma_q}} < 1 \quad \text{and} \quad \left(\boldsymbol{\lambda_p}, \boldsymbol{\mu_q}, \boldsymbol{\rho_p}, \boldsymbol{\sigma_q}\right) \in {}_p\mathbb{D}_q(Q, T).$$

The parameter space ${}_p\mathbb{D}_q(Q,T)$ is defined by ([11, p. 132, Eq. (27)])

$$
\begin{aligned}
{}_p\mathbb{D}_q(Q,T) := \Bigg\{ & \left(\boldsymbol{\lambda_p}, \boldsymbol{\mu_q}, \boldsymbol{\rho_p}, \boldsymbol{\sigma_q}\right) \in \mathbb{R}_+^{2(p+q)} : \prod_{j=1}^{p} Q\left(\lambda_j + \rho_j, \rho_j\right) \leq \prod_{j=1}^{q} Q\left(\mu_j + \sigma_j, \sigma_j\right) \\
& \text{and} \quad \prod_{j=1}^{q} T\left(\mu_j, \mu_j + 2\sigma_j\right) \leq \prod_{j=1}^{p} T\left(\lambda_j, \lambda_j + 2\rho_j\right) \Bigg\},
\end{aligned}
$$

where $Q(u,v)$ denotes the Gautschi quotient defined by (see [5])

$$
Q(u,v) := \frac{\Gamma(u+v)}{\Gamma(u)} =: [u]_v \quad (\min\{u,v\} > 0),
$$

and

$$
T(u,v) := \frac{\Gamma(u)\Gamma(v)}{\left[\Gamma\left(\frac{u+v}{2}\right)\right]^2} \quad (\min\{u,v\} > 0)
$$

is the Gurland ratio (see [5]). It may be pointed out here that (11) and (12) are simple extensions to the inequality given by Pogány and Srivastava [11, p. 133, Theorem 4].

Next, we consider the Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series which is defined formally by ([10, Eq. (2)])

$$
S(\varrho,\mu,\boldsymbol{a},\boldsymbol{\lambda}) = \sum_{n=0}^{\infty} \frac{a(n)}{(\lambda(n)+\varrho)^{\mu}} \quad (\mu,\varrho > 0).
$$

The series is tacitly assumed to be convergent and the sequences $\boldsymbol{a} := \{a(n)\}_{n\in\mathbb{N}_0}$ and $\boldsymbol{\lambda} := \{\lambda(n)\}_{n\in\mathbb{N}_0}$ are assumed to be nonnegative. We also require that the sequence $\{\lambda(n)\}_{n\in\mathbb{N}_0}$ is steadily increasing to infinity, that is,

$$
\boldsymbol{\lambda} : 0 \leq \lambda(0) < \lambda(1) < \cdots \quad \text{and} \quad \lim_{n\to\infty} \lambda(n) = \infty. \tag{13}
$$

More generally, we can define two functions $a : [0,\infty) \to [0,\infty)$ and $\lambda : [0,\infty) \to [0,\infty)$ such that $a|_{\mathbb{N}_0} = \boldsymbol{a}$ and $\lambda|_{\mathbb{N}_0} = \boldsymbol{\lambda}$ and the inverse function λ^{-1} of λ is well-defined. One of the most important results for the Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series is its integral representation which has been established by Pogány [10, p. 687, Theorem 1]. In the following theorem, we provide an *improved* version for this result.

Theorem 2.6. *Let $\mu,\varrho > 0$, and let $a \in C^1[0,\infty)$ be a nonnegative function such that*

$$
\mathcal{A}(t)\, e^{-xt} \to 0 \quad as \quad t \to \infty,
$$

where $x > 0$ and $\mathcal{A}(t)$ is defined below by (20).

Suppose that $\lambda : [0,\infty) \to [0,\infty)$ is a function such that (13) *is satisfied. Then, for convergent $S(\varrho,\mu,\boldsymbol{a},\boldsymbol{\lambda})$, we have*

$$
S(\varrho,\mu,\boldsymbol{a},\boldsymbol{\lambda}) = \frac{a(0)}{(\lambda(0)+\varrho)^{\mu}} + \mu \int_{\lambda(1)}^{\infty} \int_{0}^{[\lambda^{-1}(t)]} \frac{\mathfrak{d}_u a(u)}{(t+\varrho)^{\mu+1}} \mathrm{d}u \mathrm{d}t, \tag{14}
$$

where the operator $\mathfrak{d}_u$ is defined by

$$
\mathfrak{d}_u := 1 + \{u\} \frac{\mathrm{d}}{\mathrm{d}u},
$$

and $\{u\} = u - [u]$ and $[u]$, respectively, stand for the fractional and the integer parts of u.

