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Commutativity of Banach Algebras Characterized by Primitive Ideals and Spectra

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This paper is dedicated to Dr. Mehdi Mohammadzadeh Karizaki

Abstract. This study is an attempt to prove the following main results.
Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{A} = \mathcal{A} \bigoplus \mathbb{C}$ be its unitization. By $\prod_c(\mathfrak{A})$, we denote the set of all primitive ideals $\mathcal{P}$ of $\mathfrak{A}$ such that the quotient algebra $\frac{\mathfrak{A}}{\mathcal{P}}$ is commutative. We prove that if $\mathcal{A}$ is semi-prime and $\dim(\bigcap_{\mathcal{P}\in\prod_c(\mathfrak{A})} \mathcal{P}) \leq 1$, then $\mathcal{A}$ is commutative. Moreover, we prove the following:
Let $\mathcal{A}$ be a semi-simple Banach algebra. Then, $\mathcal{A}$ is commutative if and only if $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$ or $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\}$ for every $a \in \mathcal{A}$, where $\mathfrak{S}(a)$ and $\Phi_{\mathcal{A}}$ denote the spectrum of an element $a \in \mathcal{A}$, and the set of all non-zero multiplicative linear functionals on $\mathcal{A}$, respectively.

1. Introduction and Preliminaries

Throughout this paper, $\mathcal{A}$ denotes a Banach algebra over the complex field $\mathbb{C}$. If $\mathcal{A}$ is unital, then $\mathbf{1}$ stands for its unit element. We denote the center of $\mathcal{A}$ by $Z(\mathcal{A})$, i.e. $Z(\mathcal{A}) = \{x \in \mathcal{A} \mid ax = xa \; for \; all \; a \in \mathcal{A}\}$. Moreover, $\mathcal{A}$ is called semi-prime if $a\mathcal{A}a = \{0\}$ implies that $a = 0$. Recall that a linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a derivation if it satisfies the Leibnitz rule $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. We call d an inner derivation if there exists an element $x \in \mathcal{A}$ such that $d(a) = [x, a] = xa - ax$ for all $a \in \mathcal{A}$.

A non-zero linear functional φ on $\mathcal{A}$ is called a *character* if $\varphi(ab) = \varphi(a)\varphi(b)$ holds for every $a, b \in \mathcal{A}$. By $\Phi_{\mathcal{A}}$ we denote the set of all characters on $\mathcal{A}$. It is well known that, $\ker\varphi$ the kernel of φ is a maximal ideal of $\mathcal{A}$, where φ is an arbitrary element of $\Phi_{\mathcal{A}}$. If $\mathcal{A}$ is a Banach $*$-algebra, then we denote the set of all projections in $\mathcal{A}$ by $\mathcal{P}_{\mathcal{A}}$ (,i.e. $\mathcal{P}_{\mathcal{A}} = \{p \in \mathcal{A} \mid p^2 = p, \; p^* = p\}$), and by $\mathcal{S}_{\mathcal{A}}$ we denote the set of all self-adjoint elements of $\mathcal{A}$ (,i.e. $\mathcal{S}_{\mathcal{A}} = \{a \in \mathcal{A} \mid a^* = a\}$). The set of those elements in $\mathcal{A}$ which can be represented as finite real-linear combinations of mutually orthogonal projections, is denoted by $\mathcal{O}_{\mathcal{A}}$. Hence, we have $\mathcal{P}_{\mathcal{A}} \subseteq \mathcal{O}_{\mathcal{A}} \subseteq \mathcal{S}_{\mathcal{A}}$. Note that if $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{O}_{\mathcal{A}}$ is norm dense in $\mathcal{S}_{\mathcal{A}}$. More generally, the same is true for AW^*-algebras. Recall that a W^*-algebra is a weakly closed self-adjoint algebra of operators on a Hilbert space, and an AW^*-algebra is a C^*-algebra satisfying:
(i) In the partially ordered set of projections, any set of orthogonal projections has a least upper bound (LUB),
(ii) Any maximal commutative self-adjoint subalgebra is generated by its projections. That is, it is equal to

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the smallest closed subalgebra containing its projections.
The above-mentioned definitions and results can be found in [6, 10, 17]. This paper, has been motivated by [7, 8, 15]. An algebra $\mathcal{A}$ can always be embedded into an algebra with identity as follows. Let $\mathfrak{A}$ denote the set of all pairs (x, λ), $x \in \mathcal{A}$, $\lambda \in \mathbb{C}$, that is, $\mathfrak{A} = \mathcal{A} \bigoplus \mathbb{C}$. Then $\mathfrak{A}$ becomes an algebra if the linear space operations and multiplication are defined by $(x, \lambda) + (y, \mu) = (x + y, \lambda + \mu)$, $\mu(x, \lambda) = (\mu x, \mu\lambda)$ and $(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$ for $x, y \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. A simple calculation shows that the element $\mathbf{e} = (0, 1) \in \mathfrak{A}$ is an identity for $\mathfrak{A}$. Moreover, the mapping $x \to (x, 0)$ is an algebra isomorphism of $\mathcal{A}$ onto an ideal of codimension one in $\mathfrak{A}$. Obviously, $\mathfrak{A}$ is commutative if and only if $\mathcal{A}$ is commutative.

