Filomat 28:9 (2014), 1871–1880
DOI 10.2298/FIL1409871A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Groups with the Same Set of Orders of Maximal Abelian Subgroups

Neda Ahanjideh[a], Ali Iranmanesh[b]

[a]*Department of Pure Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P.O.Box 115, Shahrekord, Iran*
[b]*Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-137, Tehran, Iran*

Abstract. Let $n > 3$ be an even number. In this paper, we show how the orders of maximal abelian subgroups of the finite group G can influence on the structure of G. More precisely, we show that if for a finite group G, $M(G) = M(B_n(q))$, then $G \cong B_n(q)$. Note that $M(G)$ is the set of orders of maximal abelian subgroups of G. Let $\Gamma(G)$ denote the non-commuting graph of G. As a consequence of our result, we show that if G is a finite group with $\Gamma(G) \cong \Gamma(B_n(q))$, then $G \cong B_n(q)$.

1. Introduction

For an integer $z > 1$, we denote by $\pi(z)$ the set of all prime divisors of z. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The *prime graph (or Gruenberg-Kegel graph)* $GK(G)$ of a group G is the graph with vertex set $\pi(G)$ where two distinct primes p and q are joined by an edge (we write $(p, q) \in GK(G)$) if G contains an element of order pq. Let $s(G)$ be the number of connected components of $GK(G)$. A list of all finite simple groups with disconnected prime graph has been obtained in [11] and [20]. A finite group G is said to be *characterizable by the set of orders of its maximal abelian subgroups*, if G is uniquely determined by the orders of its maximal abelian subgroups. More precisely, a finite group G is called *characterizable by the set of orders of its maximal abelian subgroups*, if each finite group H with $M(G) = M(H)$ is necessarily isomorphic to G. Recall that a simple group S is a K_3-group if $|\pi(S)| = 3$. It is known that if G is any K_3-group, the alternating group A_n (where n and $n-2$ are primes or $n \leq 10$), $PSL_2(2^n)$, $Sz(2^{2m+1})$, $B_n(q)$, where $n = 2^m \geq 4$ and any sporadic simple group, then G is characterizable by the set of orders of its maximal abelian subgroups (see [1, 6, 19]). Let $M(G) = \{|N| : N \textit{ is a maximal abelian subgroup of } G\}$. In this paper, we have proved that:

Theorem 1. *Let $n > 3$ be an even natural number and let q be a prime power. If G is a finite group with $M(G) = M(B_n(q))$, then $G \cong B_n(q)$.*

For every n and q, the simple groups $B_n(q)$ and $C_n(q)$ have the same order. These groups are isomorphic if $n = 2$ or q is even. Also, $s(B_n(q)) \neq 1$ if and only if $n = |n|_2$ or n is prime and $q \in \{2, 3\}$.

2. Preliminaries

In this paper, fix: the subset of vertices of a graph is called an independent set if its vertices are pairwise non-adjacent. For a finite group G, we write $\rho(G)$ ($\rho(r, G)$) for some independent set in $GK(G)$ (containing

2010 *Mathematics Subject Classification*. Primary 20D06; Secondary 20D20
Keywords. Maximal abelian subgroup, simple group of Lie type, prime graph, automorphism groups, maximal independent set.
Received: 08 August 2013; Revised: 14 November 2013; Accepted: 08 December 2013
Communicated by Francesco Belardo
Research supported partially by Shahrekord University
Email addresses: ahanjideh.neda@sci.sku.ac.ir (Neda Ahanjideh), iranmanesh@modares.ac.ir (Ali Iranmanesh)

a prime r) with a maximal number of vertices, which is named the maximal independent set and put $t(G) = |\rho(G)|$ (and $t(r, G) = |\rho(r, G)|$). Throughout this paper, by $[x]$ we denote the integral part of x and by $\gcd(m, n)$ we denote the greatest common divisor of m and n. If m is a natural number and r is prime, the r-part of m is denoted by $|m|_r$, i.e., $|m|_r = r^t$ if $r^t \| m$. The notation for groups of Lie type is according to [10]. Given $\alpha \in \mathbb{S}_k$, we define α_d to be the $dk \times dk$ permutation matrix that permutes blocks of dimension d. For example,

$$(1,2,3)_2 = \begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ I_2 & 0 & 0 \end{pmatrix}.$$

We refer to the block matrices that arise from permutations in this way as standard permutation matrices. Given $X = \langle x_1, ..., x_m \rangle \leq GL_d(q)$ and $W = \langle \alpha_1, ..., \alpha_l \rangle \leq \mathbb{S}_k$, we define the wreath product $X \wr W$ to be the subgroup of $GL_{dk}(q)$ generated the matrices $\mathrm{Diag}(x_1, I_d, \ldots, I_d), \ldots, \mathrm{Diag}(x_m, I_d, \ldots, I_d)$, $\mathrm{Diag}\,(I_d, x_1, I_d, \ldots, I_d)$, $\ldots, \mathrm{Diag}(I_d, \ldots, x_m)$, $\alpha_{1d}, \ldots, \alpha_{ld}$ (see [14]). In the whole paper, we assume that n is an even natural number such that $n \geq 4$, q is a prime power ($q = p^k$) and p is prime. By $GF(q)$, we denote the finite field with q-elements. For a finite group G, we denote the maximum element of $M(G)$ by $a(G)$ (see [18]) and we denote the maximum of the orders of abelian subgroups of s-Sylow subgroup of G by $a_s(G)$, where $s \in \pi(G)$. All further unexplained notations are standard and can be found in [5] and [9].

Lemma 2.1. [18, Table 2] *If S is a finite simple group of Lie type in characteristic p such that $S \neq A_1(p^\alpha)$ (where $p = 2$), $A_2(p^\alpha)$ (where $\gcd(p^\alpha - 1, 3) = 1$), ${}^2A_2(p^\alpha)$ (where $\gcd(p^\alpha + 1, 3) = 1$), ${}^2A_3(2)$ and ${}^2F_4(q)$, then $a(S) = a_p(S)$. In particular,*

- *if $n \geq 4$ and q is odd, then $a(B_n(q)) = q^{n(n-1)/2+1}$;*
- *$a(C_n(q)) = q^{n(n+1)/2}$, except for $C_2(2)$;*
- *for $n \geq 5$, $a({}^2D_n(q)) = q^{(n-1)(n-2)/2+2}$;*
- *$a(A_n(q)) = q^{[(n+1)^2/4]}$, except for $A_1(q)$, where q is even and $A_2(q)$, where $(3, q-1) = 1$.*

Lemma 2.2. [3] *Let G be a finite group and $N \lhd G$. If $r \mid |G/N|$, $r \nmid |N|$ (r is prime and $r \neq p$), and if in addition $p^e \| |N|$ and $p^t \| |C_N(R)|$, where $R \in \mathrm{Syl}_r(G)$, then $r \mid p^{e-t} - 1$.*