To prove (14), we make use of the Abel's summation formula instead of the Laplace integral expression of Dirichlet series. Many different versions of Abel's summation formula can be found in literature. Here, we adopt the following version given by Chandrasekharan (see [2, p. 78, Theorem 6] and [3, p. 22]).

Theorem 2.7. (Abel's summation formula) *Let $0 \le \lambda(1) \le \lambda(2) \le \cdots$ be a sequence of real numbers such that $\lambda(n) \to \infty$ as $n \to \infty$, and let $\{a(n)\}$ be a sequence of complex numbers. Let $A(x) = \sum_{\lambda(n)\le x} a(n)$ and $\varphi(x)$ be a complex-valued function defined for $x \ge 0$. Then*

$$\sum_{n=1}^{k} a(n)\varphi(\lambda(n)) = A(\lambda(k))\varphi(\lambda(k)) - \sum_{n=1}^{k-1} A(\lambda(n))\left[\varphi(\lambda(n+1)) - \varphi(\lambda(n))\right]. \tag{15}$$

If φ has a continuous derivative in $(0,\infty)$, and $x \ge \lambda(1)$, then (15) can be written as

$$\sum_{\lambda(n)\le x} a(n)\varphi(\lambda(n)) = A(x)\varphi(x) - \int_{\lambda(1)}^{x} A(t)\varphi'(t)\,\mathrm{d}t.$$

If, in addition, $A(x)\varphi(x) \to 0$ as $x \to \infty$, then

$$\sum_{n=1}^{\infty} a(n)\varphi(\lambda(n)) = -\int_{\lambda(1)}^{\infty} A(t)\varphi'(t)\,\mathrm{d}t, \tag{16}$$

provided that either side is convergent.

Proof of Theorem 2.6. It is easy to observe that

$$\begin{aligned} S(\varrho,\mu,\boldsymbol{a},\boldsymbol{\lambda}) &= \frac{1}{\Gamma(\mu)}\int_0^\infty x^{\mu-1}e^{-\varrho x}\left(\sum_{n=0}^{\infty} a(n)e^{-\lambda(n)x}\right)\mathrm{d}x \\ &= \frac{a(0)}{(\lambda(0)+\varrho)^{\mu}} + \frac{1}{\Gamma(\mu)}\int_0^\infty x^{\mu-1}e^{-\varrho x}\left(\sum_{n=1}^{\infty} a(n)e^{-\lambda(n)x}\right)\mathrm{d}x. \end{aligned} \tag{17}$$

The series occurring in the last integral in (17) can be summed up by using the Abel's summation formula (16). Suppose that $\varphi(t) = e^{-xt}$ and $\mathcal{A}(t) = \sum_{\lambda(n)\le t} a(n)$, then we have

$$\sum_{n=1}^{\infty} a(n)e^{-\lambda(n)x} = -\int_{\lambda(1)}^{\infty}\mathcal{A}(t)\varphi'(t)\,\mathrm{d}t = x\int_{\lambda(1)}^{\infty} e^{-xt}\mathcal{A}(t)\,\mathrm{d}t. \tag{18}$$

Substituting (18) into (17) and integrating with respect to x, we get

$$S(\varrho,\mu,\boldsymbol{a},\boldsymbol{\lambda}) = \frac{a(0)}{(\lambda(0)+\varrho)^{\mu}} + \mu\int_{\lambda(1)}^{\infty}\frac{\mathcal{A}(t)}{(t+\varrho)^{\mu+1}}\mathrm{d}t. \tag{19}$$

Since the sequence $\{\lambda(n)\}_{n\in\mathbb{N}}$ satisfies $0 < \lambda(1) < \lambda(2) < \cdots$, the function $\mathcal{A}(t)$ can be expressed as

$$\mathcal{A}(t) = \sum_{n:\lambda(n)\le t} a(n) = \sum_{n=1}^{[\lambda^{-1}(t)]} a(n). \tag{20}$$

