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# On Modified $\alpha$-$\phi$-Asymmetric Meir-Keeler Contractive Mappings

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**Abstract.** Samet et al. [Nonlinear Anal. 75:2154-2165, 2012] introduced and studied $\alpha$-$\psi$-contractive mappings. More recently Salimi, et al. [Fixed Point Theory Appl., 2013:151] modified the notion of $\alpha$-$\psi$-contractive mappings and improved the fixed point theorems in [20, 32]. Here we utilize these notions to establish fixed point results for modified $\alpha$-$\phi$-asymmetric Meir-Keeler contractions and triangular $\alpha$-admissible mappings defined on $G$-metric and cone $G$-metric spaces. Several interesting consequences of our theorems are also provided here to illustrate the usability of the obtained results.

## 1. Introduction and Preliminaries

The study of fixed points of given mappings satisfying certain contractive conditions in various abstract spaces has been at the center of vigorous research activity. Banach contraction mapping principle has attracted attention of many authors to generalize, extend and improve the metric fixed point theory. For this purpose, the authors considered to extend metric fixed point theory to different abstract spaces such as symmetric spaces, quasi-metric spaces, fuzzy metric spaces, partial metric spaces, probabilistic metric spaces, (ordered)$G$-metric space (see, e.g. [1–4, 8, 10, 13, 14, 20, 23]).

Here, we collect some notions and notations which will be used throughout the rest of this work.

**Definition 1.1.** *Let $X$ be a non-empty set. A function $G : X \times X \times X \longrightarrow \mathbb{R}_+$ is called a $G$-metric if the following conditions are satisfied:*

*(G1) If $x = y = z$, then $G(x, y, z) = 0$;*

*(G2) $0 < G(x, y, y)$, for any $x, y \in X$ with $x \neq y$;*

*(G3) $G(x, x, y) \leq G(x, y, z)$ for any points $x, y, z \in X$, with $y \neq z$;*

*(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;*

*(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for any $x, y, z, a \in X$.*

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*Then the pair $(X, G)$ is called a G-metric space.*

**Definition 1.2.** *Let $(X, G)$ be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to+\infty} G(x, x_m, x_n) = 0$, and we say that the sequence $\{x_n\}$ is G-convergent to x and denote it by $x_n \longrightarrow x$.*

We have the following useful results.

**Proposition 1.3.** *(see [27]) Let $(X, G)$ be a G-metric space. Then the following are equivalent:*

*(1)* $\{x_n\}$ *is G-convergent to x;*

*(2)* $\lim\limits_{n\to+\infty} G(x_n, x_n, x) = 0$;

*(3)* $\lim\limits_{n\to+\infty} G(x_n, x, x) = 0$.

**Definition 1.4.** *([27]) Let $(X, G)$ be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq k$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.*

**Proposition 1.5.** *([27]) Let $(X, G)$ be a G-metric space. Then the following are equivalent:*

1. *the sequence $\{x_n\}$ is G-Cauchy;*
2. *for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq k$.*

**Definition 1.6.** *([27]) A G-metric space $(X, G)$ is called G-complete if every G-Cauchy sequence in $(X, G)$ is G-convergent in $(X, G)$.*

**Proposition 1.7.** *(see [27]) Let $(X, G)$ be a G-metric space. Then, for any $x, y, z, a \in X$, it follows that:*

*(i)* *If $G(x, y, z) = 0$ then $x = y = z$;*

*(ii)* $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;

*(iii)* $G(x, y, y) \leq 2G(y, x, x)$;

*(iv)* $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;

*(v)* $G(x, y, z) \leq \dfrac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$;

*(vi)* $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

**Proposition 1.8.** *(see [27]) Let $(X, G)$ be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

In 2012, Samet et al. [32] introduced the concepts of $\alpha$-$\psi$-contractive and $\alpha$-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces(see also [20]). More recently Salimi et al. [30] modified the notions of $\alpha$-$\psi$-contractive mappings and established fixed point theorems which are proper generalizations of the recent results in [32].

Samet et al. [32] defined the notion of $\alpha$-admissible mapping as follows.

**Definition 1.9.** *Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an $\alpha$-admissible mapping if*

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Hussain et al. [15] introduced the notion of $G$-$(\alpha, \psi)$-Meir-Keeler contractive mapping and proved some fixed point theorems for this class of mapping in the setting of $G$-metric spaces.