Now suppose that $\mathcal{A}$ is a normed algebra. We introduce a norm on $\mathfrak{A}$ by $\|(x, \lambda)\| = \|x\| + |\lambda|$, for $x \in \mathcal{A}$, $\lambda \in \mathbb{C}$. It is straightforward that this turns $\mathfrak{A}$ into a normed algebra. Clearly, if $\mathcal{A}$ is a Banach algebra, then $\mathfrak{A}$ is a Banach algebra, too. Some authors call $\mathfrak{A}$ the unitization of $\mathcal{A}$.

Let B be a subset of $\mathcal{A}$, the *commutant* of B is denoted by B' and defined by $B' = \{a \in \mathcal{A} \,|\, ab = ba \text{ for every } b \in B\}$. The double commutant of B is denoted by B'', and we have $B'' = \{a \in \mathcal{A} \mid ax = xa \text{ for every } x \in B'\}$. A straightforward verification shows that B' is a closed subalgebra of $\mathcal{A}$, $B \subseteq B''$, and if B is a commutative set, then so is B''. Indeed, if B is commutative, then B'' is a commutative Banach algebra (see p. 293 of [16]).

The spectrum of an element a is the set $\mathfrak{S}(a) = \{\lambda \in \mathbb{C} \mid \lambda\mathbf{1} - a \text{ is not invertible}\}$. The spectral radius of a is $r(a) = \sup\{|\lambda| \ : \ \lambda \in \mathfrak{S}(a)\}$. The element a is said to be quasi-nilpotent if $r(a) = 0$. We shall henceforth find it convenient to write $\lambda\mathbf{1}$ simply as λ.

Let $\mathcal{A}$ be a commutative Banach algebra. It follows from Theorem 1.3.4 of [14] that
(1) if $\mathcal{A}$ is unital, then $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\}$,
(2) if $\mathcal{A}$ is non-unital, then $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$.
In this article, we are going to study the converse of this result. Indeed, we will show that if $\mathcal{A}$ is a semi-simple Banach algebra, then $\mathcal{A}$ is commutative if and only if $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\}$ or $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$ for every $a \in \mathcal{A}$. Moreover, we prove that if $\delta : \mathcal{A} \to \mathcal{A}$ is a bounded derivation such that $\mathfrak{S}(\delta(a)) = \{\varphi(\delta(a)) \mid \varphi \in \Phi_{\mathcal{A}}\}$ or $\mathfrak{S}(\delta(a)) = \{\varphi(\delta(a)) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$ for every $a \in \mathcal{A}$, then $\delta(\mathcal{A}) \subseteq rad(\mathcal{A})$, where $rad(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$. By $\prod_c(\mathfrak{A})$, we denote the set of all primitive ideals $\mathcal{P}$ of $\mathfrak{A}$ such that the quotient algebra $\frac{\mathfrak{A}}{\mathcal{P}}$ is commutative. Moreover, the set of all maximal ideals $\mathcal{M}$ of $\mathfrak{A}$ such that the quotient algebra $\frac{\mathfrak{A}}{\mathcal{M}}$ is commutative, is denoted by $\mathfrak{M}_c(\mathfrak{A})$. We prove that if $\mathcal{A}$ is semi-prime and $\dim(\bigcap_{\mathcal{P} \in \prod_c(\mathfrak{A})} \mathcal{P}) \leq 1$, then $\mathcal{A}$ is commutative.

2. Results and Proofs

We begin with the following theorems which will be used to prove our main results.

Theorem 2.1. [[19], Theorem 4.4] *Let $\mathcal{A}$ be a commutative Banach algebra and $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation. Then, $\delta(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

Theorem 2.2. [[11], page 246] *Let d be a derivation on a Banach algebra $\mathcal{A}$. Then, the following three conditions are equivalent:*
(i) $[a, d(a)] \in rad(\mathcal{A})$ for all $a \in \mathcal{A}$;
(ii) d is spectrally bounded;
(iii) $d(\mathcal{A}) \subseteq rad(\mathcal{A})$;

Note that each member of $\Phi_{\mathcal{A}}$ is continuous (see Proposition 5.1.1 of [5]). In this study, we assume that $\Phi_{\mathcal{A}}$ is a non-empty set. The following theorem is motivated by [7, 8, 15].