Corollary 2.3. *Let G be a finite group and $N \lhd G$. If $r \mid |G/N|$, $r \nmid |N|$ (r is prime and $r \neq p$), and if in addition $p^e \| |N|$ and $(p, r) \notin GK(G)$, then $r \mid p^e - 1$.*

Proof. Straightforward. □

Lemma 2.4. [12, Corollary 11] *Let H be a finite group such that $2, s \in \pi(H)$. If $(2, s) \notin GK(H)$, then s-Sylow subgroup of H is abelian.*

Lemma 2.5. [6] *Let G and H be two finite groups such that $M(G) = M(H)$. Then G and H have the same prime graph.*

Lemma 2.6. [1] *Let $|n|_2 = n$. If G is a finite group such that $M(G) = M(B_n(q))$, then $G \cong B_n(q)$.*

For an integer n, by $\nu(n)$, $\eta(n)$ and $\eta'(n)$, we denote the following functions:

$$\nu(n) = \begin{cases} n & if\ n \equiv 0 \pmod 4; \\ \dfrac{n}{2} & if\ n \equiv 2 \pmod 4; \\ 2n & if\ n \equiv 1 \pmod 4. \end{cases}, \quad \eta(n) = \begin{cases} n & \text{if } n \text{ is odd}; \\ \dfrac{n}{2} & \text{otherwise}. \end{cases}, \tag{1}$$

$$\eta'(n) \;=\; \begin{cases} 2n & \text{if } n \text{ is odd}; \\ n & \text{otherwise}. \end{cases}.$$

Lemma 2.7. [15, Theorem 1] *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:*

1. *There exists a finite non-abelian simple group S such that $S \leq \bar{G} = G/K \leq Aut(S)$ for the maximal normal solvable subgroup K of G.*
2. *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K|.|\bar{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*
3. *One of the following holds:*
   (a) *every prime $r \in \pi(G)$ nonadjacent to 2 in $GK(G)$ does not divide the product $|K|.|\bar{G}/S|$; in particular, $t(2, S) \geq t(2, G)$;*
   (b) *there exists a prime $r \in \pi(K)$ nonadjacent to 2 in $GK(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong A_7$ or $A_1(q)$ for some odd q.*

Lemma 2.8. [16, Proposition 1.1] *Let $G = A_n$ be an alternating group of degree n.*

1. *Let $r, s \in \pi(G)$ be odd primes. Then r and s are nonadjacent if and only if $r + s > n$;*
2. *let $r \in \pi(G)$ be an odd prime. Then 2 and r are nonadjacent if and only if $r + 4 > n$.*

If a is a natural number, r is an odd prime and $\gcd(r, a) = 1$, then by $\exp_r(a)$ we denote the smallest natural number m such that $a^m \equiv 1 \pmod r$. Obviously by Fermat's little theorem it follows that $\exp_r(a) \mid (r-1)$. Also, if $a^n \equiv 1 \pmod r$, then $\exp_r(a) \mid n$. If a is odd, we put $\exp_2(a) = 1$ if $a \equiv 1 \pmod 4$, and $\exp_2(a) = 2$ otherwise.

Lemma 2.9. [8, Corollary to Zsigmondy's theorem] *Let a be a natural number greater than 1. For every natural number m there exists a prime r with $\exp_r(a) = m$, unless $a = 2$ and $m = 1$, $a = 3$ and $m = 1$, and $a = 2$ and $m = 6$.*

The prime r with $\exp_r(q) = m$ is called a *primitive prime divisor* of $q^m - 1$. It is obvious that $q^m - 1$ can have more than one primitive prime divisor. We denote by $r_m(q)$ some primitive prime divisor of $q^m - 1$. If there is no ambiguous, we write r_m instead of $r_m(q)$. Also, let $Z_m(q)$ denote the set of primitive prime divisors of $q^m - 1$. One can easily check the following corollary:

Corollary 2.10. *Let a, b and c be natural numbers and let s be a prime.*

(i) *If $\exp_s(p) = ab$, then $\exp_s(q^a) = b$;*

(ii) *if $c \mid a$ and $\gcd(c, b) = 1$, then $Z_b(p^{a/c}) \subseteq Z_b(p^a)$;*

(iii) *if $2 \mid a$, then $Z_{2b}(p^{a/2}) \subseteq Z_b(p^a)$.*

Lemma 2.11. [16, Propositions 2.1 and 2.2] and [17, Propositions 2.4, 2.5 and 2.7(5)] *Let $G = B_n(q)$ or $C_n(q)$. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = \exp_r(q)$ and $l = \exp_s(q)$. If $1 \leq \eta(k) \leq \eta(l)$, then r and s are nonadjacent if and only if $\eta(k) + \eta(l) > n$ and $\frac{l}{k}$ is not an odd natural number.*

Lemma 2.12. [16, Proposition 3.1] *Let $G = B_n(q)$ or $C_n(q)$, and let $r \in \pi(G)$ and $r \neq p$. Then r and p are nonadjacent if and only if $\eta(\exp_r(q)) > n - 1$.*

Lemma 2.13. [16, Proposition 4.3] *Let $G = B_n(q)$ or $G = C_n(q)$. Let r be an odd prime divisor of $|G|$, $r \neq p$, and $k = \exp_r(q)$. Then r and 2 are nonadjacent if and only if $\eta(k) = n$ and one of the following holds:*

1. *n is odd and $k = (3 - \exp_2(q))n$;*
2. *n is even and $k = 2n$.*

Lemma 2.14. [1, Corollary 3.3, Corollary 3.6 and the proof of Lemma 3.7] *Let n be an even number and $\alpha \in M(B_n(q))$.*

(i) *If $\pi(\alpha) \cap Z_{2n}(q) \neq \emptyset$, then $\alpha = \dfrac{q^n + 1}{\gcd(2, q-1)}$;*

(ii) *If* $\pi(\alpha) \cap Z_{2(n-1)}(q) \neq \emptyset$, *then* $\dfrac{q^{n-1}+1}{\gcd(2,q-1)} \mid \alpha$ *and* $\alpha \mid \dfrac{q(q^2-1)(q^{n-1}+1)}{\gcd(2,q-1)}$.

(iii) *If* $\pi(\alpha) \cap Z_{n-1}(q) \neq \emptyset$, *then* $\dfrac{q^{n-1}-1}{\gcd(2,q-1)} \mid \alpha$ *and* $\alpha \mid \dfrac{q(q^2-1)(q^{n-1}-1)}{\gcd(2,q-1)}$.

Lemma 2.15. *Let* $r \in \pi(G) - \{p\}$ *and* $R \in \mathrm{Syl}_r(SO_{2n+1}(q))$.