Salimi et al. [30] modified and generalized the notions of $\alpha$-$\psi$-contractive mappings and $\alpha$-admissible mappings as follows.

**Definition 1.10.** *[30] Let $T$ be a self-mapping on $X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that $T$ is an $\alpha$-admissible mapping with respect to $\eta$ if*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

*Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 1.9. Also, if we take, $\alpha(x, y) = 1$ then we say that $T$ is an $\eta$-subadmissible mapping.*

Recently Karapinar et al. [19] introduced the notion of triangular $\alpha$-admissible mapping as follows.

**Definition 1.11.** *[19] Let $T : X \to X$ and $\alpha : X \times X \to (-\infty, +\infty)$. We say that $T$ is a triangular $\alpha$-admissible mapping if*

*(T1)* $\alpha(x, y) \geq 1$ *implies* $\alpha(Tx, Ty) \geq 1, \quad x, y \in X,$

*(T2)* $\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases}$ *implies* $\alpha(x, y) \geq 1.$

On the other hand recently Hussain et al. [12] modified $\alpha$-$\psi$-Meir-Keeler contractive mapping as follows.

**Definition 1.12.** *[12] Let $(X, d)$ be a metric space and let $\psi$ be non-decreasing, subadditive altering distance function. Suppose that $f : X \to X$ is a triangular $\alpha$-admissible mapping satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \psi(d(x, y)) < \varepsilon + \delta \quad \text{implies} \quad \psi(d(fx, fy)) < \varepsilon \tag{1}$$

*for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Then $f$ is called a modified $\alpha$-$\psi$-Meir-Keeler contractive mapping.*

**Lemma 1.13.** *[19] Let $f$ be a triangular $\alpha$-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

## 2. Modified $\alpha$-$\phi$-Asymmetric Meir-Keeler Contractive Mapping

Denote with $\Psi$ the family of continuous nondecreasing functions $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(t) = 0$ if and only if $t = 0$.

We start this section with the following definition.

**Definition 2.1.** *Let $(X, G)$ be a G-metric space and $\phi \in \Psi$. Suppose that $T : X \to X$ satisfies the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \phi(G(x, Tx, y)) < \varepsilon + \delta \Rightarrow \phi(G(Tx, T^2x, Ty)) < \varepsilon \tag{2}$$

*for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Then $T$ is called modified $\alpha$-$\phi$-asymmetric Meir-Keeler contractive mapping.*

**Remark 2.2.** *If $T : X \to X$ is a modified $\alpha$-$\phi$-asymmetric Meir-Keeler contractive mapping such that $x \neq Tx$, then*

$$\phi(G(Tx, T^2x, Ty)) < \phi(G(x, Tx, y)) \tag{3}$$

*for all $(x, y) \in X^2$ with $\alpha(x, y) \geq 1$.*

Now, we are ready to state and prove our main result of this section.

**Theorem 2.3.** *Let $(X, G)$ be a $G$-complete $G$-metric space and $\phi \in \Psi$. Suppose that $T : X \to X$ is triangular $\alpha$-admissible and modified $\alpha$-$\phi$-asymmetric Meir-Keeler contractive mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $T$ is continuous, then $T$ has a fixed point.*

*Proof.* Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We construct a sequence $\{x_n\}_{n=0}^{\infty}$ of points in $X$ in the following way:

$$x_{n+1} = Tx_n \text{ for all } n = 0, 1, 2, \cdots .$$

Notice that, if $x_{n'} = x_{n'+1}$ for some $n' \in \mathbb{N}$, then obviously $T$ has a fixed point. Thus, we suppose that

$$x_n \neq x_{n+1} \tag{4}$$

for all $n \in \mathbb{N}$.

