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Existence of Solutions for a Second Order Boundary Value Problem with the Clarke Subdifferential

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Abstract. In this paper, we prove a theorem on the existence of solutions for a second order differential inclusion governed by the Clarke subdifferential of a Lipschitzian function and by a mixed semicontinuous perturbation.

1. Introduction

The existence of solutions for the first order differential inclusion in a separable Hilbert space H of the form

$$(\mathcal{P}_M)\begin{cases} -\dot{x}(t) \in \partial_c\varphi(x(t)) + M(t, x(t)), \quad a.e.t \in [0, T] \\ x(0) = x_0, \end{cases}$$

has been studied in [5], where $\partial_c(\varphi(\cdot))$ is the Clarke subdifferential of the proper lower semicontinous inf compact convex function $\varphi(\cdot)$, $M : [0, T] \times H \rightrightarrows H$ is an upper semicontinuous with respect to the second variable multimapping with closed convex values.

The authors in [17] investigated the same evolution inclusion with $\varphi(.)$ a proper convex and lower semicontinous function, in both cases where the perturbation $M(., .)$ has convex or nonconvex values.

Evolution differential inclusions governed by the subdifferential of proper convex l.s.c functions appears often in problems of optimal control theory (Cesari [9], Clarke [10], and Rockafellar [23]), of mechanics (Moreau [19] and Donchev [12]), and of mathematics economics (Cornet [11] and Henry [15]).

It is worth mentioning, that when $\varphi(.)$ is the indicator function of a closed convex moving set $C(t)$, the subdifferential of $\varphi(.)$ is the normal cone at $C(t)$, and problem $(\mathcal{P}_F)$ is a perturbed sweeping process. Numerical aspects of the sweeping process can be found in [21], applications include the dynamics of machines [13] and the vast area of numerical simulation in granular mechanics (see [20] and references therein for a review). Frictional contact may be somewhat regularized through the introduction of local elastic micro-deformation ([18]) and of viscosity-like effects [25, 26]. Such perturbed processes have been thoroughly studied in many papers, see for example ([2, 24, 27, 29]).

In the present work, we study, in the finite-dimensional space $\mathbb{R}^n$, the existence of solutions of the second order boundary value problem of the form

$$(\mathcal{P}_F)\begin{cases} -\ddot{x}(t) \in \partial_c\varphi(x(t)) + F(t, x(t), \dot{x}(t)), a.e.\ t \in [0, 1] \\ x(0) = x(1) = 0, \end{cases}$$

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where $F : [0,1]\times\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a nonempty closed valued multimapping measurable on $[0,1]$ and mixed semicontinuous, that is, for almost every t$\in [0,1]$, at each $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $F(t,x,y)$ is convex, the multimapping $F(t,.,.)$ is upper semicontinous on $\mathbb{R}^n \times \mathbb{R}^n$ and whenever $F(t,x,y)$ is not convex, the multimapping $F(t,.,.)$ is lower semicontinous on some neighborhood of (x,y). We refer the reader to [3] for mixed semicontinuous perturbation to a second order boundary value problem governed by a maximal monotone operator.

2. Definitions and Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with scalar product $\langle .,.\rangle$ and norm $\|.\|$.
$\mathbf{B}_{\mathbb{R}^n}(0,r)$ and $\overline{\mathbf{B}}_{\mathbb{R}^n}(0,r)$ are the closed balls of $\mathbb{R}^n$ with center 0 and radius $r > 0$, for $r = 1$ we will write $\mathbf{B}_{\mathbb{R}^n}$ and $\overline{\mathbf{B}}_{\mathbb{R}^n}$. $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue measurable sets of $[0,1]$, dt is the Lebesgue measure on $[0,1]$, and $\mathcal{B}(\mathbb{R}^n)$ is the σ-algebra of Borel subsets of $\mathbb{R}^n$. By $\mathbf{L}^1_{\mathbb{R}^n}([0,1])$ we denote the space of all Lebesgue-Bochner integrable $\mathbb{R}^n$-valued mappings defined on $[0,1]$.
Let $\mathbf{C}_{\mathbb{R}^n}([0,1])$ be the Banach space of all continuous mappings $x : [0,1] \to \mathbb{R}^n$, endowed with the sup norm $\|.\|_{\mathbf{C}}$, and $\mathbf{C}^1_{\mathbb{R}^n}([0,1])$ be the Banach space of all continuous mappings $x : [0,1] \to \mathbb{R}^n$ with continuous derivative, equipped with the norm

$$\|x\|_{\mathbf{C}^1} = \max\{\max_{t\in[0,1]}\|x(t)\|, \max_{t\in[0,1]}\|\dot{x}(t)\|\}.$$