Theorem 2.3. *Let $\delta : \mathcal{A} \to \mathcal{A}$ be a bounded derivation. Then, $\delta(\mathcal{A}) \subseteq \bigcap_{\mathcal{P} \in \prod_c(\mathfrak{A})} \mathcal{P} \subseteq \bigcap_{\mathcal{M} \in \mathfrak{M}_c(\mathfrak{A})} \mathcal{M} \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker\varphi$. In particular, if $\mathcal{A}$ is semi-prime and $\dim(\bigcap_{\mathcal{P} \in \prod_c(\mathfrak{A})} \mathcal{P}) \leq 1$, then $\delta = 0$.*

Proof. First, we define $\Delta : \mathfrak{A} \to \mathfrak{A}$ by $\Delta(a, \alpha) = (\delta(a), 0) = \delta(a)$. Clearly, Δ is a bounded derivation. Hence, if $\mathcal{P}$ is an arbitrary primitive ideal of $\mathfrak{A}$, then $\Delta(\mathcal{P}) \subseteq \mathcal{P}$ (see Theorem 6.2.3 of [5]). Assume that $\mathcal{P}$ is an arbitrary element of $\prod_c(\mathfrak{A})$. It means that $\frac{\mathfrak{A}}{\mathcal{P}}$ is commutative. Furthermore, according to Proposition 1.4.44

(ii) of [6], $\frac{\mathfrak{A}}{\mathcal{P}}$ is a primitive algebra, and so $\frac{\mathfrak{A}}{\mathcal{P}}$ is semi-simple. Now, we define the linear map $D : \frac{\mathfrak{A}}{\mathcal{P}} \to \frac{\mathfrak{A}}{\mathcal{P}}$ by $D((a,\alpha)+\mathcal{P}) = \Delta(a,\alpha)+\mathcal{P}$. If $(a,\alpha)+\mathcal{P} = (b,\beta)+\mathcal{P}$, then $(a-b,\alpha-\beta) \in \mathcal{P}$. Since $\Delta(\mathcal{P}) \subseteq \mathcal{P}$, $\Delta(a-b,\alpha-\beta) \in \mathcal{P}$. Hence, $\Delta(a,\alpha)+\mathcal{P} = \Delta(b,\beta)+\mathcal{P}$ and it means that D is well-defined. For convenience, (a,λ) is denoted by a_λ for all $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$. A straightforward verification shows that D is a derivation. It follows from Theorem 2.1 that $D(\frac{\mathfrak{A}}{\mathcal{P}}) \subseteq rad(\frac{\mathfrak{A}}{\mathcal{P}}) = \{0\}$, and it implies that $\Delta(\mathfrak{A}) \subseteq \bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P}$. Since $\mathfrak{A}$ is unital, every (maximal) ideal of $\mathfrak{A}$ is a (maximal) modular ideal (see the last paragraph of page 4 of [14]). Moreover, it follows from Proposition 1.4.34 (iv) of [6] that each maximal modular ideal in $\mathfrak{A}$ is a primitive ideal. Therefore, $\mathfrak{M}_c(\mathfrak{A}) \subseteq \Pi_c(\mathfrak{A})$ and it implies that $\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P} \subseteq \bigcap_{\mathcal{M}\in\mathfrak{M}_c(\mathfrak{A})}\mathcal{M}$, where $\Pi_c(\mathfrak{A})$ and $\mathfrak{M}_c(\mathfrak{A})$ were introduced in the introduction. According to Proposition 3.1.2 of [5], $ker\widetilde{\varphi}$ is a maximal ideal of $\mathfrak{A}$ for every $\widetilde{\varphi} \in \Phi_{\mathfrak{A}}$. Note that $\widetilde{\varphi}(a_\alpha b_\beta) = \widetilde{\varphi}(a_\alpha)\widetilde{\varphi}(b_\beta) = \widetilde{\varphi}(b_\beta)\widetilde{\varphi}(a_\alpha) = \widetilde{\varphi}(b_\beta a_\alpha)$ for all $a_\alpha, b_\beta \in \mathfrak{A}$. Hence, $a_\alpha b_\beta - b_\beta a_\alpha \in ker\widetilde{\varphi}$. Thus, $(a_\alpha + ker\widetilde{\varphi})(b_\beta + ker\widetilde{\varphi}) = (b_\beta + ker\widetilde{\varphi})(a_\alpha + ker\widetilde{\varphi})$, and it means that $\frac{\mathfrak{A}}{ker\widetilde{\varphi}}$ is a commutative algebra. Hence, $\{ker\widetilde{\varphi} \mid \widetilde{\varphi} \in \Phi_{\mathfrak{A}}\} \subseteq \mathfrak{M}_c(\mathfrak{A})$ and it is concluded that $\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P} \subseteq \bigcap_{\mathcal{M}\in\mathfrak{M}_c(\mathfrak{A})}\mathcal{M} \subseteq \bigcap_{\widetilde{\varphi}\in\Phi_{\mathfrak{A}}} ker\widetilde{\varphi}$. Therefore, we have $\Delta(\mathfrak{A}) \subseteq \bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P} \subseteq \bigcap_{\mathcal{M}\in\mathfrak{M}_c(\mathfrak{A})}\mathcal{M} \subseteq \bigcap_{\widetilde{\varphi}\in\Phi_{\mathfrak{A}}} ker\widetilde{\varphi}$. Based on the offered discussion in the first paragraph of page 15 of [14], we obtain that $\bigcap_{\widetilde{\varphi}\in\Phi_{\mathfrak{A}}} ker\widetilde{\varphi} = \bigcap_{\varphi\in\Phi_{\mathcal{A}}} ker\varphi$. Hence, $\delta(\mathcal{A}) \subseteq \bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P} \subseteq \bigcap_{\mathcal{M}\in\mathfrak{M}_c(\mathfrak{A})}\mathcal{M} \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} ker\varphi$, and it completes the first part of our proof.