By $(G2)$, we have

$$G(x_n, x_{n+1}, x_{n+1}) > 0 \tag{5}$$

for all $n = 0, 1, 2, \ldots$. By applying Lemma 1.13 we get,

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n$$

Define $t_n = \phi(G(x_n, x_{n+1}, x_{n+1}))$. By (3), we observe that for all $n = 0, 1, 2, \ldots$,

$$\begin{aligned} \phi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \phi(G(Tx_n, T^2x_n, Tx_{n+1})) \\ &< \phi(G(x_n, Tx_n, x_{n+1})) \\ &= \phi(G(x_n, x_{n+1}, x_{n+1})) \end{aligned} \tag{6}$$

In view of (6), the sequence $\{t_n\}$ is decreasing sequence in $\mathbb{R}^+$ and thus it is convergent, say $t \in \mathbb{R}^+$. We claim that $t = 0$. Suppose, to the contrary, that $t > 0$. Thus, we have

$$0 < t \leq \phi(G(x_n, x_{n+1}, x_{n+1})) \text{ for all } n = 0, 1, 2, \ldots. \tag{7}$$

Assume $\varepsilon = t > 0$. Then by hypothesis, there exists a convenient $\delta(\varepsilon) > 0$ such that (2) holds. On the other hand, due to the definition of $\varepsilon$, there exists $n_0 \in \mathbb{N}$ such that

$$\varepsilon \leq t_{n_0} = \phi(G(x_{n_0}, x_{n_0+1}, x_{n_0+1})) = \phi(G(x_{n_0}, Tx_{n_0}, x_{n_0+1})) < \varepsilon + \delta. \tag{8}$$

Taking the condition (2) into account, the expression (8) yields that

$$t_{n_0+1} = \phi(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})) = \phi(G(Tx_{n_0}, T^2x_{n_0}, Tx_{n_0+1})) < \varepsilon = t \tag{9}$$

which contradicts (7). Hence $t = 0$, that is, $\lim_{n\to\infty} t_n = 0$. Since $\phi$ is continuous and $\phi(t) = 0$ if and only if $t = 0$, we conclude

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Define $s_n = G(x_n, x_{n+1}, x_{n+1})$. Then $\{s_n\}$ is a decreasing sequence. Suppose, to the contrary, that there exists $n_0$ such that

$$G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2}) < G(x_{n_0}, x_{n_0+1}, x_{n_0+1}).$$

Then by property of the function $\phi$ we deduce,

$$\phi(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2}) \leq \phi(G(x_{n_0}, x_{n_0+1}, x_{n_0+1})).$$

It contradicts the inequality (6). Hence, $\{s_n\}$ is a decreasing sequence. We shall show that $\{x_n\}_{n=0}^{\infty}$ is a $G$-Cauchy sequence. For this purpose, at first we show that

$$\lim_{m,n\to\infty} G(x_m, Tx_m, x_n) = 0. \tag{10}$$

Suppose, to the contrary, that there exists $\varepsilon > 0$, and a subsequence $x_{n(k)}$ of $x_n$ such that

$$G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}) \geq \varepsilon \tag{11}$$

with $n(k) \geq m(k) > k$. Since $\phi$ is altering distance function (that is, nondecreasing), we have

$$\phi(G(x_{m(k)}, Tx_{m(k)}, x_{n(k)})) \geq \phi(\varepsilon) = \varepsilon_0 > 0. \tag{12}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (11). Hence,

$$G(x_{m(k)}, Tx_{m(k)}, x_{n(k)-1}) < \varepsilon \tag{13}$$

By Proposition 1.7 (iii) and (G5) we have

$$\begin{aligned}
\varepsilon &\leq G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}) = G(x_{n(k)}, x_{m(k)}, Tx_{m(k)}) \\
&\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, Tx_{m(k)}, x_{m(k)}) \\
&\leq G(x_{m(k)}, Tx_{m(k)}, x_{n(k)-1}) + 2s_{n(k)-1} \\
&\leq \varepsilon + 2s_{n(k)-1}.
\end{aligned} \tag{14}$$

Letting $k \to \infty$ in (14) we derive that

$$\lim_{k\to\infty} G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}) = \varepsilon. \tag{15}$$

Also, by Proposition 1.7 (iii) and (G5) we obtain the following inequalities

$$\begin{aligned}
G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}) &\leq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, Tx_{m(k)}, x_{n(k)}) \\
&= G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{n(k)}, x_{m(k)-1}, Tx_{m(k)}) \\
&\leq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) \\
&+G(x_{n(k)-1}, x_{m(k)-1}, Tx_{m(k)}) \\
&\leq 2s_{m(k)-1} + 2s_{n(k)-1} + G(x_{n(k)-1}, x_{m(k)-1}, Tx_{m(k)})
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
G(x_{n(k)-1}, x_{m(k)-1}, Tx_{m(k)}) &\leq G(x_{n(k)-1}, x_n, x_n) + G(x_{n(k)}, x_{m(k)-1}, Tx_{m(k)}) \\
&= G(x_{n(k)-1}, x_n, x_n) + G(x_{m(k)-1}, Tx_{m(k)}, x_{n(k)}) \\
&\leq G(x_{n(k)-1}, x_n, x_n) + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) \\
&+G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}) \\
&= s_{n(k)-1} + s_{m(k)-1} + G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}).
\end{aligned} \tag{17}$$

