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Fixed Point Theorems for g-Monotone Maps on Partially Ordered S-Metric Spaces

Nguyen Van Dung[a], Nguyen Trung Hieu[a], Slobodan Radojević[b]

[a]*Faculty of Mathematics and Information Technology Teacher Education, Dong Thap University, Cao Lanh City, Dong Thap Province, Viet Nam*
[b]*Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia*

Abstract. In this paper, we prove some fixed point theorems for g-monotone maps on partially ordered S-metric spaces. Our results generalize fixed point theorems in [1] and [7] for maps on metric spaces to the structure of S-metric spaces. Also, we give examples to demonstrate the validity of the results.

1. Introduction and Preliminaries

The fixed point theory in generalized metric spaced were investigated by many authors. In 2012, Sedghi *et al.* [23] introduced the notion of an S-metric space and proved that this notion is a generalization of a metric space. Also, they proved some properties of S-metric spaces and stated some fixed point theorems on such spaces. An interesting work naturally rises is to transport certain results in metric spaces and known generalized metric spaces to S-metric spaces. After that, Sedghi and Dung [22] proved a general fixed point theorem in S-metric spaces which is a generalization of [23, Theorem 3.1] and obtained many analogues of fixed point theorems in metric spaces for S-metric spaces. In 2013, Dung [8] used the notion of a mixed weakly monotone pair of maps to state a coupled common fixed point theorem for maps on partially ordered S-metric spaces and generalized the main results of [6], [10], [15] into the structure of S-metric spaces. In recent times, Hieu *et al.* [11] proved a fixed point theorem for a class of maps depending on another map on S-metric spaces and obtained the fixed point theorems in [16] and [23]. Very recent, An *et al.* [4] showed some relations between S-metric spaces and metric-type space in the sense of Khamsi [17].

In 2008, Ćirić *et al.* [7] introduced the concept of a g-monotone map and proved some common fixed point theorems for g-monotone generalized nonlinear contractions in partially ordered complete metric spaces. These results give rise to stating analogous fixed point theorems for maps on partially ordered S-metric spaces.

In this paper, we prove some fixed point theorems for g-monotone maps on partially ordered S-metric spaces and generalize fixed point theorems in [1] and [7] on metric spaces to the structure of S-metric spaces. Also, we give examples to demonstrate the validity of the results.

First, we recall some notions and lemmas which will be useful in what follows.

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Email addresses: nvdung@dthu.edu.vn (Nguyen Van Dung), ngtrunghieu@dthu.edu.vn (Nguyen Trung Hieu), sradojevic@mas.bg.ac.rs (Slobodan Radojević)

Definition 1.1 ([23], Definition 2.1). *Let X be a non-empty set and $S : X \times X \times X \longrightarrow [0, \infty)$ be a function such that for all $x, y, z, a \in X$,*

1. $S(x, y, z) = 0$ *if and only if* $x = y = z$.
2. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

*Then S is called an S-metric on X and (X, S) is called an S-*metric space.

The following is the intuitive geometric example for S-metric spaces.

Example 1.2 ([23], Example 2.4). *Let $X = \mathbb{R}^2$ and d be the ordinary metric on X. Put*

$$S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$$

for all $x, y, z \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.

Lemma 1.3 ([23], Lemma 2.5). *Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Lemma 1.4 ([8], Lemma 1.6). *Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z) \text{ and } S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$$

for all $x, y, z \in X$.

Proof. It is a direct consequence of Definition 1.1 and Lemma 1.3. □

Definition 1.5 ([23]). *Let (X, S) be an S-metric space.*

1. *A sequence $\{x_n\}$ is called* convergent *to x in (X, S), written $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} S(x_n, x_n, x) = 0$.*
2. *A sequence $\{x_n\}$ is called* Cauchy *in (X, S) if $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$.*
3. *(X, S) is called* complete *if every Cauchy sequence in (X, S) is a convergent sequence in (X, S).*

From [23, Examples in page 260], we have the following.

Example 1.6.
1. *Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$, is an S-metric on $\mathbb{R}$. This S-metric is called the* usual S-metric *on $\mathbb{R}$. Furthermore, the usual S-metric space $\mathbb{R}$ is complete.*
2. *Let Y be a non-empty set of $\mathbb{R}$. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in Y$, is an S-metric on Y. If Y is a closed subset of the usual metric space $\mathbb{R}$, then the S-metric space Y is complete.*

Lemma 1.7 ([23], Lemma 2.12). *Let (X, S) be an S-metric space. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.*

The following lemma shows that every metric space is an S-metric space.

Lemma 1.8 ([8], Lemma 1.10). *Let (X, d) be a metric space. Then we have*

1. *$S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$\lim_{n\to\infty} x_n = x$ in (X, d) if and only if $\lim_{n\to\infty} x_n = x$ in (X, S_d).*
3. *$\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, S_d).*
4. *(X, d) is complete if and only if (X, S_d) is complete.*

The following example proves that the inversion of Lemma 1.8 does not hold.

Example 1.9 ([8], Example 1.10). *Let $X = \mathbb{R}$ and let $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. By [23, Example (1), page 260], (X, S) is an S-metric space. We prove that there does not exist any metric d such that $S(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$. Indeed, suppose to the contrary that there exists a metric d with $S(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then $d(x, z) = \frac{1}{2}S(x, x, z) = |x - z|$ and $d(x, y) = S(x, y, y) = 2|x - y|$ for all $x, y, z \in X$. It is a contradiction.*

Definition 1.10 ([7], Definition 2.1). *Let $(X, \preceq)$ be a partially ordered set and let $F, g : X \longrightarrow X$ be two maps.*

1. *F is called* g-non-decreasing *if $gx \preceq gy$ implies $Fx \preceq Fy$ for all $x, y \in X$.*
2. *F is called* g-non-increasing *if $gx \preceq gy$ implies $Fy \preceq Fx$ for all $x, y \in X$.*

Definition 1.11. *Let X be a non-empty set and let $f, g : X \longrightarrow X$ be two maps.*

1. *f and g are called to* commute *at $x \in X$ if $f(gx) = g(fx)$.*
2. *f and g are called to* commute *[14] if $f(gx) = g(fx)$ for all $x \in X$.*

In 2006, Mustafa and Sims [18] introduced the notion of a G-metric. Then, fixed point theory in G-metric spaces were investigated by many authors [2], [5], [9], [19], [20].