(i) [1, Lemma 3.18] *If* $n = 2^t$, $p \neq 2$ *and* $r = 2$, *then* $a(R) = (|q \pm 1|_2)^{2^t}$;

(ii) [4, Corollary before Theorem 2] *let* $p \neq 2$, $r = 2$ *and* $2n = 2^{r_1} + \dots + 2^{r_t}$ *with* $r_1 < \dots < r_t$. *If* $q^{2^{r_i-1}} \equiv \delta_i \pmod 4$, *for all* $i \in \{1, \dots, t\}$ *and* $R_i \in \mathrm{Syl}_2(GO^{\varepsilon_i}_{2^{r_i}}(q))$, *where* $\varepsilon_i = +$, *if* $\delta_i = +1$ *and* $\varepsilon_i = -$, *if* $\delta_i = -1$, *then* $R \cong R_1 \times \dots \times R_t$;

(iii) [21] *let* $r \neq 2$, $\exp_r(q) = m$ *and* $n_0 = \left[\dfrac{2n}{\eta'(m)}\right]$. *If* $n_0 = a_0 + a_1 r + \dots + a_u r^u$, $R_1 \in \mathrm{Syl}_r(GO^{\varepsilon}_{\eta'(m)}(q))$, *where* $\varepsilon = -$, *if* $(-1)^{m-1} = -1$ *and* $\varepsilon = +$, *otherwise and for all* $i \in \{2, \dots, u\}$, $S_i \in \mathrm{Syl}_r(\mathbb{S}_{r^i})$, *then*

$$R \cong \underbrace{((R_1 \wr S_2) \times \dots \times (R_1 \wr S_2))}_{a_1-times} \times \dots \times \underbrace{((R_1 \wr S_u) \times \dots \times (R_1 \wr S_u))}_{a_u-times}.$$

The following lemma is a known fact and for an example one can extract it from [10].

Lemma 2.16. *For the natural number* m,

(i) *if* m *is odd*, $r \in Z_m(q)$ *and* $R \in \mathrm{Syl}_r(GO^+_{2m}(q))$, *then* R *is abelian and* $|R| = |q^m - 1|_r$;

(ii) *if* $r \in Z_{2m}(q)$ *and* $R \in \mathrm{Syl}_r(GO^-_{2m}(q))$, *then* R *is abelian and* $|R| = |q^m + 1|_r$.

Lemma 2.17. *Let* G *be a finite group such that* $M(G) = M(B_n(q))$ *and let the functions* η *and* η' *be defined as in* (1). *If* $r \in \pi(G) - \{p\}$, *then:*

(i) *if* $r = 2$, *then* $a_r(G) \mid (|q^2 - 1|_2)^n$;

(ii) *if* $r \neq 2$, $\exp_r(q) = m$ *and* $n_0 = \left[\dfrac{2n}{\eta'(m)}\right]$, *then* $a_r(G) \mid (|q^{\eta(m)} + (-1)^m|_r)^{n_0}$.

Proof. Let $R \in \mathrm{Syl}_r(B_n(q))$ and $R' \in \mathrm{Syl}_r(SO_{2n+1}(q))$. Since $M(G) = M(B_n(q))$, we conclude that $a_r(G) = a_r(B_n(q)) = a(R)$ which divides $a(R')$.
(i) If $r = 2$, then we may assume that $2n = 2^{r_1} + \dots + 2^{r_t}$ such that $r_1 < \dots < r_t$. For $i \in \{1, \dots, t\}$, put $\varepsilon_i = +$, if $\delta_i = +1$ and put $\varepsilon_i = -$, if $\delta_i = -1$, where $q^{2^{r_i-1}} \equiv \delta_i \pmod 4$. Also, let $R_i \in \mathrm{Syl}_2(GO^{\varepsilon_i}_{2^{r_i}}(q))$. Then by Lemma 2.15(ii), $R' \cong R_1 \times \dots \times R_t$ an hence, $a(R') = a(R_1) \dots a(R_t)$. Now Lemma 2.15(i) completes the proof of (i).
(ii) If $r \neq 2$, then we can assume that $R \in \mathrm{Syl}_r(SO_{2n+1}(q))$ and $n_0 = a_0 + a_1 r + \dots + a_u r^u$. Thus by Lemma 2.15(iii), we have

$$R \cong \underbrace{((R_1 \wr S_2) \times \dots \times (R_1 \wr S_2))}_{a_1-times} \times \dots \times \underbrace{((R_1 \wr S_u) \times \dots \times (R_1 \wr S_u))}_{a_u-times},$$

where for $i \in \{2, \dots, u\}$, $S_i \in \mathrm{Syl}_r(\mathbb{S}_{r^i})$ and $R_1 \in \mathrm{Syl}_r(GO^{\varepsilon}_{\eta'(m)}(q))$, such that $\varepsilon = -$, if $(-1)^{m-1} = -1$ and $\varepsilon = +$, otherwise. Thus

$$a(R) = \underbrace{(a(R_1 \wr S_2) \dots a(R_1 \wr S_2))}_{a_1-times} \dots \underbrace{(a(R_1 \wr S_u) \dots a(R_1 \wr S_u))}_{a_u-times}.$$

Now we can see that for all $i \in \{1, \dots, u\}$, $a(R_1 \wr S_i) = (a(R_1))^{r^i}$. But by Lemma 2.16, R_1 is abelian and $a(R_1) = |R_1| = |q^{\eta(m)} + (-1)^m|_r$. This completes the proof of (ii). □

Lemma 2.18. *Let* N *be a normal subgroup of the finite group* G *and* $r, t \in \pi(N)$. *If* $r \in \rho(t, N)$, $r \notin \rho(t, G)$ *and* $\rho(t, G) \cap \pi(G/N) = \emptyset$, *then* $r \in \pi(G/N)$.

Proof. The proof is straightforward. □

3. Main Results

We are going to prove the main theorem in the following:

3.1. On the Maximal Abelian Subgroups of the almost Simple Groups Containing $B_n(q)$

Let $2 \mid q$ and $S = C_n(q)$. We denote by ϕ the field automorphism of S with $(a_{ij}) \longrightarrow (a_{ij}^p)$ as its map. Applying [10], if $n \neq 2$, then $S(\langle\phi\rangle) = \mathrm{Aut}(S)$.

Let $2 \nmid q$ and $S = B_n(q)$. We denote by δ the diagonal automorphism of S which is conjugate to the diagonal matrix $\mathrm{diag}(\lambda/\lambda', (\lambda/\lambda')^{-1}, I)$, where 2λ and $2\lambda'$ are a square element and a non-square element of $GF(q)$, respectively and by ϕ the field automorphism of S with $(a_{ij}) \longrightarrow (a_{ij}^p)$ as its map. Applying [10], if $n > 2$ is even, then $S(\langle\phi\rangle\langle\delta\rangle) = \mathrm{Aut}(S)$. Note that if $p = 2$, then $C_n(q) \cong B_n(q)$.