Suppose that $\mathcal{A}$ is semi-prime and $\dim(\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P}) \leq 1$. It is obvious that if $\dim(\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P}) = 0$, then $\delta(\mathcal{A}) = \{0\}$. Now, assume that $\dim(\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P}) = 1$. Since $\dim(\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P}) = 1$, there exists a non-zero element x_λ of $\mathfrak{A}$ such that $\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P} = \{\alpha x_\lambda \mid \alpha \in \mathbb{C}\}$. Since $\delta(\mathcal{A}) \subseteq \bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})}\mathcal{P}$, we can consider the function $\psi : \mathcal{A} \to \mathbb{C}$ such that $\delta(a) = (\delta(a), 0) = \psi(a)x_\lambda = \psi(a)(x,\lambda) = (\psi(a)x, \psi(a)\lambda)$ for all $a \in \mathcal{A}$. So, $\psi(a)\lambda = 0$, and it implies that either $\psi(a) = 0$ or $\lambda = 0$. If $\lambda \neq 0$, then $\psi(a) = 0$ for every $a \in \mathcal{A}$, and consequently, δ is zero. In this case, our goal is achieved. Now, we suppose $\lambda = 0$. We want to show that δ is identically zero. To obtain a contradiction, assume δ is a non-zero derivation. Therefore, there is an element a_0 of $\mathcal{A}$ such that $\delta(a_0) \neq 0$. Clearly, $\psi(a_0) \neq 0$, too. Thus, we have $\delta(a_0) = \psi(a_0)x$. Putting $b = \frac{1}{\psi(a_0)}a_0$, we obtain $\delta(b) = \delta(\frac{1}{\psi(a_0)}a_0) = \frac{1}{\psi(a_0)}\psi(a_0)x = x$ and it implies that $\psi(b) = 1$. We will show that $ax + xa$ is a scalar multiple of x for any a in $\mathcal{A}$. Let a be an arbitrary element of $\mathcal{A}$. Then, $\delta(a^2) = \psi(a^2)x$ (*). On the other hand, we have $\delta(a^2) = \delta(a)a + a\delta(a) = \psi(a)xa + a\psi(a)x = \psi(a)(xa + ax)$ (**). Comparing (*) and (**) , we find that $\psi(a^2)x = \psi(a)(ax + xa)$. If $\psi(a) \neq 0$, then $ax + xa = \frac{\psi(a^2)}{\psi(a)}x$. If $\psi(a) = 0$, then

$$\begin{aligned}\psi(ab + ba)x &= \delta(ab + ba)\\ &= \delta(a)b + a\delta(b) + \delta(b)a + b\delta(a)\\ &= \psi(a)xb + a\psi(b)x + \psi(b)xa + b\psi(a)x\\ &= ax + xa\end{aligned}$$

and this proves that $ax + xa$ is a scalar multiple of x for any a in $\mathcal{A}$. Next, it will be shown that $x^2 = 0$. Suppose that $\psi(x) = 0$. We have $\psi(b^2)x = \delta(b^2) = \delta(b)b + b\delta(b) = \psi(b)xb + b\psi(b)x = xb + bx$. Applying δ on this equality and then using the fact that $\delta(x) = \psi(x)x = 0$, we obtain that $x^2 = 0$. Now, suppose $\psi(x) \neq 0$. Therefore, we have

$$\psi(x^2)x = \delta(x^2) = \delta(x)x + x\delta(x) = \psi(x)x^2 + \psi(x)x^2 = 2\psi(x)x^2. \tag{1}$$

If $\psi(x^2) = 0$, then it follows from previous equality that $x^2 = 0$. Assume that $\psi(x^2) \neq 0$; so $x^2 = \frac{\psi(x^2)}{2\psi(x)}x$. Simplifying the notation, we put $\gamma = \frac{\psi(x^2)}{2\psi(x)}$. Replacing x^2 by γx in $2\psi(x)x^2 = \delta(x^2)$, we have $2\psi(x)\gamma x = \gamma\delta(x) = \gamma\psi(x)x$. Since $\psi(x) \neq 0$, $\gamma x = 0$ and it implies that either $\gamma = 0$ or $x = 0$, which is a contradiction. This contradiction shows that $\psi(x^2) = 0$ and by using (1) it is obtained that $x^2 = 0$. We know that $xa + ax = \mu x$, where $\mu \in \mathbb{C}$. Multiplying the previous equality by x and using the fact that $x^2 = 0$, we see that $xax = 0$ for any a in $\mathcal{A}$. Since $\mathcal{A}$ is semi-prime, $x = 0$. This contradiction shows that δ must be zero. □

We are now ready for the following conclusions.