Definition 1.12 ([18], Definition 3). *Let X be a non-empty set and $G : X \times X \times X \longrightarrow [0, \infty)$ be a function such that for all $x, y, z, a \in X$,*

(G1) *$G(x, y, z) = 0$ if $x = y = z$.*
(G2) *$0 < G(x, x, y)$ if $x \neq y$.*
(G3) *$G(x, x, y) \leq G(x, y, z)$ if $y \neq z$.*
(G4) *The symmetry on three variables*

$$G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x).$$

(G5) *The rectangle inequality $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$.*

Then G is called a G-metric *on X and the pair (X, G) is called a* G-metric space.

2. Main Results

In 2012, Sedghi *et al.* [23] asserted that an S-metric is a generalization of a G-metric, that is, every G-metric is an S-metric, see [23, Remarks 1.3] and [23, Remarks 2.2]. The following Example 2.1 and Example 2.2 show that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct.

Example 2.1. *There exists a G-metric which is not an S-metric.*

Proof. Let X be the G-metric space in [18, Example 1]. Then we have

$$2 = G(a, b, b) > 1 = G(a, a, b) + G(b, b, b) + G(b, b, b).$$

This proves that G is not an S-metric on X. □

Example 2.2. *There exists an S-metric which is not a G-metric.*

Proof. Let (X, S) be the S-metric space in Example 1.9. We have

$$S(1, 0, 2) = |0 + 2 - 2| + |0 - 2| = 2$$

$$S(2, 0, 1) = |0 + 1 - 4| + |0 - 1| = 4.$$

Then $S(1, 0, 2) \neq S(2, 0, 1)$. This proves that S is not a G-metric. □

Also in 2012, Jeli and Samet [12] showed that a G-metric is not a real generalization of a metric. Further, they proved that the fixed point theorems proved in G-metric spaces can be obtained by usual metric arguments. The similar approach may be found in [3]. The key of that approach is the following lemma.

Lemma 2.3 ([12]). *Let (X, G) be a G-metric space. Then we have*

1. $d(x, y) = \max\left\{G(x, y, y), G(y, x, x)\right\}$ *for all $x, y \in X$ is a metric on X.*
2. $d(x, y) = G(x, y, y)$ *for all $x, y \in X$ is a quasi-metric on X.*

The following example shows that Lemma 2.3 does not hold if the G-metric is replaced by an S-metric space. Then, in general, arguments in [3], [12] are not applicable to S-metric spaces.

Example 2.4. 1. *There exists an S-metric space (X, S) such that*

$$d(x, y) = \max\left\{S(x, y, y), S(y, x, x)\right\}$$

for all $x, y \in X$ is not a metric on X.

2. *There exists an S-metric space (X, S) such that $d(x, y) = S(x, y, y)$ for all $x, y \in X$ is not a quasi-metric on X.*

Proof. (1). Let $X = \{1, 2, 3\}$ and let S be defined as follows.

$$\begin{aligned}
&S(1,1,1) = S(2,2,2) = S(3,3,3) = 0,\\
&S(1,2,3) = S(1,3,2) = S(2,1,3) = S(3,1,2) = 4,\\
&S(2,3,1) = S(3,2,1) = S(1,1,2) = S(1,1,3) = S(2,2,1) = S(3,3,1) = 2,\\
&S(2,2,3) = S(3,3,2) = 6,\\
&S(2,3,2) = S(3,2,2) = S(3,2,3) = S(2,3,3) = 3,\\
&S(1,2,1) = S(2,1,1) = S(1,3,1) = S(3,1,1) = S(2,1,2) = S(1,2,2) = S(3,1,3) = S(1,3,3) = 1.
\end{aligned}$$

We have $S(x, y, z) \geq 0$ for all $x, y, z \in X$ and $S(x, y, z) = 0$ if and only if $x = y = z$. By simple calculations as in Table 1, we see that the inequality

$$S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$$

holds for all $x, y, z, a \in X$. Then S is an S-metric on X.

$S(x,y,z)$	a	$S(x,x,a) + S(y,y,a) + S(z,z,a)$
$S(1,2,3) = 4$	1	$S(1,1,1) + S(2,2,1) + S(3,3,1) = 0 + 2 + 2 = 4$
	2	$S(1,1,2) + S(2,2,2) + S(3,3,2) = 2 + 0 + 6 = 8$
	3	$S(1,1,3) + S(2,2,3) + S(3,3,3) = 2 + 6 + 0 = 8$
$S(1,3,2) = 4$	1	$S(1,1,1) + S(3,3,1) + S(2,2,1) = 0 + 2 + 2 = 4$
	2	$S(1,1,2) + S(3,3,2) + S(2,2,2) = 2 + 6 + 0 = 8$
	3	$S(1,1,3) + S(3,3,3) + S(2,2,3) = 2 + 0 + 6 = 8$
$S(2,1,3) = 4$	1	$S(2,2,1) + S(1,1,1) + S(3,3,1) = 2 + 0 + 2 = 4$
	2	$S(2,2,2) + S(1,1,2) + S(3,3,2) = 0 + 2 + 6 = 8$
	3	$S(2,2,3) + S(1,1,3) + S(3,3,3) = 6 + 2 + 0 = 8$
$S(2,3,1) = 2$	1	$S(2,2,1) + S(3,3,1) + S(1,1,1) = 2 + 2 + 0 = 4$
	2	$S(2,2,2) + S(3,3,2) + S(1,1,2) = 0 + 6 + 2 = 8$
	3	$S(2,2,3) + S(3,3,3) + S(1,1,3) = 6 + 0 + 2 = 8$
$S(3,1,2) = 4$	1	$S(3,3,1) + S(1,1,1) + S(2,2,1) = 2 + 0 + 2 = 4$
	2	$S(3,3,2) + S(1,1,2) + S(2,2,2) = 6 + 2 + 0 = 8$
	3	$S(3,3,3) + S(1,1,3) + S(2,2,3) = 0 + 2 + 6 = 8$