Lemma 3.1. *Let $G = S.T$, where $T \leq \mathrm{Out}(S)$ and $r, r_1 \in \pi(G)$ such that $\exp_r(q) = 2n$ and $\exp_{r_1}(q) = 2(n-1)$.*

(i) *If G contains a field automorphism ψ of order t, then $C_S(\psi) \cong B_n(q^{1/t})$;*

(ii) *if $2 \nmid q$ and G contains an automorphism $\delta\psi$, where ψ is a field automorphism of G of order 2, then there is an element $\alpha \in M(C_S(\delta\psi))$ such that $(p^{k(n-1)/2} + 1) \mid \alpha$ and if $q \neq 9$, then $\alpha \mid 2(p^{k/2} - 1)(p^{k(n-1)/2} + 1)$ and otherwise, $\alpha \mid 16(p^{k(n-1)/2} + 1)$.*

Proof. (i) is a known fact (for details see [1, Proof of Lemma 3.14]) and (ii) goes back to Lemma 3.17 in [1]. □

Lemma 3.2. *Let $S \trianglelefteq G \leq \mathrm{Aut}(S)$. If M is a maximal abelain subgroup of G, then $[M : M \cap S] \mid \gcd(2, q-1)k$.*

Proof. Since $MS \leq G \leq \mathrm{Aut}(S)$ and $Z(S) = 1$, we deduce that

$$\frac{M}{M \cap S} \leq \frac{G}{S} \lesssim \mathrm{Out}(S).$$

But as mentioned above, $|\mathrm{Out}(S)| = \gcd(2, q-1)k$, so lemma follows. □

Theorem 3.3. *Let $n > 3$ be an even number. If G is a finite group such that $M(G) = M(S)$ and $S \trianglelefteq G \leq \mathrm{Aut}(S)$, then $G = S$.*

Proof. We are going to break the proof into cases:
Case 1. If G contains a field automorphism, then without loss of generality, we can assume that $\psi \in G$ such that ψ is a field automorphism of the prime order t, where $t \mid k$. Thus Lemma 3.1(i) implies that $C_S(\psi) \cong B_n(q^{1/t})$. Thus by Lemma 2.14(i,ii), $C_S(\psi)$ contains a maximal abelain subgroup M_0 of order β such that $\beta \in \left\{\frac{(q^{n/t}+1)}{\gcd(2,q-1)}, \frac{l(q^{(n-1)/t}+1)}{\gcd(2,q-1)}\right\}$, where $l \mid q^{1/t}(q^{2/t} - 1)$. Since M_0 is an abelain subgroup of $C_S(\psi)$, we deduce that G contains a maximal abelian subgroup M such that $M_0\langle\psi\rangle \leq M$, so $M \leq C_G(\psi)$ and $M_0 \leq M \cap S \leq C_G(\psi) \cap S = C_S(\psi)$, which implies that $M \cap S = M_0$. Thus Lemma 3.2 implies that $[M : M_0]$ divides $\gcd(2, q-1)k$ and hence, there exits $\alpha \in M(G)$ such that $\beta \mid \alpha$ and $\alpha \mid \beta \gcd(2, q-1)k$. We continue the proof in the following subcases:

Subcase 1. If t is odd and $t \nmid n$, then Corollary 2.10(ii) forces $Z_{2n}(q^{1/t}) \subseteq Z_{2n}(q)$. Let $\beta = \frac{(q^{n/t}+1)}{\gcd(2,q-1)}$. Then $\pi(\alpha) \cap Z_{2n}(q) \neq \emptyset$ and hence, by Lemma 2.14(i), $\alpha = \frac{(q^n+1)}{\gcd(2,q-1)}$. It follows that

$$\frac{(q^n+1)}{\gcd(2,q-1)} \mid \beta \gcd(2, q-1)k = (q^{n/t} + 1)k. \tag{2}$$

Since $n \geq 6$ is even and $k \neq 1$, Lemma 2.9 shows that there exists $s \in Z_{2nk}(p)$. Thus $s \nmid (q^{n/t} + 1)$ and so by (2), $s \mid k$. On the other hand, Fermat's little theorem shows that $2nk \mid s - 1$, which is a contradiction.

Subcase 2. If t is odd and $t \mid n$, then $t \nmid n-1$ and hence, Corollary 2.10(ii) forces $Z_{2(n-1)}(q^{1/t}) \subseteq Z_{2(n-1)}(q)$. Let $\beta = \dfrac{l(q^{(n-1)/t}+1)}{\gcd(2,q-1)}$, where $l \mid q^{1/t}(q^{2/t}-1)$. Then $\pi(\alpha) \cap Z_{2(n-1)}(q) \neq \emptyset$ and hence, by Lemma 2.14(ii), $\dfrac{(q^{n-1}+1)}{\gcd(2,q-1)} \mid \alpha$. It follows that

$$\frac{(q^{n-1}+1)}{\gcd(2,q-1)} \mid q^{1/t}(q^{2/t}-1)(q^{(n-1)/t}+1)k. \tag{3}$$

Since $n \geq 6$ and $k \neq 1$, Lemma 2.9 shows that there exists $s \in Z_{2(n-1)k}(p)$. Thus $s \nmid q^{1/t}(q^{2/t}-1)(q^{(n-1)/t}+1)$ and so by (3), $s \mid k$. On the other hand, Fermat's little theorem shows that $2(n-1)k \mid s-1$, which is a contradiction.

Subcase 3. If t is even, then Corollary 2.10(iii) forces $Z_{2(n-1)}(q^{1/t}) \subseteq Z_{n-1}(q)$. Let $\beta = \dfrac{l(q^{(n-1)/2}+1)}{\gcd(2,q-1)}$, where $l \mid q^{1/2}(q-1)$. Then $\pi(\alpha) \cap Z_{n-1}(q) \neq \emptyset$ and hence, by Lemma 2.14(iii), $\dfrac{(q^{n-1}-1)}{\gcd(2,q-1)} \mid \alpha$. It follows that

$$\frac{(q^{n-1}-1)}{\gcd(2,q-1)} \mid q^{1/2}(q-1)(q^{(n-1)/2}+1)k. \tag{4}$$

Since $n \geq 6$ and k are even, Lemma 2.9 shows that there exists $s \in Z_{(n-1)k/2}(p)$. Thus $s \nmid q^{1/2}(q-1)(q^{(n-1)/2}+1)$ and so by (4), $s \mid k$. On the other hand, Fermat's little theorem shows that $(n-1)k/2 \mid s-1$, which is a contradiction.

Case 2. If $2 \nmid q$ and $\delta^j\psi \in G$, where ψ is a field automorphism of order 2, then $2 \mid k$ and by Lemma 3.1(ii), $C_{B_n(q)}(\delta^j\psi)$ contains a maximal abelain subgroup M_0 of order β such that $(q^{(n-1)/2}+1) \mid \beta$ and $\beta \mid l(q^{(n-1)/2}+1)$, where if $q \neq 9$, then $l = 2(p^{k/2}-1)$ and otherwise, $l = 16$. Now, the same reasoning as in Subcase 3 in Case 1 leads us to get a contradiction.

Case 3. If $2 \nmid q$ and $\delta^j \in G$, then by Cases 1 and 2, G does not contain any field automorphism and $\delta^j\psi \notin G$ and hence, $G = SO_{2n+1}(q)$. It follows that $(q^n+1) \in M(G)$, which is a contradiction.