The finite sum $\mathcal{A}(t)$ given by (20) can be further summed up by using the Euler-Maclaurin formula [12, p. 2365, Eq. (3)]:

$$\sum_{j=k+1}^{m} a_j = \int_k^m \mathfrak{d}_u a(u)\,\mathrm{d}u \quad \left(\mathfrak{d}_u := 1 + \{u\}\frac{\mathrm{d}}{\mathrm{d}u}\right),$$

that is,

$$\mathcal{A}(t) = \int_0^{[\lambda^{-1}(t)]} \mathfrak{d}_u a(u)\,\mathrm{d}u. \tag{21}$$

The integral representation (14) now follows by substituting (21) into (19). □

Remark 2.8. *If $\lambda(0) = 0$, then our result coincides with the one derived by Pogány [10, p. 687, Theorem 1], that is,*

$$S(\varrho, \mu, \boldsymbol{a}, \boldsymbol{\lambda}) = \frac{a(0)}{\varrho^{\mu}} + \mu \int_{\lambda(1)}^{\infty} \int_0^{[\lambda^{-1}(t)]} \frac{\mathfrak{d}_u a(u)}{(t+\varrho)^{\mu+1}} \mathrm{d}u\mathrm{d}t.$$

However, it is interesting to note that our result Theorem 2.6 above holds also for $\lambda(0) \neq 0$.

We now apply Theorem 2.6 to Srivastava's λ-generalized Hurwitz-Lerch Zeta function defined by (6). For convenience, we write

$$\mathfrak{g}(u) := \frac{\prod_{j=1}^{p} \Gamma(\lambda_j + u\rho_j)}{\prod_{j=1}^{q} \Gamma(\mu_j + u\sigma_j)} Z_{1/\lambda}^{s/\lambda}\left((a+u)^{\lambda} b\right) \frac{z^u}{\Gamma(u+1)} \tag{22}$$

and

$$\mathfrak{a}(u) := \frac{\prod_{j=1}^{q} \Gamma(\mu_j)}{\prod_{j=1}^{p} \Gamma(\lambda_j)} \mathfrak{g}(u), \tag{23}$$

then (6) can be expressed as

$$\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z, s, a; b, \lambda) = \frac{1}{\lambda\Gamma(s)} \sum_{n=0}^{\infty} \frac{\mathfrak{a}(n)}{(a+n)^s} \quad \left(\mathfrak{a}(0) = Z_{1/\lambda}^{s/\lambda}\left(a^{\lambda} b\right)\right). \tag{24}$$

By specializing the formula (14) with $\varrho = a$, $\mu = s$, $\lambda(u) = u$, $\lambda^{-1}(u) = u$ and $a(u) = \mathfrak{a}(u)$, then in view of Theorem 2.6 and Eqns. (22) to (24), we obtain the following theorem.

Theorem 2.9. *Let a, b, λ, s be positive real numbers, and also let the conditions in* (7) *be satisfied. Then we have*

$$\begin{aligned}\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z, s, a; b, \lambda) = &\frac{Z_{1/\lambda}^{s/\lambda}\left(a^{\lambda} b\right)}{\lambda\Gamma(s)\,a^s} + \frac{s}{\lambda\Gamma(s)} \frac{\prod_{j=1}^{q} \Gamma(\mu_j)}{\prod_{j=1}^{p} \Gamma(\lambda_j)} \int_1^{\infty} \int_0^{[t]} \frac{\mathfrak{g}(u)\,\mathrm{d}u\mathrm{d}t}{(a+t)^{s+1}} \\ &+ \frac{s}{\lambda\Gamma(s)} \frac{\prod_{j=1}^{q} \Gamma(\mu_j)}{\prod_{j=1}^{p} \Gamma(\lambda_j)} \int_1^{\infty} \int_0^{[t]} \{u\} \frac{\mathrm{d}}{\mathrm{d}u} \mathfrak{g}(u) \frac{\mathrm{d}u\mathrm{d}t}{(a+t)^{s+1}},\end{aligned} \tag{25}$$

where the function $\mathfrak{g}(u)$ is given by (22)*.*

3. Two-Sided Inequalities for Srivastava's λ-Generalized Hurwitz-Lerch Zeta Function

In this section, we establish certain two-sided inequalities for Srivastava's λ-generalized Hurwitz-Lerch Zeta function. The methods applied here are inherited partially from [4] and [18], and we present below more general results.