By $\mathbf{W}^{2,1}_{\mathbb{R}^n}([0,1])$ we denote the space of all continuous mappings $x \in \mathbf{C}_{\mathbb{R}^n}([0,1])$ such that their first usual derivatives are continuous and scalarly derivable and such that $\ddot{x} \in \mathbf{L}^1_{\mathbb{R}^n}([0,1])$.

If E is a Banach space, we denote by E' its topological dual space endowed with the norm

$$\|\xi\|_* := \sup\{\langle \xi, v\rangle : v \in E, \|v\| \le 1\},$$

$\sigma(E,E')$ is the weak topology on E and $\sigma(E',E)$ is the weak* topology on E'.

For a set $A \subset \mathbb{R}^n$, $\overline{co}(A)$ is the closed convex hull of A.

The theorem below is a result characterizing the closed convex hull of a subset of a linear space E.

Theorem 2.1. *(see [6]) Let K be a nonempty subset of E. Then*

$$\overline{co}(K) = \{x \in E : \forall\, x' \in E',\ \langle x', x\rangle \le \delta^*(x', K)\},$$

where,

$$\delta^*(x', K) = \sup_{y\in K}\langle x', y\rangle$$

stands for the support function of K at $x' \in E'$.

Lemma 2.2. *(see [8]) Let E be a Banach space, and C be a closed convex subset of E, then*

$$d(x, C) = \sup_{x'\in\overline{B}_{E'}}\Big(\langle x', x\rangle - \delta^*(x', C)\Big).$$

Theorem 2.3. *(See [7]) Let E be Banach space and C be a convex subset of E, then C is weakly closed if and only if it is strongly closed.*

Theorem 2.4. *(Banach-Mazur's Lemma, see [16]) If E is a Banach space and (x_n) is a sequence of elements of E converging weakly to x, then some sequences of convex combinations of the elements x_n converges to x in the norm topology of E.*

We recall the following definitions.

Definition 2.5. *Let E be a Banach space. Let Y be a subset of E and $f : Y \to \mathbb{R}$, we shall say that f is Lipschitz (of rank L) near x if, for some $\varepsilon > 0$, f satisfies the Lipschitz condition (of rank L) on the set $x + \varepsilon \mathbf{B}_E$.*

Definition 2.6. *(see [10]) Let E be a Banach space. Let $f : E \to \mathbb{R}$ be Lipschitzian near a given point $x_0 \in E$, and v any other vector in E. The generalized directional derivative of f at x_0 in the direction v, denoted by $f^\circ(x_0, v)$, is defined as follows*

$$f^\circ(x_0, v) = \limsup_{\substack{y \to x_0 \\ t \to 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in E and t is a positive scalar.
The Clarke subdifferential of f at x_0, denoted by $\partial_c f(x_0)$, is the subset of E' defined by

$$\partial_c f(x_0) := \{\xi \in E' : \langle \xi, v \rangle \leq f^\circ(x_0, v) \quad \text{for all} \quad v \in E\}.$$

Proposition 2.7. *(see [10]) Let $f : E \to \mathbb{R}$ be Lipschitzian of rank L near x. Then,*
(**a**) *$\partial_c f(x)$ is a nonempty convex and weakly*-compact subset of E', and $\|\xi\|_* \leq L$ for every ξ in $\partial_c f(x)$;*
(**b**) *for every v in E, one has*

$$f(x, v) = \max\{\langle \xi, v \rangle : \xi \in \partial_c f(x)\}.$$

Lemma 2.8. *(see [22]) The Clarke subdifferential mapping $\partial_c f : E \rightrightarrows E'$ is norm-to-weak* upper semicontinuous.*

3. Main Results

We begin by giving a proposition which summarizes some properties of some Green type function needed in the proof of our main result (see [1] and [14]).