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra and $\delta : \mathcal{A} \to \mathcal{A}$ be a bounded derivation. If $\mathfrak{S}(\delta(a)) = \{\varphi(\delta(a)) \mid \varphi \in \Phi_{\mathcal{A}}\}$ or $\mathfrak{S}(\delta(a)) = \{\varphi(\delta(a)) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$ for every $a \in \mathcal{A}$, then $\delta(\mathcal{A}) \subseteq rad(\mathcal{A})$. In particular, if $\mathcal{A}$ is semi-simple, then δ is zero.*

Proof. It follows from Theorem 2.3 that $\delta(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker\varphi$. This fact and our assumption concerning $\mathfrak{S}(\delta(a))$ imply that $\mathfrak{S}(\delta(a)) = \{0\}$ for every $a \in \mathcal{A}$. It means that δ is spectrally bounded. At this moment, Theorem 2.2 completes the proof. □

Remark 2.5. *Let $\{d_n\}$ be a higher derivation on an algebra $\mathcal{A}$ with $d_0 = I$, where I is the identity mapping on $\mathcal{A}$. Based on Proposition 2.1 of [12] there is a sequence $\{\delta_n\}$ of derivations on $\mathcal{A}$ such that*

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k}$$

for each non-negative integer n. Therefore, we have

$$\begin{aligned}
&d_0 = I,\\
&d_1 = \delta_1,\\
&2d_2 = \delta_1 d_1 + \delta_2 d_0 = \delta_1\delta_1 + \delta_2,\\
&d_2 = \frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2,\\
&3d_3 = \delta_1 d_2 + \delta_2 d_1 + \delta_3 d_0 = \delta_1(\frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2) + \delta_2\delta_1 + \delta_3,\\
&d_3 = \frac{1}{6}\delta_1^3 + \frac{1}{6}\delta_1\delta_2 + \frac{1}{3}\delta_2\delta_1 + \frac{1}{3}\delta_3.
\end{aligned}$$

Now, assume that $\{d_n\}$ is a bounded higher derivation (,i.e. d_n is a bounded linear map for every non-negative integer n). Obviously, $\delta_1 = d_1$ is bounded. Hence, $\delta_2 = 2d_2 - \delta_1^2$ is also bounded. Based on the d_3 formula, we have $\delta_3 = 3d_3 - \frac{1}{2}\delta_1^3 - \frac{1}{2}\delta_1\delta_2 - \delta_2\delta_1$. Using the boundedness of d_3, δ_1 and δ_2, we obtain that δ_3 is a bounded derivation. In the next step, we will show that every δ_n is a bounded derivation for every $n \in \mathbb{N}$. To reach this aim, we use induction on n. According to the above-mentioned discussion, δ_1, δ_2 and δ_3 are bounded derivations. Now, suppose that δ_k is a bounded derivation for $k \leq n$. We will show that δ_{n+1} is also a bounded derivation. Based on the proof of Theorem 2.3 in [12], we have

$$\delta_{n+1} = (n+1)d_{n+1} - \sum_{i=2}^{n+1} \Big(\sum_{\sum_{j=1}^{i} r_j = n+1} (n+1) a_{r_1,\ldots,r_i} \delta_{r_1} \ldots \delta_{r_i} \Big) \tag{2}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n+1$. From $\sum_{j=1}^{i} r_j = r_1 + r_2 + \ldots + r_i = n+1$ along with the condition that r_j is a positive integer for every $1 \leq j \leq i$, we find that $1 \leq r_j \leq n$ for every $1 \leq j \leq i$. Since we are assuming d_n and δ_k are bounded linear mappings for all non-negative integer n and $k \leq n$, it follows from (2) that δ_{n+1} is a bounded derivation.

Corollary 2.6. *Let $\mathcal{A}$ be a Banach algebra such that $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$ or $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\}$ for every $a \in \mathcal{A}$. If $\{d_n\}$ is a bounded higher derivation (that means d_n is a bounded linear map for every n), then $d_n(\mathcal{A}) \subseteq rad(\mathcal{A})$ for every $n \geq 1$.*

Proof. This is an immediate conclusion from Corollary 2.4, Remark 2.5, and Theorem 2.3 of [12]. □

In the next corollary, we offer a spectrum criterion for the commutativity of Banach algebras.