Theorem 3.1. *Let a, b, λ, s be positive real numbers, and let $\left(\boldsymbol{\lambda_p}, \boldsymbol{\mu_q}, \boldsymbol{\rho_p}, \boldsymbol{\sigma_q}\right) \in {}_p\mathbb{D}_q(Q, T)$. For $z > 0$, we have*

$$\frac{Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right)}{\lambda a^s \Gamma(s)} + \Omega^* z \frac{Z_{1/\lambda}^{s/\lambda}\left((a+1)^\lambda b\right)}{\lambda (a+1)^s \Gamma(s)} \le \Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z, s, a; b, \lambda) \le \frac{Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right)}{\lambda a^s \Gamma(s)} \left(1 - \Omega^* + \Omega^* e^z\right) \tag{26}$$

and

$$\Omega^* e^{-z} \frac{Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right)}{\lambda a^s \Gamma(s)} \le \Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(-z, s, a; b, \lambda) \le \frac{Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right)}{\lambda a^s \Gamma(s)}. \tag{27}$$

Proof. We first prove (26). For the upper bound, we use (2), (4) and (11) to find that

$$\begin{aligned}
\Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\boldsymbol{\rho},\boldsymbol{\sigma}}(z, s, a, b, \lambda) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} {}_p\Psi_q^* \left[\begin{matrix} ((\lambda_p, \rho_p)) \\ ((\mu_p, \sigma_p)) \end{matrix}; ze^{-t} \right] \mathrm{d}t \\
&\le \frac{1-\Omega^*}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} \mathrm{d}t + \frac{\Omega^*}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} e^{ze^{-t}} \mathrm{d}t \\
&= \frac{1-\Omega^*}{\lambda a^s \Gamma(s)} Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right) + \frac{\Omega^*}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} e^{ze^{-t}} \mathrm{d}t.
\end{aligned} \tag{28}$$

Since the function $e^{ze^{-t}}$ $(z > 0)$ is a deceasing function of t on $(0, \infty)$ and is such that $e^{ze^{-t}} \in (1, e^z)$, hence the last integral in (28) can be estimated in the following manner:

$$\int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} e^{ze^{-t}} \mathrm{d}t \le e^z \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} \mathrm{d}t = \frac{e^z}{\lambda a^s} Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right). \tag{29}$$

Combining (28) with the estimate (29), we get

$$\Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\boldsymbol{\rho},\boldsymbol{\sigma}}(z, s, a, b, \lambda) \le \frac{1-\Omega^*}{\lambda a^s \Gamma(s)} Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right) + \frac{\Omega^* e^z}{\lambda a^s \Gamma(s)} Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right).$$

By using (2), (11) and the elementary inequality that $e^x \ge 1 + x$ $(x \in \mathbb{R})$, we obtain that

$$\begin{aligned}
\Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\boldsymbol{\rho},\boldsymbol{\sigma}}(z, s, a, b, \lambda) &\ge \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} e^{\Omega^* ze^{-t}} \mathrm{d}t \ge \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} \left(1 + \Omega^* ze^{-t}\right) \mathrm{d}t \\
&= \frac{Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right)}{\lambda a^s \Gamma(s)} + \Omega^* z \frac{Z_{1/\lambda}^{s/\lambda}\left((a+1)^\lambda b\right)}{\lambda (a+1)^s \Gamma(s)},
\end{aligned}$$

which proves the lower bound of (26).

The derivation of (27) is similar. Making use of (2), (4) and (12), it is easy to see that

$$\Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\boldsymbol{\rho},\boldsymbol{\sigma}}(-z, s, a, b, \lambda) \le \frac{1-\Omega^*}{\lambda a^s \Gamma(s)} Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right) + \frac{\Omega^*}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} e^{-ze^{-t}} \mathrm{d}t.$$

Since the function $e^{-ze^{-t}}$ $(z > 0)$ is a increasing function of t on $(0, \infty)$ and satisfies $e^{-ze^{-t}} \in (e^{-z}, 1)$, we have

$$\Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\boldsymbol{\rho},\boldsymbol{\sigma}}(-z, s, a, b, \lambda) \le \frac{1-\Omega^*}{\lambda a^s \Gamma(s)} Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right) + \frac{\Omega^*}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} \mathrm{d}t = \frac{Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right)}{\lambda a^s \Gamma(s)}.$$