Proposition 3.1. *Let E be a separable Banach space, and let $G : [0,1] \times [0,1] \to \mathbb{R}$ be the function defined by*

$$G(t, s) = \begin{cases} (t-1)s & \text{if } 0 \leq s \leq t, \\ t(s-1) & \text{if } t \leq s \leq 1. \end{cases}$$

Then the following assertions hold.
(**a**) *If $u \in \mathbf{W}^{2,1}_E([0,1])$ with $u(0) = u(1) = 0$, then*

$$u(t) = \int_0^1 G(t, s)\ddot{u}(s)ds, \quad \forall t \in [0,1].$$

(**b**) *$G(., s)$ is derivable on $[0,1]$, for every $s \in [0,1]$ except on the diagonal and its derivative is given by*

$$\frac{\partial G}{\partial t}(t, s) = \begin{cases} s & \text{if } 0 \leq s < t, \\ (s-1) & \text{if } t < s \leq 1. \end{cases}$$

(**c**) *$G(.,.)$ and $\frac{\partial G}{\partial t}(.,.)$ satisfy*

$$\sup_{t \in [0,1]} |G(t,s)| \leq 1, \quad \sup_{\substack{t \in [0,1] \\ t \neq s}} |\frac{\partial G}{\partial t}(t,s)| \leq 1.$$

(**d**) *For $f \in \mathbf{L}^1_E([0,1])$ and for the mapping $u_f : [0,1] \to E$ defined by*

$$u_f(t) = \int_0^1 G(t,s) f(s) ds, \quad \forall\, t \in [0,1],$$

one has $u_f(0) = u_f(1) = 0$. Furthermore, the mapping u_f is derivable, and its derivative $\dot{u}_f$ satisfies

$$\lim_{h \to 0} \frac{u_f(t+h) - u_f(t)}{h} = \dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s) ds.$$

(**e**) *The mapping $\dot{u}_f$ is scalarly derivable, that is, there exists a mapping $\ddot{u}_f : [0,1] \to E$ such that, for every $x' \in E'$, the scalar function $\langle x', \dot{u}_f(.) \rangle$ is derivable, with $\frac{d}{dt}\langle x', \dot{u}_f(t) \rangle = \langle x', \ddot{u}_f(t) \rangle$. Furthermore, $\ddot{u}_f = f$ a.e. on $[0,1]$.*

Let us mention a useful consequence of Proposition 3.1.

Proposition 3.2. *Let E be a separable Banach space and let $f : [0,1] \to E$ be a continuous mapping (respectively, a mapping in $\mathbf{L}^1_E([0,1])$). Then the mapping $u_f : [0,1] \to E$ defined by*

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \quad \forall\, t \in [0,1],$$

is the unique $\mathbf{C}^2_E([0,1])$-solution (respectively, $\mathbf{W}^{2,1}_E([0,1])$-solution) of the differential equation

$$\begin{cases} \ddot{u}(t) = f(t), \quad \forall\, t \in [0,1], \\ u(0) = u(1) = 0. \end{cases}$$

For the proof of our theorem we will also need the following theorem and we refer the reader to [28] and [4] for its proof.

Theorem 3.3. *Let $M : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed valued multimapping satisfying the following hypotheses.*
*(**i**) M is $\mathcal{L}([0,1]) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n)$-measurable;*
*(**ii**) for every $t \in [0,1]$, at each $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $M(t,x,y)$ is convex, the multimapping $M(t,.,.)$ is upper semicontinuous, and whenever $M(t,x,y)$ is not convex, $M(t,.,.)$ is lower semicontinuous on some neighborhood of (x,y);*
*(**iii**) there exists a positive function $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}$ of Carathéodory type which is integrably bounded on bounded subsets of $\mathbb{R}^n$ such that*

$$M(t,x,y) \cap \overline{\mathbf{B}}_{\mathbb{R}^n}(0, f(t,x,y)) \neq \emptyset,$$

for all $(t,x,y) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$.
Then for any $\varepsilon > 0$ and any compact set $\mathcal{K} \subset \mathbf{C}^1_{\mathbb{R}^n}([0,1])$, there is a nonempty closed convex valued multimapping $\Phi : \mathcal{K} \rightrightarrows \mathbf{L}^1_{\mathbb{R}^n}([0,1])$ which has a strongly-weakly sequentially closed graph such that for any $x \in \mathcal{K}$ and $\phi \in \Phi(x)$ for almost every $t \in [0,1]$, one has,

$$\phi(t) \in M(t, x(t), \dot{x}(t)),$$
$$\|\phi(t)\| \leq f(t, x(t), \dot{x}(t)) + \varepsilon.$$

Now we are able to prove our main result.