$S(3,2,1) = 2$	1	$S(3,3,1) + S(2,2,1) + S(1,1,1) = 2 + 2 + 0 = 4$
	2	$S(3,3,2) + S(2,2,2) + S(1,1,2) = 6 + 0 + 2 = 8$
	3	$S(3,3,3) + S(2,2,3) + S(1,1,3) = 0 + 6 + 2 = 8$
$S(1,1,2) = 2$	1	$S(1,1,1) + S(1,1,1) + S(2,2,1) = 0 + 0 + 2 = 2$
	2	$S(1,1,2) + S(1,1,2) + S(2,2,2) = 2 + 2 + 0 = 4$
	3	$S(1,1,3) + S(1,1,3) + S(2,2,3) = 2 + 2 + 6 = 10$
$S(1,2,1) = 1$	1	$S(1,1,1) + S(2,2,1) + S(1,1,1) = 0 + 2 + 0 = 2$
	2	$S(1,1,2) + S(2,2,2) + S(1,1,2) = 2 + 0 + 2 = 4$
	3	$S(1,1,3) + S(2,2,3) + S(1,1,3) = 2 + 6 + 2 = 10$
$S(2,1,1) = 1$	1	$S(2,2,1) + S(1,1,1) + S(1,1,1) = 2 + 0 + 0 = 2$
	2	$S(2,2,2) + S(1,1,2) + S(1,1,2) = 0 + 2 + 2 = 4$
	3	$S(2,2,3) + S(1,1,3) + S(1,1,3) = 6 + 2 + 2 = 10$
$S(1,1,3) = 2$	1	$S(1,1,1) + S(1,1,1) + S(3,3,1) = 0 + 0 + 2 = 2$
	2	$S(1,1,2) + S(1,1,2) + S(3,3,2) = 2 + 2 + 6 = 10$
	3	$S(1,1,3) + S(1,1,3) + S(3,3,3) = 2 + 2 + 0 = 4$
$S(1,3,1) = 1$	1	$S(1,1,1) + S(3,3,1) + S(1,1,1) = 0 + 2 + 0 = 2$
	2	$S(1,1,2) + S(3,3,2) + S(1,1,2) = 2 + 6 + 2 = 10$
	3	$S(1,1,3) + S(3,3,3) + S(1,1,3) = 2 + 0 + 2 = 4$
$S(3,1,1) = 1$	1	$S(3,3,1) + S(1,1,1) + S(1,1,1) = 2 + 0 + 0 = 2$
	2	$S(3,3,2) + S(1,1,2) + S(1,1,2) = 6 + 2 + 2 = 10$
	3	$S(3,3,3) + S(1,1,3) + S(1,1,3) = 0 + 2 + 2 = 4$
$S(2,2,1) = 2$	1	$S(2,2,1) + S(2,2,1) + S(1,1,1) = 2 + 2 + 0 = 4$
	2	$S(2,2,2) + S(2,2,2) + S(1,1,2) = 0 + 0 + 2 = 4$
	3	$S(2,2,3) + S(2,2,3) + S(1,1,3) = 6 + 6 + 2 = 14$
$S(2,1,2) = 1$	1	$S(2,2,1) + S(1,1,1) + S(2,2,1) = 2 + 0 + 2 = 4$
	2	$S(2,2,2) + S(1,1,2) + S(2,2,2) = 0 + 2 + 0 = 2$
	3	$S(2,2,3) + S(1,1,3) + S(2,2,3) = 6 + 2 + 6 = 14$
$S(1,2,2) = 1$	1	$S(1,1,1) + S(2,2,1) + S(2,2,1) = 0 + 2 + 2 = 4$
	2	$S(1,1,2) + S(2,2,2) + S(2,2,2) = 2 + 0 + 0 = 2$
	3	$S(1,1,3) + S(2,2,3) + S(2,2,3) = 2 + 6 + 6 = 14$
$S(1,3,3) = 1$	1	$S(1,1,1) + S(3,3,1) + S(3,3,1) = 0 + 2 + 2 = 4$
	2	$S(1,1,2) + S(3,3,2) + S(3,3,2) = 2 + 6 + 6 = 14$
	3	$S(1,1,3) + S(3,3,3) + S(3,3,3) = 2 + 0 + 0 = 2$
$S(3,1,3) = 1$	1	$S(3,3,1) + S(1,1,1) + S(3,3,1) = 2 + 0 + 2 = 4$
	2	$S(3,3,2) + S(1,1,2) + S(3,3,2) = 6 + 2 + 6 = 14$
	3	$S(3,3,3) + S(1,1,3) + S(3,3,3) = 0 + 2 + 0 = 2$
$S(3,3,1) = 2$	1	$S(3,3,1) + S(3,3,1) + S(1,1,1) = 2 + 2 + 0 = 4$
	2	$S(3,3,2) + S(3,3,2) + S(1,1,2) = 6 + 6 + 2 = 14$
	3	$S(3,3,3) + S(3,3,3) + S(1,1,3) = 0 + 0 + 2 = 2$
$S(2,2,3) = 6$	1	$S(2,2,1) + S(2,2,1) + S(3,3,1) = 2 + 2 + 2 = 6$
	2	$S(2,2,2) + S(2,2,2) + S(3,3,2) = 0 + 0 + 6 = 6$
	3	$S(2,2,3) + S(2,2,3) + S(3,3,3) = 6 + 6 + 0 = 12$
$S(2,3,2) = 3$	1	$S(2,2,1) + S(3,3,1) + S(2,2,1) = 2 + 2 + 2 = 6$
	2	$S(2,2,2) + S(3,3,2) + S(2,2,2) = 0 + 6 + 0 = 6$
	3	$S(2,2,3) + S(3,3,3) + S(2,2,3) = 6 + 0 + 6 = 12$
$S(3,2,2) = 3$	1	$S(3,3,1) + S(2,2,1) + S(2,2,1) = 2 + 2 + 2 = 6$
	2	$S(3,3,2) + S(2,2,2) + S(2,2,2) = 6 + 0 + 0 = 6$
	3	$S(3,3,3) + S(2,2,3) + S(2,2,3) = 0 + 6 + 6 = 12$
$S(3,3,2) = 6$	1	$S(3,3,1) + S(3,3,1) + S(2,2,1) = 2 + 2 + 2 = 6$
	2	$S(3,3,2) + S(3,3,2) + S(2,2,2) = 6 + 6 + 0 = 12$