Letting $k \to \infty$ in (16) and (17) and applying (15) we find that

$$\lim_{k\to\infty} G(x_{n(k)-1}, x_{m(k)-1}, Tx_{m(k)}) = \varepsilon. \tag{18}$$

Again by Proposition 1.7 (iii) and (G5) we have,

$$\begin{aligned} G(x_{n(k)-1}, x_{m(k)-1}, Tx_{m(k)}) &= G(Tx_{m(k)}, x_{m(k)-1}, x_{n(k)-1}) \\ &= G(x_{m(k)+1}, x_{m(k)-1}, x_{n(k)-1}) \\ &\leq G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)-1}, x_{n(k)-1}) \\ &= G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) \\ &\leq 2s_{m(k)} + G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) \\ &= 2s_{m(k)} + G(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1}) \end{aligned} \tag{19}$$

and

$$\begin{aligned} G(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1}) &= G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) \\ &\leq G(x_{m(k)-1}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}) \\ &\leq G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) \\ &\quad G(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}) \\ &= s_{m(k)-1} + s_{m(k)} + G(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}) \\ &= s_{m(k)-1} + s_{m(k)} + G(x_{m(k)}, Tx_{m(k)}, x_{n(k)-1}) \\ &< s_{m(k)-1} + s_{m(k)} + \varepsilon. \end{aligned} \tag{20}$$

Taking limit as $k \to \infty$ in (19) and (20) and applying (18) we have,

$$\lim_{k\to\infty} G(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{21}$$

Since $\phi$ is continuous, we have

$$\lim_{k\to\infty} \phi(G(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1})) = \phi(\varepsilon) = \varepsilon_0. \tag{22}$$

For this $\varepsilon_0$, there exists $\delta > 0$ such that

$$\varepsilon_0 \leq \phi(G(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1})) < \varepsilon_0 + \delta \tag{23}$$

this implies that

$$\phi(G(Tx_{m(k)-1}, T^2x_{m(k)-1}, Tx_{n(k)-1})) = \phi(G(x_{m(k)}, Tx_{m(k)}, x_{n(k)})) < \varepsilon_0. \tag{24}$$

Taking limit as $n \to \infty$ in the above inequality and applying (15) we get,

$$\phi(G(x_{m(k)}, Tx_{m(k)}, x_{n(k)})) < \varepsilon_0$$

which contradicts (12). Hence, (10) holds. That is,

$$\lim_{m,n\to\infty} G(x_m, Tx_m, x_n) = \lim_{m,n\to\infty} G(x_m, x_{m+1}, x_n) = 0. \tag{25}$$

From (4) we know that $x_m \neq x_{m+1}$. Then by (G3) and Proposition 1.7 (iii) we have,

$$G(x_m, x_m, x_n) \leq 2G(x_n, x_n, x_m) \leq 2G(x_n, x_m, x_{m+1}) = 2G(x_m, x_{m+1}, x_n).$$

Taking limit as $m, n \to \infty$ in the above inequality and applying (25) we get,

$$\lim_{m,n\to\infty} G(x_m, x_m, x_n) = 0.$$

Hence, $\{x_n\}$ is a $G$-Cauchy sequence. Since $(X, G)$ is $G$-complete, there exists $w \in X$ such that $x_n \to w$ as $n \to \infty$. Since, $G$ is a continuous function, we have,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, w) = G(w, w, w) = 0. \tag{26}$$

As $T$ and $G$ are continuous, so we have,

$$G(w, Tw, w) = \lim_{n\to\infty} G(x_n, Tx_n, w) = \lim_{n\to\infty} G(x_n, x_{n+1}, w) = G(w, w, w) = 0.$$