Theorem 3.4. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitzian function of rank L and $F : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a nonempty closed valued multimapping satisfying the following hypotheses:*
($\mathbf{H_1}$) F is $\mathcal{L}([0,1]) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n)$-measurable;
($\mathbf{H_2}$) for every $t \in [0,1]$, at each (x,y) such that $F(t,x,y)$ is convex, $F(t,.,.)$ is upper semicontinuous, and whenever $F(t,x,y)$ is not convex, $F(t,.,.)$ is lower semicontinuous on some neighborhood of (x,y);
($\mathbf{H_3}$) there exists some positive Lebesgue integrable function $\rho(.)$ defined on $[0,1]$ such that

$$F(t,x,y) \cap \rho(t)\overline{\mathbf{B}}_{\mathbb{R}^n} \neq \emptyset, \ \text{ for all } \ (t,x,y) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n.$$

Then the boundary differential inclusion

$$(\mathcal{P}_F) \begin{cases} -\ddot{x}(t) \in \partial_c \varphi(x(t)) + F(t, x(t), \dot{x}(t)), \ \text{ a.e. } t \in [0,1], \\ x(0) = x(1) = 0 \end{cases}$$

has at least one solution $x(.) \in \mathbf{W}^{2,1}_{\mathbb{R}^n}([0,1])$.

Proof. **Step 1**. Remark by Proposition (2.7) that for all $x \in \mathbb{R}^n$, $\partial_c \varphi(x) \subset L\overline{\mathbf{B}}_{\mathbb{R}^n}$ since φ is Lipschitzian on $\mathbb{R}^n$. Put for all $t \in [0,1]$, $m(t) = L + \rho(t) + \dfrac{1}{2}$ and let us consider the sets

$$\mathcal{D} = \Big\{ h \in \mathbf{L}^1_{\mathbb{R}^n}([0,1]) : \|h(t)\| \leq m(t), \ \text{ a.e. on } \ [0,1] \Big\},$$

and

$$\mathcal{K} = \Big\{ x_f \in \mathbf{W}^{2,1}_{\mathbb{R}^n}([0,1]) : x_f(t) = \int_0^1 G(t,s)f(s)ds,\ \forall t \in [0,1],\ f \in \mathcal{D} \Big\}.$$

It is clear that $\mathcal{D}$ is a convex $\sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})$-compact subset of $\mathbf{L}^1_{\mathbb{R}^n}([0,1])$ and that $\mathcal{K}$ is a convex compact subset of $\mathbf{C}^1_{\mathbb{R}^n}([0,1])$. Indeed, let $(h_n(.))_n$ be a sequence of elements of $\mathcal{D}$ converging to $h(.) \in \mathbf{L}^1_{\mathbb{R}^n}([0,1])$. For all $t \in [0,1]$, let $s_n(t) = \dfrac{h_n(t)}{m(t)}$. We have $\|s_n(t)\| \leq 1$, that is, $s_n(.) \in \overline{\mathbf{B}}_{\mathbf{L}^\infty_{\mathbb{R}^n}}$, which is weakly*-compact, so by extracting a subsequence, we may suppose that $(s_n(.))_n$ $\sigma(\mathbf{L}^\infty_{\mathbb{R}^n}, \mathbf{L}^1_{\mathbb{R}^n})$-converges to a mapping $s(.) \in \mathbf{L}^\infty_{\mathbb{R}^n}([0,1])$, this implies that for all $z(.) \in \mathbf{L}^1_{\mathbb{R}^n}([0,1])$, $\langle s_n(.), z(.)\rangle \to \langle s, z\rangle$.
Let $y(.) \in \mathbf{L}^\infty_{\mathbb{R}^n}([0,1])$, then, $m(.)y(.) \in \mathbf{L}^1_{\mathbb{R}^n}([0,1])$. Consequently,