These contradictions show that $G = B_n(q)$. □

3.2. Proof of the Main Theorem

Theorem 3.4. *If G is a finite group such that $M(G) = M(B_n(q))$, then $G \cong B_n(q)$.*

Proof. If $n = |n|_2$, then Lemma 2.6 completes the proof. Thus we may assume that $n \neq |n|_2$. This allows us to assume that $n \geq 6$. Since $M(G) = M(B_n(q))$, we have $GK(G) = GK(B_n(q))$, considering Lemma 2.5. Therefore, $\pi(G) = \pi(B_n(q))$, $t(G) = t(B_n(q)) = \left[\dfrac{3n+5}{4}\right] \geq 5$ and $\rho(2,G) = \rho(2,B_n(q)) = \{2, r_{2n}(q)\}$, using [16, Tables 4, 6 and 8]. It follows by Lemma 2.7(1) that there is a finite non-abelian simple group S such that $S \leq \bar{G} = G/K \leq \mathrm{Aut}(G)$ for the maximal solvable subgroup K of G such that $t(S) \geq t(G)-1$. We continue the proof in the following steps:

Step I) $K = 1$. This implies that $S \leq G \leq \mathrm{Aut}(S)$.

Proof. Put $\rho = \{r_{2nk}(p), r_{2(n-1)k}(p), r_{(n-1)k}(p)\}$. By Corollary 2.10(i), $\exp_{r_{mk}(p)}(q) = m$ and by [16, Table 8], if $(n,q) = (6,2)$, then $\rho(G) = \rho(B_n(q)) = \{7,\ 11,\ 13,\ 17,\ 31\}$ and otherwise, $\rho(G) = \rho(B_n(q)) = \{r_{2i}(q) : \left[\dfrac{n+1}{2}\right] \leq i \leq n\} \cup \{r_i(q) : \left[\dfrac{n}{2}\right] < i \leq n,\ i \equiv 1 \pmod 2\}$. These imply that $\rho \subseteq \rho(G)$ and hence, ρ is independent. Thus by Lemma 2.7(2), there is a prime $z \in \{r_{2(n-1)k}(p), r_{(n-1)k}(p)\} \cap \pi(S)$ such that $z \notin \pi(K)$. Also, $r_{2nk}(p) \in \rho(2,S)$ and hence Lemma 2.7(3) forces $r_{2nk}(p) \in \pi(S)$ and $r_{2nk}(p) \notin \pi(K)$. Let $R \in \mathrm{Syl}_{r_{2nk}(p)}(S)$ and $R_1 \in \mathrm{Syl}_z(S)$. We have that R and R_1 act coprimely on K. We claim that $|K|_p = 1$. If not, then we deduce that K has an R-invariant p-Sylow subgroup P_1 and an R_1-invariant p-Sylow subgroup P_2. Thus $Z(P_1)R$ and $Z(P_2)R_1$ are subgroups of G. Since $\exp_{r_{2nk}(p)}(q) = 2n$, we have $(p, r_{2nk}(p)) \notin GK(B_n(q)) = GK(G)$, by Lemma 2.12. It follows that by Corollary 2.3, $r_{2nk}(p) \mid p^t - 1$, where $|Z(P_1)| = |Z(P_2)| = p^t$ and hence, $2nk \mid t$. If $|C_{Z(P_2)}(R_1)| = p^e$, then there is $\alpha \in M(G) = M(B_n(q))$ such that $zp^e \mid \alpha$. It follows by Lemma 2.14(ii,iii) that $p^e \leq p^k$. Also, $\exp_z(p) \in \{(n-1)k, 2(n-1)k\}$ and by Lemma 2.2, $z \mid p^{t-e} - 1$ and hence, $(n-1)k \mid t - e$. Since $2nk \mid t$, we conclude that there is a natural number a such that $t = 2nka$. Therefore, $(n-1)k \mid 2nka - e = 2(n-1)ka + 2ka - e$, so $(n-1)k \mid 2ka - e$. Since $e \leq k$, we have that $2ka \neq e$ and hence, $(n-1)k \leq 2ka - e \leq 2ka$. It follows that $(n-1) \leq 2a$, so $t \geq n(n-1)k$. But $Z(P_2)$ is an abelian subgroup of G and hence, $|Z(P_2)| = p^t \leq a(G) = a(B_n(q))$, which is a contradiction, because if p is even, then $a(B_n(q)) = q^{\frac{n(n+1)}{2}}$ and otherwise, $a(B_n(q)) = q^{\frac{n(n-1)}{2}+1}$, by Lemma 2.1.

Now, we show that $|K| = 1$. If this is not the case, then there is a prime $s \in \pi(K)$. Since $a(B_n(q)) \in M(G)$ is a power of p, we may assume that there is an abelian p-subgroup P of G such that $|P| = a(G)$. Also, $|K|_p = 1$, so P acts coprimely on K and hence, we can see that K has a P-invariant s-Sylow subgroup S_0. So $Z(S_0)P$ is a subgroup of G. We may assume that $Z(S_0)$ is a s-elementary abelian subgroup of G and $|Z(S_0)| = s^\alpha$. But P is abelian and $|P| = a(G)$. This implies that $C_{PZ(S_0)}(Z(S_0))$ is abelian and hence, $|C_{PZ(S_0)}(Z(S_0))| = s^\alpha p^\beta < |P|$. Also,

$$\frac{N_{PZ(S_0)}(Z(S_0))}{C_{PZ(S_0)}(Z(S_0))} \leq \mathrm{Aut}(Z(S_0)) = GL_\alpha(s).$$

Thus $GL_\alpha(s)$ has an abelian subgroup of order $|P|/p^\beta$. On the other hand, similar to the proof of Lemma 2.17 we can see that $a_p(GL_\alpha(s)) < s^\alpha$. Therefore, $|P|/p^\beta < s^\alpha$, which is a contradiction. It follows that $|K| = 1$. Thus by Lemma 2.7(1), $S \leq G \leq \mathrm{Aut}(S)$. □

Step II) $|\frac{G}{S}|_p < q^{n-|n|_2}$.

Proof. Let $p \mid |\frac{G}{S}|$. Since $t(S) \geq t(G) - 1 \geq 4$, [16, Tables 3,8], $S \not\cong A_7, A_1(q)$. Also, since $(2, r_{2n}(q)) \notin GK(G)$, Lemma 2.7(3)(a) forces $r_{2n}(q) \nmid |\frac{G}{S}|$ and so, $\mathrm{Syl}_{r_{2n}(q)}(G) = \mathrm{Syl}_{r_{2n}(q)}(S)$. Let $R \in \mathrm{Syl}_{r_{2n}(q)}(G)$. It follows by Frattini's argument that $|\frac{G}{S}|_p \mid |N_G(R)|$. Thus there is a p-subgroup Q of G such that QR is a subgroup of G and $|\frac{G}{S}|_p \mid |Q|$. Since $(p, r_{2n}(q)) \notin GK(B_n(q)) = GK(G)$, the action of Q on R is Frobenius. Therefore, $|\frac{G}{S}|_p \mid |R| - 1$. Also, $(2, r_{2n}(q)) \notin GK(G)$ and hence by Corollary 2.4, R is abelian. Thus $|R| = a(R_1)$, where $R_1 \in \mathrm{Syl}_{r_{2n}(q)}(B_n(q))$. But since $|n|_2 = 2^m \neq n$, $|B_n(q)|_{r_{2n}(q)} = \left|\frac{q^n+1}{q^{2^m}+1}\right|_{r_{2n}(q)}$. So, $|\frac{G}{S}|_p < q^{n-|n|_2}$. □

Step III) S is not isomorphic to a sporadic simple group.