	3	$S(3,3,3)+S(3,3,3)+S(2,2,3)=0+0+6=6$
$S(3,2,3)=3$	1	$S(3,3,1)+S(2,2,1)+S(3,3,1)=2+2+2=6$
	2	$S(3,3,2)+S(2,2,2)+S(3,3,2)=6+0+6=12$
	3	$S(3,3,3)+S(2,2,3)+S(3,3,3)=0+6+0=6$
$S(2,3,3)=3$	1	$S(2,2,1)+S(3,3,1)+S(3,3,1)=2+2+2=6$
	2	$S(2,2,2)+S(3,3,2)+S(3,3,2)=0+6+6=12$
	3	$S(2,2,3)+S(3,3,3)+S(3,3,3)=6+0+0=6$
$S(1,1,1)=0$	1	$S(1,1,1)+S(1,1,1)+S(1,1,1)=0+0+0=0$
	2	$S(1,1,2)+S(1,1,2)+S(1,1,2)=2+2+2=6$
	3	$S(1,1,3)+S(1,1,3)+S(1,1,3)=2+2+2=6$
$S(2,2,2)=0$	1	$S(2,2,1)+S(2,2,1)+S(2,2,1)=2+2+2=6$
	2	$S(2,2,2)+S(2,2,2)+S(2,2,2)=0+0+0=0$
	3	$S(2,2,3)+S(2,2,3)+S(2,2,3)=6+6+6=18$
$S(3,3,3)=0$	1	$S(3,3,1)+S(3,3,1)+S(3,3,1)=2+2+2=6$
	2	$S(3,3,2)+S(3,3,2)+S(3,3,2)=6+6+6=18$
	3	$S(3,3,3)+S(3,3,3)+S(3,3,3)=0+0+0=0$

Table 1: Calculations on S

On the other hand, if $d(x,y)=\max\{S(x,y,y),S(y,x,x)\}$ for all $x,y\in X$, then we have

$$d(1,1)=d(2,2)=d(3,3)=0,$$
$$d(1,2)=d(2,1)=d(1,3)=d(3,1)=1,$$
$$d(2,3)=d(3,2)=3.$$

It implies that $3=d(2,3)\geq d(2,1)+d(1,3)=1+1=2$. Then d is not a metric on X.

(2). We consider the S-metric as in (1). If $d(x,y)=S(x,y,y)$ for all $x,y\in X$, then we have

$$d(1,1)=d(2,2)=d(3,3)=0,$$
$$d(1,2)=d(2,1)=d(1,3)=d(3,1)=1,$$
$$d(2,3)=d(3,2)=3.$$

It implies that $3=d(2,3)\geq d(2,1)+d(1,3)=1+1=2$. Then d is not a quasi-metric on X. □

Now, we investigate the fixed point problem on S-metric spaces. The following result states the existence of a common fixed point of two maps F and g on partially ordered S-metric spaces. For the existence of a common fixed point of two maps F and g on partially ordered metric spaces, see [1, Theorem 2.2], [1, Theorem 2.3] and [7, Theorem 2.2].

Theorem 2.5. *Let $(X,\preceq,S)$ be a partially ordered S-metric space, $F,g:X\longrightarrow X$ be two maps and $\varphi:[0,\infty)\longrightarrow[0,\infty)$ be a function such that*

1. *X is complete.*
2. *φ is continuous and $\varphi(t)<t$ for all $t>0$.*
3. *$F(X)\subset g(X)$, F is a g-non-decreasing map, $g(X)$ is closed and $gx_0\preceq Fx_0$ for some $x_0\in X$.*
4. *For all $x,y\in X$ with $gx\preceq gy$,*
$$S(Fx,Fx,Fy)\leq\max\left\{\varphi(S(gx,gx,gy),\varphi(S(gx,gx,Fx)),\varphi(S(gy,gy,Fy)),\varphi\left(\frac{S(gx,gx,Fy)+S(gy,gy,Fx)}{3}\right)\right\}.$$
5. *If $\{gx_n\}$ is a non-decreasing sequence with $\lim_{n\to\infty}gx_n=gz$ in $g(X)$, then $gx_n\preceq gz\preceq g(gz)$ for all $n\in\mathbb{N}$.*

Then F and g have a coincidence point. Furthermore, if F and g commute at the coincidence point, then F and g have a common fixed point.

Proof. Since $F(X) \subset g(X)$, we can choose $x_1 \in X$ such that $gx_1 = Fx_0$. Again, from $F(X) \subset g(X)$ we can choose $x_2 \in X$ such that $gx_2 = Fx_1$. Continuing this process, we can choose a sequence $\{x_n\}$ in X such that

$$gx_{n+1} = Fx_n \tag{1}$$

for all $n \in \mathbb{N}$. Since $gx_0 \preceq Fx_0$ and $Fx_0 = gx_1$, we have $gx_0 \preceq gx_1$. Since F is g-non-decreasing, we get $Fx_0 \preceq Fx_1$. By using (1), we have $gx_1 \preceq gx_2$. Again, since F is g-non-decreasing, we get $Fx_1 \preceq Fx_2$, that is, $gx_2 \preceq gx_3$. Continuing this process, we obtain

$$Fx_n \preceq Fx_{n+1}, gx_n \preceq gx_{n+1} \tag{2}$$

for all $n \in \mathbb{N}$. To prove that F and g have a coincidence point, we consider two following cases.