$$\langle m(.)s_n(.), y(.)\rangle = \langle s_n(.), m(.)y(.)\rangle \to \langle s(.), m(.)y(.)\rangle = \langle m(.)s(.), y(.)\rangle,$$

that is, $(h_n(.))$ $\sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})$-converges to the mapping $h(.) := m(.)s(.)$. This shows that $\mathcal{D}$ is relatively weakly compact. Furthermore, since $\mathcal{D}$ is a strongly closed convex subset of $\mathbf{L}^1_{\mathbb{R}^n}([0,1])$, then, by Theorem(2.3), it is weakly closed. We conclude that $\mathcal{D}$ is weakly compact in $\mathbf{L}^1_{\mathbb{R}^n}([0,1])$.
Now, to see the compactness of $\mathcal{K}$ in $\mathbf{C}^1_{\mathbb{R}^n}([0,1])$, observe first that it is equicontinuous since for all $f(.) \in \mathcal{D}$ and for all t_1, $t_2 \in [0,1]$, $(t_1 < t_2)$ one has

$$\|x_f(t_2) - x_f(t_1)\| = \|\int_0^1 G(t_2,s)f(s)ds - \int_0^1 G(t_1,s)f(s)ds\|$$

$$\leq \int_0^1 |G(t_2,s) - G(t_1,s)| m(s)ds$$

and, by the assertion **(d)** in Proposition(3.1),

$$\|\dot{x}_f(t_2) - \dot{x}_f(t_1)\| = \|\int_0^1 \frac{\partial G}{\partial t_2}(t_2,s)f(s)ds - \int_0^1 \frac{\partial G}{\partial t_1}(t_1,s)f(s)ds\|$$

$$\leq \int_0^1 |\frac{\partial G}{\partial t_2}(t_2,s) - \frac{\partial G}{\partial t_1}(t_1,s)| m(s)ds.$$

Since $m(.) \in \mathbf{L}^1_{\mathbb{R}}([0,1])$ and $G(.)$ and $\dfrac{\partial G}{\partial t}(.)$ are uniformly continuous, we get the equicontinuity of $\mathcal{K}$ and of the set $\{\dot{x}(.), x(.) \in \mathcal{K}\}$.
On the other hand, for all $f(.) \in \mathcal{D}$, we have by assertion **(c)** of Proposition(3.1),

$$\|x_f(t)\| = \|\int_0^1 G(t,s)f(s)ds\| \leq \int_0^1 |G(t,s)| \|f(s)\| ds$$

$$\leq \int_0^1 m(s)ds = \|m\|_{\mathbf{L}^1_{\mathbb{R}}},$$

and

$$\|\dot{x}_f(t)\| = \|\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds\| \leq \int_0^1 |\frac{\partial G}{\partial t}(t,s)| \|f(s)\| ds$$

$$\leq \int_0^1 m(s)ds = \|m\|_{\mathbf{L}^1_{\mathbb{R}}}.$$

This shows that $\mathcal{K}(t)$ and $\{\dot{x}(t), x(.) \in \mathcal{K}\}$ are bounded in the finite-dimensional space $\mathbb{R}^n$ and hence there are relatively compact. By the Ascoli-Arzelà Theorem we conclude that $\mathcal{K}$ and $\{\dot{x}(.), x(.) \in \mathcal{K}\}$ are relatively compact in $\mathbf{C}_{\mathbb{R}^n}([0,1])$, or equivalently, $\mathcal{K}$ is relatively compact in $\mathbf{C}^1_{\mathbb{R}^n}([0,1])$.

In the following we prove that $\mathcal{K}$ is closed in $\mathbf{C}^1_{\mathbb{R}^n}([0,1])$. Let $(x_{f_n}(.))_n$ be a sequence of elements of $\mathcal{K}$ converging to $x(.) \in \mathbf{C}^1_{\mathbb{R}^n}([0,1])$, that is, for all $t \in [0,1]$, $x_{f_n}(t) = \int_0^1 G(t,s)f_n(s)ds$, and $(f_n(.))_n \subset \mathcal{D}$. Since $\mathcal{D}$ is $\sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})$-compact, we can extract from $(f_n(.))_n$ a subsequence that we do not relabel and which converges weakly to a mapping $f(.) \in \mathcal{D}$. Let $y(.) \in \mathbf{L}^\infty_{\mathbb{R}^n}([0,1])$, for all $t \in [0,1]$, we have