Proof. If S is isomorphic to a sporadic simple group, then since $Z(S) = 1$, we have by Step I, $\frac{G}{S} \leq \mathrm{Out}(S)$. But $|\mathrm{Out}(S)| \mid 2$, using [10, page 171, Table 5.1.c]. Therefore, $\rho(G) = \rho(S)$, by Lemma 2.18. So,

$$t(S) = t(G).$$

Also, $t(S) \leq 11$ and $t(G) = \left[\frac{3n+5}{4}\right]$ by [16, Tables 2 and 8]. Therefore, since $\left[\frac{3n+5}{4}\right] \leq 11$ if and only if $n \leq 13$, we conclude that $n \in \{6, 10, 12\}$. Thus, we have the following cases:

a) If $n = 6$, then $t(S) = t(G) = \left[\frac{23}{4}\right] = 5$. It follows that $S \in \{Fi_{23}, Fi'_{24}, F_3\}$ (up to isomorphism), considering [16, Table 2]. On the other hand, $\exp_{r_{2nk}(p)}(q) = 2n$ and hence by Lemma 2.13, $r_{2nk}(p) \in \rho(2, G)$. Thus since by Lemma 2.7(3)(a), $\rho(2, G) \subseteq \rho(2, S)$, we conclude that $r_{2nk}(p) \in \rho(2, S)$ and hence, Fermat's little theorem implies that there is an element $z \in \rho(2, S)$ such that $12k = 2nk \mid z - 1$, which is a contradiction, considering the elements of $\rho(2, Fi_{23})$, $\rho(2, Fi'_{24})$ and $\rho(2, F_3)$ (see [16, Table 2]).

b) If $n = 10$, then $t(S) = t(G) = \left[\frac{35}{4}\right] = 8$. It follows that $S \cong F_2$, considering [16, Table 2]. Similar to the previous argument, we can assume that there is an element $z \in \rho(2, S)$ such that $20k = 2nk \mid z - 1$, which is impossible, considering the elements of $\rho(2, F_2)$.

c) If $n = 12$, then $t(S) = t(G) = \left[\frac{41}{4}\right] = 10$, which is impossible, because there does not exist any sporadic simple group S with $t(S) = 10$. □

Step IV) S is not isomorphic to the alternating group A_x of degree x.

Proof. If S is isomorphic to A_x, then similar to the previous argument, we can see that $r_{2n}(q) \in \rho(2, S)$. Since $n \geq 6$, we have $t(G) \geq 5$ and hence, $t(S) \geq 4$. Therefore, [16, Table 3] implies that $x \geq 7$ and hence, $G \leq \mathrm{Aut}(A_x) = S_x$. But by Lemma 2.8(2), $\rho(2, S) = \{s \in \pi(A_x) : x - 3 \leq s \leq x\} \cup \{2\}$, so $x - 3 \leq r_{2n}(q) \leq x$. It follows that $r_{2n}(q) \in M(G)$, $2r_{2n}(q) \in M(G)$ or $3r_{2n}(q) \in M(G)$. Thus by Lemma 2.14(i), $\frac{q^n+1}{\gcd(2, q-1)} = dr_{2n}(q)$, where $d \in \{1, 2, 3\}$. This forces $q^{|n|_2} + 1 \in \{1, 2, 3\}$, which is impossible. □

Step V) S is isomorphic to the simple group of Lie type in characteristic p.

Proof. Using Steps (II,III,IV) and the classification theorem of finite simple groups, we conclude that S is a simple group of Lie type in characteristic p'. If $p \neq p'$, then since by Lemma 2.1, $a(G) = a(B_n(q))$ is a power of p, we can see that

$$a(G) \leq a_p(S) \left|\frac{G}{S}\right|_p. \tag{5}$$

On the other hand,

$$a_p(S) \leq a(S). \tag{6}$$

We continue the proof in the following cases:

Case 1. Let $S \ncong {}^2F_4(2^{2n+1})$, for $n \geq 1$. Since $t(S) \geq t(G) - 1 \geq 4$, we may assume that S is not isomorphism to $A_1(p'^e)$, $A_2(p'^e)$ with $(p'^e - 1, 3) = 1$, ${}^2A_3(2)$ and ${}^2A_2(p'^e)$ with $(p'^e + 1, 3) = 1$, using [16, Table 8]. Therefore, Lemma 2.1 implies that

$$a(S) = a_{p'}(S). \tag{7}$$

Also, since $p \neq p'$, we obtain by Lemma 2.17 that $a_{p'}(B_n(q)) < q^{2n}$. But $a_{p'}(S) \leq a_{p'}(G)$, so $a(G) \leq a_{p'}(G)|G/S|_p < q^{2n}q^{n-|n|_2} = q^{3n-|n|_2}$, using (5,6,7) and Step II. It follows that by Lemma 2.1, either $\left(\frac{n(n-1)}{2} + 1\right)k < 3nk - |n|_2k$ or $\left(\frac{n(n+1)}{2}\right)k < 3nk - |n|_2k$. This forces $n < 6$, which is a contradiction.

Case 2. Let $S \cong {}^2F_4(2^{2m+1})$, for $n \geq 1$. Fix $q' = 2^{2m+1}$. Then [16, Table 5] implies that $\rho(2, S) = \{2, s_1, s_2, s_3\}$, for some $s_1 \in \pi((q'^3 + 1)/(q' + 1))$ and $s_2, s_3 \in \pi((q'^6 + 1)/(q'^2 + 1))$. Without loss of generality, we can assume that $s_1 \in Z_{3(2m+1)}(2)$ and $s_2, s_3 \in Z_{6(2m+1)}(2)$. Thus Fermat's little theorem shows that

$$2m + 1 \mid s_i - 1, \text{ for } i \in \{1, 2, 3\}. \tag{8}$$

It is known that $\mathrm{Out}(S) \cong \mathbb{Z}_{2m+1}$, so "$G/S \lesssim \mathrm{Out}(S) \cong \mathbb{Z}_{2m+1}$" shows that $2 \notin \pi(G/S)$. Thus by Lemma 2.7(3), $\rho(2, G) \cap \pi(G/S) = \emptyset$. On the other hand, by [16, Tabeles 4,6], $t(2, G) = 2$, so Lemma 2.18 forces to exist $1 \leq j \leq 3$ such that $s_j \in \pi(G/S) \subseteq \pi(\mathrm{Out}(S)) = \pi(2m+1)$. This implies that $s_j \mid 2m+1$, contradicting (8). □

Step VI) $S \cong B_n(q)$ or $C_n(q)$.