Case 1. There exists n_0 such that $S(Fx_{n_0}, Fx_{n_0}, Fx_{n_0+1}) = 0$. It implies that $Fx_{n_0+1} = Fx_{n_0}$. By (1), we get

$$Fx_{n_0+1} = gx_{n_0+1}. \tag{3}$$

Therefore, x_{n_0+1} is a coincidence point of F and g.

Case 2. $S(Fx_n, Fx_n, Fx_{n+1}) > 0$ for all $n \in \mathbb{N}$. We will show that

$$S(Fx_n, Fx_n, Fx_{n+1}) < S(Fx_{n-1}, Fx_{n-1}, Fx_n) \tag{4}$$

for all $n \in \mathbb{N}$. It follows from the assumption (4) and (2) that

$$\begin{aligned} S(Fx_n, Fx_n, Fx_{n+1}) \quad \leq \quad & \max\{\varphi(S(gx_n, gx_n, gx_{n+1})), \varphi(S(gx_n, gx_n, Fx_n)), \varphi(S(gx_{n+1}, gx_{n+1}, Fx_{n+1})), \\ & \varphi\left(\frac{S(gx_n, gx_n, Fx_{n+1}) + S(gx_{n+1}, gx_{n+1}, Fx_n)}{3}\right)\}. \end{aligned}$$

Thus by (1), we get

$$\begin{aligned} S(Fx_n, Fx_n, Fx_{n+1}) \quad \leq \quad & \max\Big\{\varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \\ & \varphi(S(Fx_n, Fx_n, Fx_{n+1})), \varphi\left(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1}) + S(Fx_n, Fx_n, Fx_n)}{3}\right)\Big\} \\ = \quad & \max\left\{\varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \varphi(S(Fx_n, Fx_n, Fx_{n+1})), \varphi\left(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3}\right)\right\}. \end{aligned} \tag{5}$$

We consider three following subcases.

Subcase 2.1.

$$\max\left\{\varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \varphi(S(Fx_n, Fx_n, Fx_{n+1})), \varphi\left(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3}\right)\right\} = \varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)).$$

By (5), we have $S(Fx_n, Fx_n, Fx_{n+1}) \leq \varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n))$. Therefore, (4) holds since $\varphi(t) < t$ for $t > 0$.

Subcase 2.2.

$$\max\left\{\varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \varphi(S(Fx_n, Fx_n, Fx_{n+1})), \varphi\left(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3}\right)\right\} = \varphi(S(Fx_n, Fx_n, Fx_{n+1})).$$

By (5), we have $S(Fx_n, Fx_n, Fx_{n+1}) \leq \varphi(S(Fx_n, Fx_n, Fx_{n+1}))$. Since $\varphi(t) < t$ for $t > 0$, we get $S(Fx_n, Fx_n, Fx_{n+1}) = 0$. It is a contradiction.

Subcase 2.3.

$$\max\left\{\varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \varphi(S(Fx_n, Fx_n, Fx_{n+1})), \varphi\left(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3}\right)\right\} = \varphi\left(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3}\right).$$

Note that $\varphi(0) = \lim\limits_{n\to\infty} \varphi(1/n) \leq \lim\limits_{n\to\infty} 1/n = 0$, then $\varphi(0) = 0$.

If $\dfrac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3} = 0$, then by (5), we have $S(Fx_n, Fx_n, Fx_{n+1}) = 0$. It is a contradiction.

If $\dfrac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3} > 0$, it follows from (5), Lemma 1.4 and the fact $\varphi(t) < t$ for $t > 0$ that

$$\begin{aligned} S(Fx_n, Fx_n, Fx_{n+1}) &\leq \varphi\left(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3}\right) \\ &< \frac{1}{3}S(F(x_{n-1}, Fx_{n-1}, Fx_{n+1})) \\ &\leq \frac{1}{3}(2S(Fx_{n-1}, Fx_{n-1}, Fx_n) + S(Fx_n, Fx_n, Fx_{n+1})) \end{aligned}$$

Then we have $S(Fx_n, Fx_n, Fx_{n+1}) < S(Fx_{n-1}, Fx_{n-1}, Fx_n)$. By the conclusions of three above subcases that (4) holds.

It follows from (4) that the sequence $\{S(Fx_n, Fx_n, Fx_{n+1})\}$ of real numbers is monotone decreasing. Then there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} S(Fx_n, Fx_n, Fx_{n+1}) = \delta. \tag{6}$$

Now we shall prove that $\delta = 0$. It follows from Lemma 1.4 and (4) that

$$\begin{aligned} \frac{1}{3}S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1}) &\leq \frac{1}{3}\Big(2S(Fx_{n-1}, Fx_{n-1}, Fx_n) + S(Fx_n, Fx_n, Fx_{n+1})\Big) \\ &< \frac{1}{3}\Big(2S(Fx_{n-1}, Fx_{n-1}, Fx_n) + S(Fx_{n-1}, Fx_{n-1}, Fx_n)\Big) \\ &= S(Fx_{n-1}, Fx_{n-1}, Fx_n). \end{aligned} \tag{7}$$

Taking the upper limit as $n \to \infty$ in (7), we get

$$\limsup_{n\to\infty} \frac{1}{3}S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1}) \leq \limsup_{n\to\infty} S(Fx_{n-1}, Fx_{n-1}, Fx_n).$$

Put

$$b = \limsup_{n\to\infty} \frac{1}{3}S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1}) \tag{8}$$

then $0 \leq b \leq \delta$. Now taking the upper limit as $n \to \infty$ in (5) and note that $\varphi(t)$ is continuous, we get

$$\begin{aligned} \lim_{n\to\infty} S(Fx_n, Fx_n, Fx_{n+1}) \leq \max\Big\{&\varphi(\lim_{n\to\infty} S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \varphi(\lim_{n\to\infty} S(Fx_n, Fx_n, Fx_{n+1})), \\ &\varphi\left(\limsup_{n\to\infty} \frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3}\right)\Big\}. \end{aligned} \tag{9}$$

Using (6), (8) and (9), we have $\delta \leq \max\{\varphi(\delta), \varphi(b)\}$. If $\delta > 0$, then

$$\delta \leq \max\{\varphi(\delta), \varphi(b)\} < \max\{\delta, b\} = \delta. \tag{10}$$