Corollary 2.7. *Let $\mathcal{A}$ be a semi-simple Banach algebra. Then, $\mathcal{A}$ is commutative if and only if $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$ or $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\}$ for every $a \in \mathcal{A}$.*

Proof. Suppose that $\mathcal{A}$ is a commutative Banach algebra. It follows from Theorem 1.3.4 of [14] that $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$ or $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\}$ for every $a \in \mathcal{A}$. To prove the converse statement we assume that $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\} \bigcup \{0\}$ or $\mathfrak{S}(a) = \{\varphi(a) \mid \varphi \in \Phi_{\mathcal{A}}\}$ for every $a \in \mathcal{A}$. Evidently, $\delta_{a_0}(a) = [a, a_0]$ is a bounded derivation on $\mathcal{A}$, where a_0 is an arbitrary fixed element of $\mathcal{A}$. Corollary 2.4 then yields that δ is zero, and since a_0 is arbitrary, $\mathcal{A}$ is commutative. $\square$

Corollary 2.8. *Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation and $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$ such that $\delta(\mathcal{P}) \subseteq \mathcal{P}$. If $\mathfrak{S}(a + \mathcal{P}) = \{\varphi(a + \mathcal{P}) \mid \varphi \in \Phi_{\frac{\mathcal{A}}{\mathcal{P}}}\} \bigcup \{0\}$ or $\mathfrak{S}(a + \mathcal{P}) = \{\varphi(a + \mathcal{P}) \mid \varphi \in \Phi_{\frac{\mathcal{A}}{\mathcal{P}}}\}$ for every $a \in \mathcal{A}$, then $\delta(\mathcal{A}) \subseteq \mathcal{P}$.*

Proof. According to Proposition 1.4.44 (ii) of [6], $\frac{\mathcal{A}}{\mathcal{P}}$ is a primitive algebra, and so $\frac{\mathcal{A}}{\mathcal{P}}$ is semi-simple. Let us define $\Delta : \frac{\mathcal{A}}{\mathcal{P}} \to \frac{\mathcal{A}}{\mathcal{P}}$ by $\Delta(a+\mathcal{P}) = \delta(a)+\mathcal{P}$. One can easily show that Δ is a derivation. It follows from Theorem 2.3.2 of [18] that Δ is a bounded derivation, and so, Corollary 2.4 implies that Δ is zero. Consequently, $\delta(\mathcal{A}) \subseteq \mathcal{P}$. $\square$

In the following two corollaries, we extend Corollary 2.5 and Corollary 2.6 in [8] to any semi-prime Banach algebra.

Corollary 2.9. *Let $\mathcal{A}$ be a semi-prime Banach algebra such that $\dim(\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})} \mathcal{P}) \leq 1$. Then $\mathcal{A}$ is commutative.*

Proof. Let x_0 be a non-zero arbitrary fixed element of $\mathcal{A}$. Define $d_{x_0} : \mathcal{A} \to \mathcal{A}$ by $d_{x_0}(a) = ax_0 - x_0a$. Obviously, d_{x_0} is a bounded derivation. It follows from Theorem 2.3 that $d_{x_0}(a) = 0$, i.e. $ax_0 = x_0a$ for all $a \in \mathcal{A}$. Since x_0 is arbitrary, $\mathcal{A}$ is commutative. This is exactly what we had to prove. $\square$

Corollary 2.10. *Let $\mathcal{A}$ be a semi-prime Banach algebra, and $\{d_n\}$ be a bounded higher derivation from $\mathcal{A}$ into $\mathcal{A}$. If $\dim(\bigcap_{\mathcal{P}\in\Pi_c(\mathfrak{A})} \mathcal{P}) \leq 1$, then $d_n = 0$ for all $n \in \mathbb{N}$.*

Proof. Let $\{d_n\}$ be the above-mentioned higher derivation. According to Theorem 2.3 of [12] there exists a sequence $\{\delta_n\}$ of derivations on $\mathcal{A}$ such that

$$d_n = \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} \big(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \big) \delta_{r_1} \ldots \delta_{r_i} \Big)$$

, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. It follows from Remark 2.5 that δ_n is a bounded derivation for every positive integer n. At this moment, Theorem 2.3 completes the proof. $\square$

The question under which conditions all derivations are zero on a given Banach algebra have attracted much attention of authors (for instance, see [7, 8, 11, 15, 20]). In the following propositions, we also concentrate on this topic.

Proposition 2.11. *Let $\mathcal{A}$ be a Banach $*$-algebra such that $\overline{O_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$ and $\delta : \mathcal{A} \to \mathcal{A}$ be a bounded derivation. Suppose that $B = \{\delta(p) \mid p \in \mathcal{P}_{\mathcal{A}}\}$ is a commutative set, and furthermore, if $\varphi \in \Phi_{B''}$, then $\varphi(p)$ exists for every $p \in \mathcal{P}_{\mathcal{A}}$. Then, $\delta(\mathcal{A}) \subseteq rad(\mathcal{A})$. In particular, if $\mathcal{A}$ is semi-simple, then δ is zero.*

Proof. Since B is commutative, we have $\delta(p)\delta(q) = \delta(q)\delta(p)$ for all $p, q \in \mathcal{P}_{\mathcal{A}}$. Let a_1 and b_1 be two arbitrary elements of $\mathcal{S}_{\mathcal{A}} = \overline{O_{\mathcal{A}}}$. Hence, there are two sequences $\{x_r\}$ and $\{y_s\}$ in $O_{\mathcal{A}}$ such that $\lim_{r\to\infty} x_r = a_1$ and $\lim_{s\to\infty} y_s = b_1$. From this and using the fact that B is a commutative set, we deduce that

$$\delta(a_1)\delta(b_1) = \delta(b_1)\delta(a_1) \; for\ all\ a_1, b_1 \in \mathcal{S}_{\mathcal{A}}. \tag{3}$$