$$\lim_{n\to\infty}\langle G(t,.)f_n(.), y(.)\rangle = \lim_{n\to\infty}\langle f_n(.), G(t,.)y(.)\rangle$$

$$= \langle f(.), G(t,.)y(.)\rangle = \langle G(t,.)f(.), y(.)\rangle,$$

i.e.,

$$\lim_{n\to\infty}\int_0^1 \langle G(t,s)f_n(s), y(s)\rangle ds = \int_0^1 \langle G(t,s)f(s), y(s)\rangle ds,$$

in particular for $y(.) = \mathbb{1}_{[0,1]}(.)e_j$, with $(e_j)_j$ a basis of the space $\mathbb{R}^n$, then,

$$\langle \lim_{n\to\infty}\int_0^1 G(t,s)f_n(s)ds, e_j\rangle = \langle \int_0^1 G(t,s)f(s)ds, e_j\rangle, \forall j,$$

which ensures,

$$\lim_{n\to\infty} x_{f_n}(t) = \lim_{n\to\infty}\int_0^1 G(t,s)f_n(s)ds = \int_0^1 G(t,s)f(s)ds = x(t),$$

from assertion **(d)** of Proposition(3.1), we have

$$\lim_{n\to\infty} \dot{x}_{f_n}(t) = \lim_{n\to\infty}\int_0^1 \frac{\partial G}{\partial t}(t,s)f_n(s)ds = \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds = \dot{x}(t).$$

We conclude that the sequence $(x_n(.), \dot{x}_n(.))_n$ converges to $(x(.), \dot{x}(.)) = (x_f(.), \dot{x}_f(.))$, this implies that $\mathcal{K}$ is closed and hence it is compact in $\mathbf{C}^1_{\mathbb{R}^n}([0,1])$.

By Theorem(3.3) where we take $f(t,x,y) = \rho(t)$, there is a nonempty closed convex valued multimapping $\Phi : \mathcal{K} \rightrightarrows \mathbf{L}^1_{\mathbb{R}^n}([0,1])$ which has a strongly-weakly sequentially closed graph, such that, for all $x \in \mathcal{K}$ and $\phi \in \Phi(x)$, we have for almost all $t \in [0,1]$

$$\phi(t) \in F(t, x(t), \dot{x}(t)) \ \text{ and } \ \|\phi(t)\| \leq \rho(t) + \frac{1}{2}. \tag{3.1}$$

Step 2. Let us define the multimapping $\Gamma : \mathcal{K} \rightrightarrows \mathbf{C}^1_{\mathbb{R}^n}([0,1])$ by

$$\Gamma(x) = \Big\{ y \in \mathbf{C}^1_{\mathbb{R}^n}([0,1]) : y(t) = \int_0^1 G(t,s)w(s)ds, \forall t \in [0,1],$$

$$w(t) \in -\partial_c\varphi(x(t)) - \phi(t), a.e. t \in [0,1], \phi \in \Phi(x) \Big\}.$$

First, observe that for any $x \in \mathcal{K}$ and all $\phi \in \Phi(x)$ the multimapping $t \mapsto -\partial_c\varphi(x(t)) - \phi(t)$ is measurable. According to the theorem of the existence of measurable selection (see [8]), there is a measurable mapping $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(t) \in -\partial_c\varphi(x(t)) - \phi(t)$ for all $t \in [0,1]$. Consequently, the mapping $y(.) : [0,1] \to \mathbb{R}^n$ defined by $y(t) = \int_0^1 G(t,s)\gamma(s)ds$ belongs to $\Gamma(x)$, this shows that $\Gamma(x)$ is a nonempty set.
Fix any $x \in \mathcal{K}$ and $y \in \Gamma(x)$, by the definition of $\Gamma(x)$, there exists $\phi \in \Phi(x)$ and a Lebesgue measurable mapping $w : [0,1] \to \mathbb{R}^n$ such that

$$y(t) = \int_0^1 G(t,s)w(s)ds, \ \forall\, t \in [0,1] \ \text{ and } w(t) \in -\partial_c\varphi(x(t)) - \phi(t), a.e.\ t \in [0,1].$$

From (3.1), for almost every $t \in [0, 1]$, we get

$$\|w(t)\| \leq L + \rho(t) + \frac{1}{2} = m(t), \tag{3.2}$$

this implies that $\Gamma(x) \subset \mathcal{K}$, that is, Γ is a map from $\mathcal{K}$ into itself.
Clearly $\Gamma(x)$ is convex since the set $\Phi(x)$ and the Clarke subdifferential of $\varphi(x(\cdot))$ are convex.