Proof. By Step V, S is a simple group of Lie type in characteristic p. Since $r_{2nk}(p) \in \rho(2, G)$, Lemma 2.7(3)(a) forces

$$r_{2nk}(p) \in \pi(S). \tag{9}$$

Now, we consider all simple groups of Lie type in characteristic p one by one:

a) Let $S \cong B_m(p^e)$ or $S \cong C_m(p^e)$. Then $\max\{\exp_s(p) : s \in \pi(G) - \{p\}\} = 2nk$ and $\pi(S) \subseteq \pi(G)$. Thus by (9), $\max\{\exp_s(p) : s \in \pi(S) - \{p\}\} = 2nk$. On the other hand, $|B_m(p^e)| = |C_m(p^e)| = p^{m^2e}(p^{2e} - 1)...(p^{2me} - 1)$ and hence, $\max\{\exp_s(p) : s \in \pi(S) - \{p\}\} = 2me$. It follows that $2nk = 2me$. If $r_{2(n-1)k}(p) \notin \pi(S)$, then $r_{2(n-1)k}(p) \in \pi(\frac{G}{S})$. But $Z(S) = 1$ and $G \leq \mathrm{Aut}(S)$. So, $\frac{G}{S} \lesssim \mathrm{Out}(S)$. Since $|\mathrm{Out}(S)| \mid 2e$ (see [10, Propositions 2.4.4 and 2.6.3]), we have $r_{2(n-1)k}(p) \mid e$. Also, $2nk = 2me$ and hence, $r_{2(n-1)k}(p) \mid nk$. But Fermat's little theorem implies that $2(n-1)k \mid r_{2(n-1)k}(p) - 1$, which is a contradiction. Otherwise, $r_{2(n-1)k}(p) \in \pi(S)$. Thus $2(n-1)k = \max\{\exp_s(p) : s \in \pi(S) - (Z_{2nk}(p) \cup \{p\})\} = 2(m-1)e$. It follows that $e = k$ and $m = n$. Therefore

$S \cong B_n(q)$ or $S \cong C_n(q)$.
b) Let $S \cong {}^2D_m(p^e)$. Applying the same argument as that of in Step VI(a) shows that $2nk = 2me$ and since by [10, Proposition 2.8.2], $|\mathrm{Out}(S)| \mid 2^3e$, we get that $r_{2(n-1)k}(p) \in \pi(S)$. Thus $2(n-1)k = 2(m-1)e$. It follows that $e = k$ and $m = n$. It is evident that $a_p(G) \le a_p(S)|\frac{G}{S}|_p$ and by Lemma 2.1, $q^{\frac{n(n-1)}{2}+1} \le a_p(G)$ and $a_p(S) = q^{\frac{(n-1)(n-2)}{2}+2}$. Therefore, (II) implies that $\dfrac{n(n-1)}{2} + 1 < \dfrac{(n-1)(n-2)}{2} + n$, which is impossible.
c) Let $S \cong D_m(p^e)$. Since $t(S) = \left[\dfrac{3m+1}{4}\right]$ and $t(S) \ge t(G) - 1 \ge 4$, by [16, Table 8] and Lemma 2.7(2), respectively, we have $m \ge 5$ and hence, $\max\{\exp_s(p) : s \in \pi(S) - \{p\}\} = 2(m-1)e$ and $\max\{\exp_s(p) : s \in \pi(S) - (Z_{2(m-1)e}(p) \cup \{p\})\} = 2(m-2)e$. On the other hand, [10, Proposition 2.7.3] implies that $|\mathrm{Out}(S)| \mid 8e$ and hence, arguing as in Step VI(a) shows that $2nk = 2(m-1)e$ and $2(n-1)k = 2(m-2)e$. Therefore, $m - 1 = n$ and $e = k$. But $r_m(q) \in \pi(S)$ and hence $r_{n+1}(q) \in \pi(G) = \pi(B_n(q))$. Since $|B_n(q)| = q^{n^2}(q^2-1)...(q^{2n}-1)/(2, q-1)$, there exists a natural number f such that $1 \le f \le n$ and $r_{n+1}(q) \mid q^{2f} - 1$. It follows that $n + 1 \mid 2f$. Moreover, n is even and hence, $n + 1 \mid f$, which is a contradiction.
d) Let $S \cong A_{m-1}(p^e)$. Using Lemma 2.7(2) and [16, Table 8], $t(S) = \left[\dfrac{m+1}{2}\right] \ge t(G) - 1 \ge 4$. Thus $m \ge 7$ and hence, $\max\{\exp_s(p) : s \in \pi(S) - \{p\}\} = me$. Also, $\max\{\exp_s(p) : s \in \pi(S) - (Z_{me}(p) \cup \{p\})\} = (m-1)e$. Thus arguing as in Step VI(a) shows that $me = 2nk$ and $(m-1)e = 2(n-1)k$. Therefore, $m = n$ and $e = 2k$. But $a_p(S) \le a_p(G) \le q^{\frac{n(n+1)}{2}}$ and $a_p(S) = p^{[(m+1)^2e/4]}$, by Lemma 2.1. Therefore, $\left[\dfrac{(n+1)^2(2k)}{4}\right] \le \dfrac{n(n+1)k}{2}$, which is impossible.
e) Let $S \cong {}^2A_{m-1}(p^e)$. We denote $\max\{\exp_s(p) : s \in \pi(S) - \{p\}\}$ by α. Arguing as in Step VI(a) shows that

$$2nk = \alpha = \begin{cases} 2me & m \equiv 1 \pmod 2; \\ 2(m-1)e & \text{otherwise}. \end{cases}$$