It is a contradiction. Therefore, $\delta = 0$. It follows from (6) that

$$\lim_{n\to\infty} S(Fx_n, Fx_n, Fx_{n+1}) = 0. \tag{11}$$

Now we shall prove that $\{Fx_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{Fx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and two sequences of integers $\{n_k\}$ and $\{m_k\}$ with $m_k > n_k > k$ and

$$r_k = S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k}) \geq \varepsilon \tag{12}$$

for all $k \in \mathbb{N}$. We can choose m_k that is the smallest number with $m_k > n_k > k$ and (12) holds. Then

$$S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k-1}) < \varepsilon. \tag{13}$$

From Lemma 1.4, Lemma 1.3 and (12), (13), we have

$$\begin{aligned} \varepsilon &\leq r_k \\ &= S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k}) \\ &= S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k}) \\ &\leq 2S(Fx_{m_k}, Fx_{m_k}, Fx_{m_k-1}) + S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k-1}) \\ &< 2S(Fx_{m_k}, Fx_{m_k}, Fx_{m_k-1}) + \varepsilon. \end{aligned} \tag{14}$$

Taking the limit as $k \to \infty$ in (14) and using (11), we obtain

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{15}$$

It follows from (1) and (2) that $gx_{n_k+1} = Fx_{n_k} \preceq Fx_{m_k} = gx_{m_k+1}$. Now, by using the assumptions (4) and (1), we have

$$\begin{aligned} S(Fx_{n_k+1}, Fx_{n_k+1}, Fx_{m_k+1}) \leq \max\Big\{&\varphi(S(gx_{n_k+1}, gx_{n_k+1}, gx_{m_k+1}), \varphi(S(gx_{n_k+1}, gx_{n_k+1}, Fx_{n_k+1})), \\ &\varphi(S(gx_{m_k+1}, gx_{m_k+1}, Fx_{m_k+1})), \\ &\varphi\left(\frac{S(gx_{n_k+1}, gx_{n_k+1}, Fx_{m_k+1}) + S(gx_{m_k+1}, gx_{m_k+1}, Fx_{n_k+1})}{3}\right)\Big\} \\ = \max\Big\{&\varphi(S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k}), \varphi(S(Fx_{n_k}, Fx_{n_k}, Fx_{n_k+1})), \varphi(S(Fx_{m_k}, Fx_{m_k}, Fx_{m_k+1})), \\ &\varphi\left(\frac{S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k+1}) + S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k+1})}{3}\right)\Big\}. \end{aligned} \tag{16}$$

Denoting $\delta_n = S(Fx_n, Fx_n, Fx_{n+1})$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \delta_n = 0$ by (11). From (16), Lemma 1.3 and (12), we have

$$S(Fx_{n_k+1}, Fx_{n_k+1}, Fx_{m_k+1}) \leq \max\left\{\varphi(r_k), \varphi(\delta_{n_k}), \varphi(\delta_{m_k}), \varphi\left(\frac{S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k+1}) + S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k+1})}{3}\right)\right\}. \tag{17}$$

Using Lemma 1.4 again, we get

$$\begin{aligned} r_k &\leq 2S(Fx_{n_k}, Fx_{n_k}, Fx_{n_k+1}) + S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k+1}) \\ &\leq 2S(Fx_{n_k}, Fx_{n_k}, Fx_{n_k+1}) + 2S(Fx_{m_k}, Fx_{m_k}, Fx_{m_k+1}) + S(Fx_{n_k+1}, Fx_{n_k+1}, Fx_{m_k+1}) \\ &= 2\delta_{n_k} + 2\delta_{m_k} + S(Fx_{n_k+1}, Fx_{n_k+1}, Fx_{m_k+1}). \end{aligned} \tag{18}$$

From (12), (17) and (18), we have

$$\begin{aligned} \varepsilon &\leq r_k \\ &\leq 2\delta_{n_k} + 2\delta_{m_k} + \max\left\{\varphi(r_k), \varphi(\delta_{n_k}), \varphi(\delta_{m_k}), \varphi\left(\frac{S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k+1}) + S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k+1})}{3}\right)\right\}. \end{aligned} \tag{19}$$

Next, we will show that

$$\lim_{n\to\infty} \frac{S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k+1}) + S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k+1})}{3} = \frac{2}{3}\varepsilon. \tag{20}$$

Indeed, by using Lemma 1.4, (12) and (13), we obtain

$$\begin{aligned}\varepsilon &\leq r_k\\ &= S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k})\\ &\leq 2\delta_{m_k} + S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k+1})\end{aligned}$$

and

$$\begin{aligned}S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k+1}) &= S(Fx_{m_k+1}, Fx_{m_k+1}, Fx_{n_k})\\ &\leq 2S(Fx_{m_k+1}, Fx_{m_k+1}, Fx_{m_k-1}) + S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k-1})\\ &\leq 4S(Fx_{m_k+1}, Fx_{m_k+1}, Fx_{m_k}) + 2S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{m_k}) + S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k-1})\\ &< 4\delta_{m_k} + 2\delta_{m_k-1} + \varepsilon.\end{aligned}$$

It implies that

$$\varepsilon - 2\delta_{m_k} \leq S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k+1}) < \varepsilon + 4\delta_{m_k} + 2\delta_{m_k-1}. \tag{21}$$

Similarly to (21), we obtain

$$\varepsilon - 2\delta_{n_k} \leq S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k+1}) < \varepsilon + 4\delta_{n_k} + 2\delta_{n_k-1}. \tag{22}$$

It follows from (21) and (22) that

$$\begin{aligned}\frac{2}{3}\Big(\varepsilon - (\delta_{m_k} + \delta_{n_k})\Big) &\leq \frac{S(Fx_{n_k}, Fx_{n_k}, Fx_{m_k+1}) + S(Fx_{m_k}, Fx_{m_k}, Fx_{n_k+1})}{3}\\ &\leq \frac{2}{3}\Big(\varepsilon + 2(\delta_{m_k} + \delta_{n_k}) + \delta_{m_k-1} + \delta_{n_k-1}\Big).\end{aligned} \tag{23}$$

Using (11) and taking the limit as $n \to \infty$ in (23), we get that (20) holds.