To prove the lower bound, we note that the function $e^{-\Omega^* ze^{-t}}$ $(z > 0; 0 < \Omega^* < 1)$ is an increasing function of t on $(0, \infty)$ with the property that $e^{-\Omega^* ze^{-t}} \in \left(e^{-\Omega^* z}, 1\right)$. From this fact, it follows that

$$\begin{aligned}
\Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\boldsymbol{\rho},\boldsymbol{\sigma}}(-z, s, a, b, \lambda) &\ge \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} e^{-\Omega^* ze^{-t}} \mathrm{d}t \\
&\ge \frac{e^{-\Omega^* z}}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}} \mathrm{d}t = e^{-\Omega^* z} \frac{Z_{1/\lambda}^{s/\lambda}\left(a^\lambda b\right)}{\lambda a^s \Gamma(s)}.
\end{aligned}$$

□

In what follows, the Psi (or digamma) function $\psi(z)$ is defined by ([15, p. 24])

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t$$

and possesses the following expansion that

$$\psi(z) = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{z+k-1}\right) - \gamma \quad \left(z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

where γ is the Euler-Mascheroni constant (see [15, p. 2, Eq. (3)]).

Theorem 3.2. *Let a, b, λ, s be positive real numbers and let $\lambda_j, \mu_k, \rho_j, \sigma_k \in \mathbb{R}$ $(j = 1, \cdots, p; k = 1, \cdots, q)$. Then we have*

$$R + L < \Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z, s, a; b, \lambda) \leq R, \tag{30}$$

$$\left(z \in (0, e^{-\gamma});\ \mu_{k_j} + u\sigma_{k_j} \geq \lambda_j + u\rho_j > 0;\ \sigma_{k_j} \geq \rho_j > 0;\ \psi\left(\lambda_j + u\rho_j\right) > 0;\ p \leq q\right) \tag{31}$$

where $\left(k_1, \cdots, k_p\right)$ is a permutation of p indices $k_j \in \{1, \cdots, q\}$ and

$$R := \frac{Z^{s/\lambda}_{1/\lambda}\left(a^{\lambda}b\right)}{\lambda\Gamma(s)\,a^s} + \frac{s}{\lambda\Gamma(s)}\frac{\prod_{j=1}^{q}\Gamma(\mu_j)}{\prod_{j=1}^{p}\Gamma(\lambda_j)}\int_1^{\infty}\int_0^{[t]}\frac{\mathfrak{g}(u)\,\mathrm{d}u\mathrm{d}t}{(a+t)^{s+1}}, \tag{32}$$

$$L := \frac{s}{\lambda\Gamma(s)}\frac{\prod_{j=1}^{q}\Gamma(\mu_j)}{\prod_{j=1}^{p}\Gamma(\lambda_j)}\int_1^{\infty}\frac{\mathfrak{g}([t])}{(a+t)^{s+1}}\mathrm{d}t - \frac{Z^{s/\lambda}_{1/\lambda}\left(a^{\lambda}b\right)}{(a+1)^s\,\lambda\Gamma(s)}. \tag{33}$$

The upper bound in (30) *is sharp in the sense that* $0 \leq \{u\} < 1$.

Proof. The main part of this proof is to show that the function

$$\mathfrak{g}(u) := \frac{\prod_{j=1}^{p}\Gamma(\lambda_j + u\rho_j)}{\prod_{j=1}^{q}\Gamma(\mu_j + u\sigma_j)} Z^{s/\lambda}_{1/\lambda}\left((a+u)^{\lambda}b\right)\frac{z^u}{\Gamma(u+1)}$$

$$\left(z \in (0, e^{-\gamma});\ \mu_{k_j} + u\sigma_{k_j} \geq \lambda_j + u\rho_j > 0;\ \sigma_{k_j} \geq \rho_j > 0;\ \psi\left(\lambda_j + u\rho_j\right) > 0;\ p \leq q\right)$$