Let us prove now, that for any $x \in \mathcal{K}$, $\Gamma(x)$ is a compact subset of $\mathbf{C}^1_{\mathbb{R}^n}([0, 1])$. Since $\mathcal{K}$ is compact, it is sufficient to prove that $\Gamma(x)$ is closed. Let $(y_n(.))$ be a sequence of $\Gamma(x)$ converging to $y(.) \in \mathcal{K}$, that is, there is a sequence $(\phi_n(.)) \subset \Phi(x)$ and a sequence of Lebesgue measurable mappings $(w_n(.))$ such that for each $n \in \mathbb{N}$,

$$y_n(t) = \int_0^1 G(t, s) w_n(s) ds \quad \forall t \in [0, 1],$$

and

$$w_n(t) \in -\partial_c \varphi(x(t)) - \phi_n(t) \ \ a.e.t \in [0, 1]. \tag{3.3}$$

By (3.2), $(w_n(.))$ is included in $\mathcal{D}$ which is $\sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})$-compact, then we can extract a subsequence, that we do not relabel, $\sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})$-converging to some mapping $w(.) \in \mathbf{L}^1_{\mathbb{R}^n}([0, 1])$. Consequently, for every $t \in [0, 1]$,

$$y(t) = \lim_{n\to+\infty} \int_0^1 G(t, s) w_n(s) ds = \int_0^1 G(t, s) w(s) ds.$$

Indeed, let $z \in \mathbf{L}^\infty_{\mathbb{R}^n}([0, 1])$, for all $t \in [0, 1]$, we have

$$\lim_{n\to\infty} \langle G(t, .) w_n(.), z(.)\rangle = \lim_{n\to\infty} \langle w_n(.), G(t, .) z(.)\rangle = \langle w(.), G(t, .) z(.)\rangle = \langle G(t, .) w(.), z(.)\rangle,$$

i.e.,

$$\lim_{n\to\infty} \int_0^1 \langle G(t, s) w_n(s), z(s)\rangle ds = \int_0^1 \langle G(t, s) w(s), z(s)\rangle ds,$$

in particular for $z(.) = \mathbb{1}_{[0,1]}(.) e_j$ with (e_j) a basis of the space $\mathbb{R}^n$ we obtain our claim.
On the other hand, as $(\phi_n(.)) \subset \Phi(x)$, by (3.1), there is a subsequence also denoted $(\phi_n(.))$ which converges $\sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})$ to a mapping $\phi(.) \in \Phi(x)$ since $\Phi(x)$ is closed. Consequently, $(w_n(.) + \phi_n(.))_n \sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})-$ converges to $(w(.) + \phi(.)) \in \mathbf{L}^1_{\mathbb{R}^n}([0, 1])$. By Banach Mazur's Lemma (Theorem(2.4)), there exists a sequence $(z_n(.))_n$ which converges strongly in $\mathbf{L}^1_{\mathbb{R}^n}([0, 1])$ to $w(.) + \phi(.)$ with for each $n \in \mathbb{N}$,

$$z_n(.) \in co\{w_m(.) + \phi_m(.),\ m \geq n\}.$$

Extracting a subsequence, we may suppose that $(z_n(t))_n$ converges almost every where to $w(t) + \phi(t)$. Then,

$$w(t) + \phi(t) \in \bigcap_n \overline{co}\{w_m(t) + \phi_m(t) :\ m \geq n\} \ \ a.e.t \in [0, 1].$$

Fix such $t \in [0, 1]$ and any $z \in \mathbb{R}^n$. The relation (3.3) and Theorem(2.1) give

$$\langle z, w(t) + \phi(t)\rangle \leq \delta^*\Big(z, -\partial_c \varphi(x(t))\Big),$$

and since $\partial_c \varphi(x(t))$ is a closed convex set, we have by Lemma(2.2),

$$d\Big(w(t) + \phi(t), -\partial_c \varphi(x(t))\Big) = \sup_{z' \in \overline{\mathbf{B}}_{\mathbb{R}^n}} \Big[\langle z', w(t) + \phi(t)\rangle - \delta^*\Big(z', -\partial_c \varphi(x(t))\Big)\Big] \leq 0,$$

i.e.,

$$w(t) + \phi(t) \in -\partial_c \varphi(x(t)) \ \ a.e.t \in [0, 1].$$

This shows that $\Gamma(x)$ is a compact subset of $\mathcal{K}$.