Using (11), (15), (20) and taking the limit as $n \to \infty$ in (19) and keeping in mind properties of φ, we get

$$\varepsilon \leq \max\Big\{\varphi(\varepsilon), 0, 0, \varphi(2\varepsilon/3)\Big\} < \max\Big\{\varepsilon, 0, 0, 2\varepsilon/3\Big\} = \varepsilon.$$

It is a contradiction. Therefore, the assumption (12) is not true, that is, $\{Fx_n\}$ is a Cauchy sequence. From (1), we have $\{gx_{n+1}\}$ is also a Cauchy sequence. Since $g(X)$ is closed, there exists $z \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} Fx_n = gz. \tag{24}$$

Now we will show that z is a coincidence point of F and g. By (2), (24) and the assumption (5), we have $gx_n \leq gz$ for all $n \in \mathbb{N}$. By using Lemma 1.4 and the assumption (4), we get

$$\begin{aligned}S(gz, gz, Fz) &\leq 2S(gz, gz, Fx_n) + S(Fx_n, Fx_n, Fz)\\ &\leq 2S(gz, gz, Fx_n) + \max\Big\{\varphi(S(gx_n, gx_n, gz), \varphi(S(gx_n, gx_n, Fx_n)), \varphi(S(gz, gz, Fz)),\\ &\qquad \varphi\Big(\frac{S(gx_n, gx_n, Fz) + S(gz, gz, Fx_n)}{3}\Big)\Big\}.\end{aligned} \tag{25}$$

By using (24), the continuity of φ, Lemma 1.7 and taking the limit as $n \to \infty$ in (25), we have

$$S(gz, gz, Fz) \leq \max\Big\{\varphi(S(gz, gz, Fz)), \varphi(S(gz, gz, Fz)/3)\Big\}.$$

If $S(gz, gz, Fz) > 0$, then by the assumption (2),

$$S(gz, gz, Fz) < \max\Big\{S(gz, gz, Fz), S(gz, gz, Fz)/3\Big\} = S(gz, gz, Fz).$$

It is a contradiction. Then $S(gz, gz, Fz) = 0$, that is, $Fz = gz$. Therefore, F and g have a coincidence point z.

Furthermore, we will show that gz is a common fixed point of F and g if F and g are commutative at the coincidence point. Indeed, we have $F(gz) = g(Fz) = g(gz)$. By (2), (24) and the assumption (5), we obtain $gz \preceq g(gz)$. It follows from the assumption (4) and Lemma 1.3 that

$$\begin{aligned} S(Fz, Fz, F(gz)) &\leq \max\Big\{\varphi(S(gz, gz, g(gz))), \varphi(S(gz, gz, Fz)), \varphi(S(g(gz), g(gz), F(gz))), \\ &\qquad \varphi\Big(\frac{S(gz, gz, F(gz)) + S(g(gz), g(gz), Fz)}{3}\Big)\Big\} \\ &= \max\Big\{\varphi(S(gz, gz, g(gz))), 0, 0, \varphi\Big(\frac{S(gz, gz, g(gz)) + S(g(gz), g(gz), gz)}{3}\Big)\Big\} \\ &= \max\Big\{\varphi(S(gz, gz, g(gz))), 0, 0, \varphi\Big(\frac{2S(gz, gz, g(gz))}{3}\Big)\Big\} \\ &= \max\Big\{\varphi(S(Fz, Fz, g(gz))), \varphi\Big(\frac{2S(Fz, Fz, g(gz))}{3}\Big)\Big\}. \end{aligned} \tag{26}$$

If $S(Fz, Fz, F(gz)) > 0$, then from (26) and the assumption (2), we have

$$S(Fz, Fz, F(gz)) < \max\Big\{S(Fz, Fz, g(gz)), \frac{2S(Fz, Fz, g(gz))}{3}\Big\} = S(Fz, Fz, F(gz)).$$

It is a contradiction. Then $S(Fz, Fz, F(gz)) = 0$, that is, $F(gz) = g(gz) = Fz = gz$. This proves that gz is a common fixed point of F and g. $\square$

Remark 2.6. *The assumption 'F is g-non-decreasing' in Theorem 2.5 can be replaced by 'F is g-non-increasing' provided that '$gx_0 \preceq Fx_0$' is replaced by '$gx_0 \succeq Fx_0$'.*

From Theorem 2.5, we get following corollaries.

Corollary 2.7. *Let $(X, \preceq, S)$ be a partially ordered S-metric space, $F : X \longrightarrow X$ be a map and $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be a function such that*

1. *X is complete.*
2. *φ is continuous and $\varphi(t) < t$ for all $t > 0$.*
3. *F is a non-decreasing map and $x_0 \preceq Fx_0$ for some $x_0 \in X$.*
4. *For all $x, y \in X$ with $x \preceq y$,*
$$S(Fx, Fx, Fy) \leq \max\Big\{\varphi(S(x, x, y), \varphi(S(x, x, Fx)), \varphi(S(y, y, Fy)), \varphi\Big(\frac{S(x, x, Fy) + S(y, y, Fx)}{3}\Big)\Big\}.$$
5. *If $\{x_n\}$ is a non-decreasing sequence with $\lim_{n\to\infty} x_n = z$ in $g(X)$, then $x_n \preceq z$ for all $n \in \mathbb{N}$.*

Then F has a fixed point. Furthermore, the assumption (5) *can be replaced by 'F is continuous'.*

Proof. By taking g is the identity map in Theorem 2.5, we get F has a fixed point z. Furthermore, if F is continuous, then by (24), we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n) = F(\lim_{n\to\infty} x_n) = Fz.$$

This proves that z is a fixed point of F. $\square$

The following corollary is an analogue of [1, Theorem 2.3] for maps on partially ordered S-metric spaces.