That is, $w = Tw$. □

**Theorem 2.4.** *Let $(X, G)$ be a G-complete G-metric space and $\phi \in \Psi$. Suppose that $T : X \to X$ is $\alpha$-admissible and modified $\alpha$-$\phi$-asymmetric Meir-Keeler contractive mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Suppose that*

*(T3) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then T has a fixed point.*

*Proof.* Condition (T3) implies property (T2) in definition of triangular $\alpha$-admissible map. Indeed, if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$, then applying (T3) to $(x_n)$ defined by

$$x_1 := x,\ x_2 := y,\ x_n := z \ \ for \ \ n \geq 3,$$

we get $\alpha(x_n, z) \geq 1$ for $n \in \mathbb{N}$, and hence $\alpha(x, z) \geq 1$. Now as in the proof of Theorem 2.3 we get,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, w) = G(w, w, w) = 0. \tag{27}$$

By (T3) we get, $\alpha(x_n, w) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. By (3), we have

$$\begin{aligned} \phi(G(x_n, x_{n+1}, Tw)) &= \phi(G(Tx_n, T^2x_n, Tw)) \\ &< \phi(G(x_n, Tx_n, w)) = \phi(G(x_n, x_{n+1}, w)). \end{aligned}$$

Using continuity of $\phi$ and letting $n \to \infty$, the inequality (**??**) yields that $\phi(G(w, w, Tw)) \leq \phi(G(w, w, w)) = 0$. Consequently, we have $G(w, w, Tw) = 0$. Hence by $(G2)$, we have $Tw = w$ as required. □

**Example 2.5.** *Let $X = \mathbb{R}_+$. Define, $G : X^3 \to [0, \infty)$ by,*

$$G(x, y, z) = \begin{cases} 0, & if\, x = y = z \\ \max\{x, y\} + \max\{y, z\} + \max\{x, z\}, & otherwise \end{cases}.$$

*Clearly, $(X, G)$ is a G-complete G-metric space. Define, $T : X \to X$, $\alpha : X^2 \to (\infty, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$ by*

$$Tx = \begin{cases} \frac{1}{4}x, & if\ x \in [0, 1] \\ x^2 + 2|x - 2||x - 3| \ln x, & if\ x \in (1, \infty) \end{cases}, \quad \alpha(x, y) = \begin{cases} 8, & if\ x, y \in [0, 1] \\ 0, & otherwise \end{cases}$$

*and* $\phi(t) = t$. *Let,* $\alpha(x, y) \geq 1$, *then* $x, y \in [0, 1]$. *At first, assume that* $x \leq y$. *Then,*

$$G(x, Tx, y) = \max\{x, Tx\} + \max\{Tx, y\} + \max\{x, y\} = x + 2y$$

*and*

$$G(Tx, T^2x, Ty) = \max\{Tx, T^2x\} + \max\{T^2x, Ty\} + \max\{Tx, Ty\} = \frac{1}{4}(x + 2y).$$

*Next, assume that,* $y < x$. *Then,*

$$G(x, Tx, y) = \max\{x, Tx\} + \max\{Tx, y\} + \max\{x, y\} = 2x + \max\{\frac{1}{4}x, y\}$$

*and*

$$\begin{aligned} G(Tx, T^2x, Ty) &= \max\{Tx, T^2x\} + \max\{T^2x, Ty\} + \max\{Tx, Ty\} \\ &= \frac{1}{4}(2x + \max\{\frac{1}{4}x, y\}). \end{aligned}$$

*Let* $\varepsilon > 0$. *Then for all* $\delta = 3\varepsilon$ *condition (2) holds. Again if* $\alpha(x, y) \geq 1$, *then* $x, y \in [0, 1]$. *On the other hand, for all* $w \in [0, 1]$, *we have* $Tw \leq 1$. *Hence* $\alpha(Tx, Ty) \geq 1$. *Further, if* $\alpha(x, y) \geq 1$ *and* $\alpha(y, z) \geq 1$, *then* $x, y, z \in [0, 1]$. *Thus* $\alpha(x, z) \geq 1$. *This implies that T is a triangular* $\alpha$*-admissible mapping. Clearly,* $\alpha(0, T0) \geq 1$.