It is well-known that if a is an arbitrary element of $\mathcal{A}$, then there exist two self-adjoint elements a_1, a_2 such that $a = a_1 + ia_2$. This fact with (3) imply that $\delta(a)\delta(b) = \delta(b)\delta(a)$ for all $a, b \in \mathcal{A}$. Assume that $a_1 \in \mathcal{S}_{\mathcal{A}} = \overline{\mathcal{O}_{\mathcal{A}}}$. Then, there is a sequence $\{x_r\} \subseteq \mathcal{O}_{\mathcal{A}}$ such that $\lim_{r\to\infty} x_r = a_1$. We have

$$\begin{aligned}\delta(a_1) &= \delta(\lim_{r\to\infty} x_r) = \lim_{r\to\infty} \delta(x_r)\\ &= \lim_{r\to\infty} \delta\Big(\sum_{k_r=1}^{n_r} \alpha_{k_r} p_{k_r}\Big)\\ &= \lim_{r\to\infty} \sum_{k_r=1}^{n_r} \alpha_{k_r} \delta(p_{k_r})\end{aligned}$$

It is evident that, $\sum_{k_r=1}^{n_r} \alpha_{k_r}\delta(p_{k_r})$ is a sequence in B'' and since B'' is a commutative Banach algebra, $\lim_{r\to\infty} \sum_{k_r=1}^{n_r} \alpha_{k_r}\delta(p_{k_r}) = \delta(a_1) \in B''$. Hence, $\delta^n(a) \in B''$ for every natural number n and each $a \in \mathcal{A}$. Since B'' is a commutative Banach algebra, $\Phi_{B''}$ is a non-empty set (see Theorem 2.3.25 of [6]). If we define $d_n = \frac{\delta^n}{n!}$ with $d_0 = I$, the identity mapping on $\mathcal{A}$, then we have

$$\begin{aligned}d_n(ab) &= \frac{1}{n!}\delta^n(ab) = \frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k} \delta^{n-k}(a)\delta^k(b)\\ &= \sum_{k=0}^{n} \frac{1}{n!}.\frac{n!}{(n-k)!k!}\delta^{n-k}(a)\delta^k(b)\\ &= \sum_{k=0}^{n} d_{n-k}(a)d_k(b).\end{aligned}$$

Define $F(t) = \sum_{n=0}^{\infty} d_n(p)t^n$, where $|t| < 1$ and p is an arbitrary, non-trivial fixed element of $\mathcal{P}_{\mathcal{A}}$ (see [13]). Note that

$$\|d_n\| = \|\frac{\delta^n}{n!}\| \le \frac{1}{n!}\|\delta\|^n < \sum_{n=0}^{\infty} \frac{\|\delta\|^n}{n!} = e^{\|\delta\|}$$

It means that $\{d_n\}$ is a uniformly bounded sequence of linear mappings. Hence, we have

$$\begin{aligned}\|\sum_{n=0}^{\infty} d_n(p)t^n\| &\le \sum_{n=0}^{\infty} \|d_n(p)t^n\|\\ &= \sum_{n=0}^{\infty} \|d_n(p)\||t^n|\\ &\le \sum_{n=0}^{\infty} \|d_n\|\|p\||t^n|\\ &\le \sum_{n=0}^{\infty} e^{\|\delta\|}\|p\||t^n| = e^{\|\delta\|}\|p\|\frac{1}{1-|t|} < \infty.\end{aligned}$$

This fact ensures that F is well-defined. Hence, the m-th derivative of F exists and is given by the formula

$F^{(m)}(t)=\sum_{n=m}^{\infty}\frac{n!}{(n-m)!}d_n(p)t^{n-m}$. Furthermore, we have

$$\begin{aligned}F(t)F(t)=&\Big(\sum_{n=0}^{\infty}d_n(p)t^n\Big)\Big(\sum_{n=0}^{\infty}d_n(p)t^n\Big)\\=&\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}d_{n-k}(p)d_k(p)\Big)t^n\\=&\sum_{n=0}^{\infty}d_n(p)t^n\\=&F(t).\end{aligned}$$