Corollary 2.8. *Let $(X, \preceq, S)$ be a partially ordered S-metric space, $F : X \longrightarrow X$ be a map and $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be a function such that*

1. *X is complete.*
2. *φ is continuous and $\varphi(t) < t$ for all $t > 0$.*
3. *F is a non-decreasing map and $x_0 \preceq Fx_0$ for some $x_0 \in X$.*
4. *For all $x, y \in X$ with $x \preceq y$,*

$$S(Fx, Fx, Fy) \leq \max\Big\{\varphi(S(x, x, y), \varphi(S(x, x, Fx)), \varphi(S(y, y, Fy))\Big\}.$$

5. *If $\{x_n\}$ is a non-decreasing sequence with $\lim_{n\to\infty} x_n = z$, then $x_n \preceq z$ for all $n \in \mathbb{N}$.*

Then F has a fixed point. Furthermore, the assumption (5) *can be replaced by 'F is continuous'.*

By choosing $\varphi(t) = k.t$ for all $t \in [0, \infty)$ and some $k \in (0, 1)$ in Corollary 2.7, we get the following corollary which is an analogue of results in [13], [21].

Corollary 2.9. *Let $(X, \preceq, S)$ be a partially ordered S-metric space and $F : X \longrightarrow X$ be a map such that*

1. *X is complete.*
2. *F is a non-decreasing map and $x_0 \preceq Fx_0$ for some $x_0 \in X$.*
3. *For all $x, y \in X$ with $x \preceq y$, there exists $k \in (0, 1)$ satisfying*

$$S(Fx, Fx, Fy) \leq k \max\left\{S(x, x, y), S(x, x, Fx), S(y, y, Fy), \frac{S(x, x, Fy) + S(y, y, Fx)}{3}\right\}.$$

4. *If $\{x_n\}$ is a non-decreasing sequence with $\lim_{n\to\infty} x_n = z$, then $x_n \preceq z$ for all $n \in \mathbb{N}$.*

Then F has a fixed point. Furthermore, the assumption (4) *can be replaced by 'F is continuous'.*

Finally, we give examples to demonstrate the validity of the above results. The following example shows that Corollary 2.9 is a proper generalization of [23, Theorem 3.1].

Example 2.10. *Let $X = \{-3, -1, 0, 2, 4\}$ be a complete S-metric space with the S-metric in Example 1.6 and let $F(-3) = F(-1) = F0 = 0, F2 = -1, F4 = -3$. We have*

$$S(F2, F2, F4) = S(-1, -1, -3) = 2|-1 + 3| = 4 = S(2, 2, 4) = 2|2 - 4|.$$

Then [23, Theorem 3.1] is not applicable to F.

On the other hand, define the partial order on X as follows

$$x \preceq y \text{ if and only if } x, y \in \{-3, -1, 0\} \text{ and } x \leq y.$$

Then F is non-decreasing, $x_0 = 0 \preceq Fx_0 = F0$ and if $\{x_n\}$ is non-decreasing and $\lim_{n\to\infty} x_n = z$, then $x_n \preceq z$. We also have $S(Fx, Fx, Fy) = 0$ for all $x, y \in \{-3, -1, 0\}$. Then, Corollary 2.9 is applicable to F.

The following example shows that Corollary 2.8 is a proper generalization of Corollary 2.9.

Example 2.11. *Let $X = [0, \pi/4]$ with the S-metric defined by $S(x, y, z) = \frac{1}{2}(|x - z| + |y - z|)$ for all $x, y, z \in X$. Define the partial order on X by $x \preceq y$ if and only if $x \geq y$, where $\leq$ is the usual order on $\mathbb{R}$. Then $(X, \preceq, S)$ is a complete, partially ordered S-metric space. For each $x \in X$, put $Fx = \sin x$. For all $x \neq y$ and any $k \in (0, 1)$, we have*

$$S(Fx, Fx, Fy) \quad = \quad S(\sin x, \sin x, \sin y) = |\sin x - \sin y|$$

and

$$
\begin{aligned}
& k\max\left\{S(x,x,y),S(x,x,Fx),S(y,y,Fy),\frac{S(x,x,Fy)+S(y,y,Fx)}{3}\right\}\\
&= k\max\left\{S(x,x,y),S(x,x,\sin x),S(y,y,\sin y),\frac{S(x,x,\sin y)+S(y,y,\sin x)}{3}\right\}\\
&= k\max\left\{|x-y|,x-\sin x,y-\sin y,\frac{|x-\sin y|+|y-\sin x|}{3}\right\}.
\end{aligned}
$$

For $y = 0 \geq x$, we have $S(Fx,Fx,Fy) = \sin x$ and

$$k\max\left\{S(x,x,y),S(x,x,Fx),S(y,y,Fy),\frac{S(x,x,Fy)+S(y,y,Fx)}{3}\right\} = kx.$$

Since $\sin x \leq kx$ is not true for all $x \in X$ and $k \in (0,1)$, Corollary 2.9 is not applicable to F.

On the other hand, put $\varphi(t) = \sin t$ for all $t \in [0,\infty)$, then $\varphi(t) < t$ for all $t > 0$. We have that for all $x \leq y$,

$$S(Fx,Fx,Fy) = \sin x - \sin y \leq \sin(x-y) = \varphi(S(x,x,y)) \leq \max\left\{\varphi(S(x,x,y),\varphi(S(x,x,Fx)),\varphi(S(y,y,Fy))\right\}. \quad (27)$$

Note that $x_0 = 0 \leq F0 = Fx_0$ and if $\{x_n\}$ is non-decreasing and $\lim_{n\to\infty} x_n = z$, then $x_n \leq z$. Moreover, F is also continuous. Therefore, Corollary 2.8 is applicable to F.

The following example shows that our results can not be derived from the techniques used in [12], see Lemma 2.3, even for trivial maps.

Example 2.12. *Let (X,S) be an S-metric space in the proof of Example 2.4 with the usual order and let $F, g : X \longrightarrow X$ be defined by $Fx = gx = 1$ for all $x \in X$. Then all assumptions of Theorem 2.5 are satisfied. Then Theorem 2.5 is applicable to F and g on (X,S).*

It follows from Example 2.4 that the techniques used in [12] are not applicable to F and g on (X,S).

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