*Now, if* $\{x_n\}$ *is a sequence in X such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$. *Then* $\{x_n\} \subseteq [0, 1]$ *and hence* $x \in [0, 1]$. *This implies that* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *Thus all of the conditions of Theorem 2.4 hold and T has a fixed point. Now if,*

$$\epsilon \leq \phi(d(0, 1)) = 1 < \epsilon + \delta$$

*for an* $\epsilon > 0$ *and* $\delta > 0$ *where d is a usual metric on X. Then,*

$$\alpha(0, 1)\phi(d(T0, T1)) = 8.\frac{1}{4} = 2 > 1 = \phi(d(0, 1)) \geq \epsilon.$$

*That is, Theorems 6 and 8 of [19] can not be applied to this Example.*

From Theorem 2.3, we can deduce the following corollary.

**Corollary 2.6.** *Let* $(X, G)$ *be a G-complete G-metric space. Suppose that* $T : X \to X$ *is a triangular* $\alpha$*-admissible mapping satisfying the following condition:*
*for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \Rightarrow G(Tx, T^2x, Ty) < \varepsilon \tag{28}$$

*for all* $x, y \in X$ *with* $\alpha(x, y) \geq 1$. *If there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and T is continuous, then T has a fixed point.*

From Theorem 2.4, we can deduce the following result.

**Corollary 2.7.** *Let* $(X, G)$ *be a G-complete G-metric space. Suppose that* $T : X \to X$ *is an* $\alpha$*-admissible mapping satisfying the following condition:*
*for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \Rightarrow G(Tx, T^2x, Ty) < \varepsilon \tag{29}$$

*for all* $x, y \in X$ *with* $\alpha(x, y) \geq 1$. *If there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and (T3) holds, then T has a fixed point.*

By taking $\alpha(x, y) = 1$ for all $x, y \in X$, in the above corollary we deduce the following result.

**Corollary 2.8.** *Let* $(X, G)$ *be a G-complete G-metric space. Suppose that* $T : X \to X$ *satisfies the following condition:*
*for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \Rightarrow G(Tx, T^2x, Ty) < \varepsilon \tag{30}$$

*for all* $x, y \in X$. *Then, T has a fixed point.*

## 3. Fixed Point Results in Partially Ordered G-Metric Spaces

Existence of fixed points in partially ordered metric and $G$-metric spaces has been considered recently by many authors (see, [1, 13, 16, 33, 35–37]). In this section, as an application of obtained results, we prove some fixed point results in partially ordered G-metric spaces.

Recall that if $(X, \preceq)$ is a partially ordered set and $T : X \to X$ is such that for $x, y \in X$, $x \preceq y$ implies $Tx \preceq Ty$, then mapping $T$ is said to be non-decreasing.

**Theorem 3.1.** *Let $(X, G, \preceq)$ be a partially ordered G-complete G-metric space and $T : X \to X$ be a non-decreasing mapping. Assume that given $\varepsilon > 0$ there exist $\phi \in \Psi$ and $\delta > 0$ such that*

$$\varepsilon \leq \phi(G(x, Tx, y)) < \varepsilon + \delta \Rightarrow \phi(G(Tx, T^2x, Ty)) < \varepsilon \tag{31}$$

*for all $x, y \in X$ with $x \preceq y$. Suppose there exists $x_0$ in $X$ such that, $x_0 \preceq Tx_0$. Now if $T$ is continuous, then $T$ has a fixed point.*

*Proof.* Define the mapping $\alpha : X \times X \to \mathbb{R}_+$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

From (31), for given $\varepsilon > 0$ there exist $\phi \in \Psi$ and $\delta > 0$ such that

$$\varepsilon \leq \phi(G(x, Tx, y)) < \varepsilon + \delta \Rightarrow \phi(G(Tx, T^2x, Ty)) < \varepsilon$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Again let $x, y \in X$ such that $\alpha(x, y) \geq 1$. This implies that $x \preceq y$. As the mapping $T$ is non-decreasing, we deduce that $Tx \preceq Ty$ and hence $\alpha(Tx, Ty) \geq 1$. Also, let, $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then, $x \preceq z$ and $z \preceq y$. So from transitivity we have, $x \preceq y$. That is, $\alpha(x, y) \geq 1$. Thus $T$ is a triangular $\alpha$-admissible mapping. Since $x_0 \preceq Tx_0$ then $\alpha(x_0, Tx_0) \geq 1$. Therefore, all the hypotheses of Theorem 2.3 are satisfied and so $T$ has a fixed point in $X$. □