Let φ be an arbitrary fixed element of $\Phi_{B''}$. It is clear that the function $G=\varphi F:(-1,1)\to\mathbb{C}$ defined by $G(t)=\varphi F(t)=\varphi(F(t))=\varphi(\sum_{n=0}^{\infty}d_n(p)t^n)=\sum_{n=0}^{\infty}\varphi(d_n(p))t^n$ is continuous on $|t|<1$. Hence, $G(t)^2=(\varphi(F(t)))^2=\varphi(F(t))=G(t)$ implies that $G(t)=0$ or $G(t)=1$. It is observed that $G(t)$ is a power series in $\mathbb{C}$. Thus, the m-th derivative of G exists and is given by $G^{(m)}(t)=\sum_{n=m}^{\infty}\frac{n!}{(n-m)!}\varphi(d_n(p))t^{n-m}$. We continue the proof by using the presented argument in Theorem 2.2 of [8]. Since the function G is constant, we have $G^{(m)}(t)=0$ for every $m\in\mathbb{N}\backslash\{0\}$ and every $|t|<1$. So, $\varphi(d_1(p))+2\varphi(d_2(p))t+3\varphi(d_3(p))t^2+4\varphi(d_4(p))t^3+...=G^{(1)}(t)=0$. Putting $t=0$ in the former equation, it is obtained that $\varphi(d_1(p))=0$. Using an argument similar to what was described concerning $\varphi(d_1(p))$, we conclude that $\varphi(d_2(p))=0$. By continuing this procedure, it is proved that $\varphi(d_n(p))=0$ for all $n\geq 1$. Our next task is to show that $\varphi(d_n(a))=0$ for every $a\in\mathcal{A}$. Let x be an arbitrary element of $O_{\mathcal{A}}$. Hence, $x=\sum_{i=1}^{m}r_ip_i$, where $p_1,p_2,...,p_m$ are mutually orthogonal projections and $r_1,r_2,...,r_m$ are real numbers. We have $\varphi(d_n(x))=\varphi(d_n(\sum_{i=1}^{m}r_ip_i))=\sum_{i=1}^{m}r_i\varphi(d_n(p_i))=0$. Since $\overline{O_{\mathcal{A}}}=\mathcal{S}_{\mathcal{A}}$, $\varphi(d_n(a))=0$ for every $a\in\mathcal{S}_{\mathcal{A}}$. It is well-known that each a in $\mathcal{A}$ can be represented as $a=a_1+ia_2$, $a_1,a_2\in\mathcal{S}_{\mathcal{A}}$; therefore, $\varphi(d_n(a))=\varphi(d_n(a_1+ia_2))=\varphi(d_n(a_1))+i\varphi(d_n(a_2))=0$ for all $n\geq 1$, $a\in\mathcal{A}$ and $\varphi\in\Phi_{B''}$. It means that $d_n(\mathcal{A})\subseteq\bigcap_{\varphi\in\Phi''_B}\ker\varphi$. According to Theorem 11.22 of [16] and Theorem 1.3.4 of [14], it is achieved that $\mathfrak{S}_{\mathcal{A}}(d_n(a))=\mathfrak{S}_{B''}(d_n(a))=\{\varphi(d_n(a))\mid\varphi\in\Phi_{B''}\}\bigcup\{0\}$ $(or=\{\varphi(d_n(a))\mid\varphi\in\Phi_{B''}\})=\{0\}$. Hence, $r(d_n(a))=0$ for all $n\geq 1$ and $a\in\mathcal{A}$. It means that d_n is spectrally bounded for every $n\geq 1$. Since $d_1=\delta$ is spectrally bounded, Theorem 2.2 shows that $\delta(\mathcal{A})\subseteq rad(\mathcal{A})$. Evidently, if $\mathcal{A}$ is semi-simple, i.e. $rad(\mathcal{A})=\{0\}$, then δ is zero. □

Before proving Proposition 2.13, we define the socle of $\mathcal{A}$. Let $\mathcal{A}$ be a semi-simple Banach algebra. Then the sum of all the minimal left ideals of $\mathcal{A}$ coincides with the sum of all the minimal right ideals of $\mathcal{A}$, is called the socle of $\mathcal{A}$, and it will be denoted by $soc(\mathcal{A})$. We refer the reader to [2–4] for more information on the socle of a Banach algebra.

Proposition 2.12. *Let $\mathcal{A}$ be a semi-simple Banach algebra, and let d be a derivation on $\mathcal{A}$ satisfying $\sharp\mathfrak{S}(d(a))=1$ for all $a\in\mathcal{A}$. Here, $\sharp\mathfrak{S}(x)$ denotes the cardinality of the spectrum of x. Then, d is zero.*

Proof. It follows from Theorem 1.2 of [4] that d is an inner derivation induced by an element $u\in soc(\mathcal{A})$. It means that $d(a)=[u,a]=ua-au$ for all $a\in\mathcal{A}$. According to the aforementioned assumption, we have $1=\sharp\mathfrak{S}(d(a))=\sharp\mathfrak{S}(ua-au)$ for all $a\in\mathcal{A}$. Now, Theorem 5.2.1 of [1] implies that $u\in Z(\mathcal{A})$, and consequently, d is zero. □

Now, the article is ended with a problem which has attracted the author's attention .

Problem 2.13. *Let d be a derivation on a given Banach algebra $\mathcal{A}$. Under which conditions,* $\sharp\mathfrak{S}(d(a))=1$ *for all $a\in\mathcal{A}$?*

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