(defined above by (22)) is monotonically decreasing for $u > 0$. Let us rewrite the function $\mathfrak{g}(u)$ as

$$\mathfrak{g}(u) = \mathfrak{g}_1(u)\,\mathfrak{g}_2(u),$$

where

$$\mathfrak{g}_1(u) = \frac{\prod_{j=1}^{p}\Gamma\left(\lambda_j + u\rho_j\right)}{\prod_{j=1}^{q}\Gamma\left(\mu_j + u\sigma_j\right)}\frac{z^u}{\Gamma(u+1)} \quad \text{and} \quad \mathfrak{g}_2(u) = Z^{s/\lambda}_{1/\lambda}\left((a+u)^{\lambda}b\right)$$

are positive functions on $(0, \infty)$. In Lemma 2.1, we showed that the function $Z^{s/\lambda}_{1/\lambda}\left(u^{\lambda}b\right)$ is a decreasing function of u on $(0, \infty)$ which implies that the function $\mathfrak{g}_2(u)$ is also decreasing on $(0, \infty)$, that is, $\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}_2(u) < 0$. The monotonicity of $\mathfrak{g}_1(u)$ has been proved in [18, p. 521], and so we have

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}_1(u) < 0$$

$$\left(z \in (0, e^{-\gamma})\,;\;\; \mu_{k_j} + u\sigma_{k_j} \geq \lambda_j + u\rho_j > 0;\;\; \sigma_{k_j} \geq \rho_j > 0;\;\; \psi\left(\lambda_j + u\rho_j\right) > 0;\;\; p \leq q\right).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}\,(u) = \mathfrak{g}_2\,(u)\,\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}_1\,(u) + \mathfrak{g}_1\,(u)\,\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}_2\,(u) < 0,$$

that is, $\mathfrak{g}\,(u)$ is monotonically decreasing on $(0,\infty)$.

To prove now the inequality (30), let us denote the second integral in (25) by L^*, then (25) becomes $\Phi^{(\rho,\sigma)}_{\lambda;\mu}\,(z,s,a;b,\lambda) = R + L^*$, where R is given by (32). It remains to prove that $L < L^* \leq 0$, where L is given by (33).

From the inequalities: $0 \leq \{u\} < 1$ and $\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}\,(u) < 0$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}\,(u) < \{u\}\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}\,(u) \leq 0.$$

Integrating this estimate over $(0,[t])$, we have

$$\mathfrak{g}\,([t]) - \mathfrak{g}\,(0) < \int_0^{[t]}\{u\}\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}\,(u)\,\mathrm{d}u \leq 0. \tag{34}$$

Integrating also the inequality (34) over $(0,\infty)$ with respect to $(a+t)^{-s-1}\,\mathrm{d}t$, we get

$$\begin{aligned} 0 \geq \int_0^\infty \int_0^{[t]}\{u\}\frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{g}\,(u)\,\frac{\mathrm{d}u\mathrm{d}t}{(a+t)^{s+1}} &> \int_0^\infty \frac{\mathfrak{g}\,([t]) - \mathfrak{g}\,(0)}{(a+t)^{s+1}}\mathrm{d}t \\ &= \int_1^\infty \frac{\mathfrak{g}\,([t]) - \mathfrak{g}\,(0)}{(a+t)^{s+1}}\mathrm{d}t \quad (\mathfrak{g}\,([t]) \equiv \mathfrak{g}\,(0)\,, 0 < t < 1) \\ &= \int_1^\infty \frac{\mathfrak{g}\,([t])}{(a+t)^{s+1}}\mathrm{d}t - \frac{\prod_{j=1}^p \Gamma(\lambda_j)}{\prod_{j=1}^q \Gamma(\mu_j)}\frac{Z^{s/\lambda}_{1/\lambda}\left(a^\lambda b\right)}{s\,(a+1)^s}. \end{aligned}$$

After a little simplification, we get

$$0 \geq L^* > \frac{s}{\lambda\Gamma\,(s)}\frac{\prod_{j=1}^q \Gamma(\mu_j)}{\prod_{j=1}^p \Gamma(\lambda_j)}\int_1^\infty \frac{\mathfrak{g}\,([t])}{(a+t)^{s+1}}\mathrm{d}t - \frac{Z^{s/\lambda}_{1/\lambda}\left(a^\lambda b\right)}{(a+1)^s\,\lambda\Gamma\,(s)} = L,$$

and the proof of inequality (30) is complete.

Finally, we need to check that the condition (31) does not violate the condition stated in (7). In fact, from $\sigma_{k_j} \geq \rho_j > 0$, we have

$$\sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j = \sum_{j=1, j\neq k_1,\cdots,k_p}^q \sigma_j + \sum_{j=1}^p \left(\sigma_{k_j} - \rho_j\right) > 0 > -1.$$

This ends the proof of the theorem. □

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