Finally we will show that Γ is upper semi-continuous or equivalently that the graph of Γ

$$\mathbf{gph}(\Gamma) = \{(x, y) \in \mathcal{K} \times \mathcal{K} : y \in \Gamma(x)\}$$

is closed for $\mathcal{K}$ equipped with the topology of uniform convergence.

Let $(x_n(.), y_n(.))_n$ be a sequence in $\mathbf{gph}(\Gamma)$ converging to $(x(.), y(.)) \in \mathcal{K} \times \mathcal{K}$. i.e., for all $n \in \mathbb{N}$, there exists $\phi_n(.) \in \Phi(x_n(.))$ and $w_n(.) \in -\partial_c\varphi(x_n(.)) - \phi_n(.)$, such that

$$y_n(t) = \int_0^1 G(t, s)w_n(s)ds. \tag{3.4}$$

By (3.2), $(w_n(.))_n$ is included in $\mathcal{D}$, and hence we can extract a subsequence that we do not relabel and which converges $\sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})$ to some mapping $w(.) \in \mathcal{D}$, i.e., $\|w(t)\| \le m(t)$ for almost every t$\in [0, 1]$.

Furthermore, since $(\phi_n(.))_n \subset \Phi(x_n(.)) \subset (\rho(.) + \frac{1}{2})\overline{\mathbf{B}}_{\mathbf{L}^\infty_{\mathbb{R}^n}}$, we can extract a subsequence $\sigma(\mathbf{L}^1_{\mathbb{R}^n}, \mathbf{L}^\infty_{\mathbb{R}^n})$-converging to some mapping $\phi(.) \in \Phi(x(.))$ since $\mathbf{gph}(\Phi)$ is strongly-weakly sequentially closed. As $w_n(t) + \phi_n(t) \in -\partial_c\varphi(x_n(t))$, and as the convex compact valued multimapping $-\partial_c\varphi(.)$ is upper semicontinuous on $\mathbb{R}^n$ (see Lemma(2.8)), by applying Theorem VI.4 in [8], we obtain

$$w(t) + \phi(t) \in -\partial_c\varphi(x(t)),\ a.e.t \in [0, 1].$$

Furthermore, for every $t \in [0, 1]$

$$\lim_{n\to\infty} \int_0^1 G(t, s)w_n(s)ds = \int_0^1 G(t, s)w(s)ds,$$

and hence, according to (3.4) $y(t) = \int_0^t G(t, s)w(s)ds$. Consequently, $(x(.), y(.)) \in \mathbf{gph}(\Gamma)$, that is, the graph of Γ is closed and hence Γ is upper semicontinuous because $\mathcal{K}$ is compact for the topology of uniform convergence. An application of the Kakutani fixed point Theorem to the multimaping Γ gives some mapping $x(.) \in \mathcal{K}$ such that $x(.) \in \Gamma(x(.))$, i.e. $x(t) = \int_0^1 G(t, s)w(s)ds$ for all $t \in [0, 1]$, with for almost every $t \in [0, 1]$ $w(t) \in -\partial_c\varphi(x(t)) - \phi(t)$, and $\phi(.) \in \Phi(x(.))$. As $\phi(t) \in F(t, x(t), \dot{x}(t))$, a.e. $t \in [0, 1]$, we get

$$\ddot{x}(t) \in -\partial_c\varphi(x(t)) + F(t, x(t), \dot{x}(t)),\ a.e.\ t \in [0, 1],$$

that is, $x(.)$ is a solution in $\mathbf{W}^{2,1}_{\mathbb{R}^n}([0, 1])$ of our problem $(\mathcal{P}_F)$.

The proof is then complete. □

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