and if $m \equiv 1 \pmod 2$, then $\max\{\exp_s(p) : s \in \pi(S) - (Z_{2me}(p) \cup \{p\})\} = 2(m-2)e$ and otherwise, $\max\{\exp_s(p) : s \in \pi(S) - (Z_{2(m-1)e}(p) \cup \{p\})\} = 2(m-3)e$. On the other hand, $\max\{\exp_s(p) : s \in \pi(S) - (Z_{2nk}(p) \cup \{p\})\} = 2(n-1)k$. Therefore, we can see that if $m \equiv 1 \pmod 2$, then $m = 2n$ and $2e = k$, and hence m is even, which is a contradiction. Also, if $m \equiv 0 \pmod 2$, then we can assume that $m - 1 = 2n$ and $2e = k$, and hence m is odd, which is a contradiction.
f) If $p = 2$, $e = 2f + 1$ and $S \cong {}^2F_4(p^e)$, then similar to the previous argument and by the order of ${}^2F_4(p^e)$ we can see that $12e = 2nk$ and $6e = 2(n-1)k$. It follows that $k = 3e$ and $n = 2$, which is a contradiction.
g) If $S \cong E_7(p^e)$, then $|\mathrm{Out}(S)| = (2, q-1)e$, considering [10, page 170, Table 5.1.B] and hence similar to the previous argument, we can see that $r_{2(n-1)k}(p) \in \pi(S)$. Also, by the order of $E_7(p^e)$ we can see that $\max\{\exp_s(p) : s \in \pi(S) - \{p\}\} = 18e$ and $\max\{\exp_s(p) : s \in \pi(S) - (Z_{18e}(p) \cup \{p\})\} = 14e$. Again, similar to the previous argument we can conclude that $18e = 2nk$ and $14e = 2(n-1)k$. It follows that $2n = 9$, which is impossible.
h) If $S \cong E_8(p^e)$, then $|\mathrm{Out}(S)| = e$, considering [10, page 170, Table 5.1.B]. Similar to the previous argument, we may assume that $30e = 2nk$ and $2(n-1)k = 24e$. It follows that $n = 5$, which is a contradiction.
i) If $S \in \{F_4(p^e), E_6(p^e), {}^2E_6(p^e), {}^2B_2(2^{2e+1}), {}^2G_2(3^{2e+1})\}$, then the same argument as that of in the previous case shows that $\{\exp_s(p) : s \in \pi(G) - \{p\}\} = 2nk$ and $\{\exp_s(p) : s \in \pi(G) - (Z_{2nk}(p) \cup \{p\})\} = 2(n-1)k$. Therefore, we can see that $n < 4$, which is a contradiction.
j) If $S \in \{G_2(p^e), {}^3D_4(p^e), {}^2F_4(2)'\}$, then [16, Table 9] implies that $t(S) \le 3$, which is a contradiction.
Therefore, we conclude that $S \cong B_n(q)$ or $S \cong C_n(q)$. □

Step VII) $S \cong B_n(q)$.
Proof. By Step VI, it is enough to show that if $p \ne 2$, then $S \not\cong C_n(q)$. If not, then $a(S) = a(C_n(q)) \mid a(G) = a(B_n(q))$, because $M(G) = M(B_n(q))$. Therefore, by Lemma 2.1, $q^{\frac{n(n+1)}{2}} \mid q^{\frac{n(n-1)}{2}+1}$, which is impossible and hence $S \not\cong C_n(q)$. If $p = 2$, then $C_n(q) \cong B_n(q)$. It follows that $S \cong B_n(q)$. □

Step VIII) $G = S \cong B_n(q)$.

Proof. Using Theorem 3.3 and the previous steps, we conclude that $G = S \cong B_n(q)$, as claimed. □

AAM's Conjecture. Given an arbitrary non-abelian group H, associate a graph $\Gamma(H)$ to H which is called the *non-commuting graph* of H. The vertex set $V(\Gamma(H))$ is $H - Z(H)$ and the edge set $E(\Gamma(H))$ consists of (x, y), where x and y are distinct non-central elements of H such that $xy \neq yx$. AAM's conjecture implies that if S is a non-abelian finite simple group and H is a group such that $\Gamma(H) \cong \Gamma(S)$, then $H \cong S$.

Lemma 3.5. *If S is a finite simple group and H is a finite group such that $\Gamma(S) \cong \Gamma(H)$, then*

1. [7] $Z(H) = 1$;
2. [2, Theorem 2.5] $M(H) = M(S)$.

In [13], authors prove that AAM's conjecture holds for finite simple groups. As a consequence of the main theorem, we prove the following corollary. It is worth mentioning that our proof is different from [13].

Corollary 3.6. *Let $n > 3$ be an even natural number and let q be a prime power. If G is a finite group such that $\Gamma(G) \cong \Gamma(B_n(q))$, then $G \cong B_n(q)$.*

Proof. By Lemma 3.5, $\Gamma(G) \cong \Gamma(B_n(q))$ gives that $M(G) = M(B_n(q))$. Therefore, Theorem 3.4 completes the proof. □

Acknowledgment.

The first author would like to thank Shahrekord University for financial support.

Partial support by the Center of Excellence of Algebraic Hyperstructures and its Applications of Tarbiat Modares University (CEAHA) is gratefully acknowledged by the second author (AI).

References

[1] N. Ahanjideh, A. Iranmanesh, A characterization of $B_n(q)$ by the set of orders of maximal abelian subgroups, Int. J. Algebra and Computation 19 (2) (2009) 191–211.
[2] N. Ahanjideh, A. Iranmanesh, On the relation between the non-commuting graph and the prime graph, Int. J. Group Theory 1 (1) (2012) 25–28.
[3] J. Bi, A characterization of $L_n(q)$ by the normalizers' orders of their Sylow subgroups, Acta Math. Sinica (New Ser.) 11 (3) (1995) 300–306.
[4] R. Carter, P. Fong, The Sylow 2-subgroup of the classical groups, J. Algebra 1 (1964) 139–151.
[5] R.W. Carter, Finite groups of Lie type: conjugacy classes and complex characters, Wiley, New York, 1985.
[6] G.Y. Chen, A characterization of alternating groups by the set of orders of their maximal abelian subgroups, Siberian Math. J. 47 (3) (2006) 594–596.
[7] M.R. Darafsheh, Groups with the same non-commuting graph, Discrete applied math. 157 (2009) 833–837.
[8] W. Feit, On large Zsigmondy primes, Proc. Amer. Math. Soc. 102 (1) (1988) 29–33.
[9] B. Huppert, Endliche Gruppen, Springer-Verlag, 1967.
[10] P. Kleidman, M. Liebeck, The subgroup structure of finite classical groups, Cambridge University Press, 1990.
[11] A.S. Kondrat´ev, Prime graph components of finite simple groups, Math. USSRSb. 67 (1) (1990) 235–247.
[12] G. Malle, A. Moreto, G. Navarro, Element orders and Sylow structure of finite groups, Mathematische Zeitschrift 252 (1) (2006) 223–230.
[13] R.M. Solomon, A.J. Woldar, Simple groups are characterized by their non-commuting graphs, J. Group Theory 16 (2013) 793–824.
[14] M. Stather, Constructive Sylow theorems for the classical groups, J. Algebra 316 (2007) 536–559.
[15] A.V. Vasil'ev, I.B. Gorshkov, On recognition of finite simple groups with connected prime graph, Siberian Math. J. 50 (2) (2009) 233–238.
[16] A.V. Vasil'ev, E.P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, Algebra and Logic 44 (6) (2005) 381–406.
[17] A.V. Vasil'ev, E.P. Vdovin, Cocliques of maximal size in the prime graph of a finite simple group, Algebra and Logic 50(4) (2011) 291–322.
[18] E.P. Vdovin, Maximal orders of abelian subgroups in finite Chevalley groups, Matematicheskie zametki 69 (4) (2001) 524–549.
[19] L. Wang, A new characterization of simple K_3−groups, Northeast. Math. J. 17 (2) (2001) 205–209.
[20] J.S. Willams, Prime graph components of finite groups, J. Algebra 69 (2) (1981) 487–513.
[21] A.J. Weir, Sylow p-subgroups of the classical groups over finite fields with characteristic prime to p, Proc. Amer. Math. Soc. 6 (1955) 529–533.