**Theorem 3.2.** *Let $(X, G, \preceq)$, $T$,* (31) *and $x_0$ be as in Theorem 3.1. If instead of continuity of $T$ we assume that for any non-decreasing sequence $\{x_n\}$ in $X$ with $x_n \to x$ as $n \to \infty$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$. Then, $T$ has a fixed point.*

*Proof.* Define, $\alpha : X \times X \to (-\infty, +\infty)$ as in proof of Theorem 3.1. Let, $\{x_n\}$ be a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. So, $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Then we have $x_n \preceq x$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Thus (T3) holds. All other conditions can be proved as in proof of Theorem 3.1. Consequently, $T$ has a fixed point in $X$ by Theorem 2.4. □

We can deduce the following corollaries from the above theorems.

**Corollary 3.3.** *Let $(X, G, \preceq)$ be a partially ordered G-complete G-metric space. Suppose that $T : X \to X$ is a non-decreasing mapping. Assume that given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \Rightarrow G(Tx, T^2x, Ty) < \varepsilon \tag{32}$$

*for all $x, y \in X$ with $x \preceq y$. Suppose there exists $x_0$ in $X$ such that, $x_0 \preceq Tx_0$ and $T$ is continuous, then $T$ has a fixed point.*

**Corollary 3.4.** *Let $(X, G, \preceq)$ be a partially ordered G-complete G-metric space. Suppose that $T : X \to X$ is a non-decreasing mapping. Assume that given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \Rightarrow G(Tx, T^2x, Ty) < \varepsilon \tag{33}$$

*for all $x, y \in X$ with $x \preceq y$. Suppose there exists $x_0$ in $X$ such that, $x_0 \preceq Tx_0$ and for any non-decreasing sequence $\{x_n\}$ in $X$ with $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Then $T$ has a fixed point in $X$.*

## 4. Fixed Point Results for Orbitally $G$-Continuous Mappings

In 1971, Ćirić [7] introduced orbitally continuous maps on metric spaces as follows.

**Definition 4.1.** *Let $(X, d)$ be a metric space. A mapping $T$ on $X$ is orbitally continuous if* $\lim_{i\to\infty} T^{n_i}x = u$ *implies* $\lim_{i\to\infty} TT^{n_i}x = Tu$ *for each* $x \in X$.

We define the notion of orbital continuity in the context of $G$-metric space as follows.

**Definition 4.2.** *Let $(X, G)$ be a G-metric space and $T : X \to X$ be a self map. We say that $T$ is orbitally G-continuous whenever* $\lim_{i\to\infty} G(T^{n_i}x, z, z) = 0$ *implies that* $\lim_{i\to\infty} G(TT^{n_i}x, Tz, Tz) = 0$ *for each* $x \in X$.

It is clear that $G$-continuous mappings are orbitally $G$-continuous. For $x \in X$, the set $O_T(x) = \{x, Tx, T^2x, ...\}$ is called an orbit of $x$ with respect to the operator $T$. The closure of $O_T(x)$ with respect to the topology of $G$-metric is denoted by $\overline{O_T(x)}$.

**Theorem 4.3.** *Let $(X, G)$ be a G-metric space and $T : X \to X$ be a self-mapping. Assume that given $\varepsilon > 0$ there exist $\phi \in \Psi$ and $\delta > 0$ such that*

$$\varepsilon \le \phi(G(x, Tx, y)) < \varepsilon + \delta \Rightarrow \phi(G(Tx, T^2x, Ty)) < \varepsilon \tag{34}$$

*for all distinct $x, y \in \overline{O_T(x)}$ with $Tx = y$. Suppose also that*

(C) *for some $x_0 \in X$, the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$.*

*Then, $z$ is a fixed point of $T$ in $\overline{O_T(x_0)}$ provided that $T$ is orbitally continuous at $z$.*

*Proof.* We construct a sequence $\{x_n\}_{n=0}^{\infty}$ of points in $X$ in the following way:

$$x_{n+1} = Tx_n \text{ for all } n = 0, 1, 2, \cdots .$$

Notice that, if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then obviously $T$ has a fixed point. Thus, we suppose that

$$x_n \neq x_{n+1} \tag{35}$$

for all $n \in \mathbb{N}$. By $(G2)$, we have

$$G(x_n, x_{n+1}, x_{n+1}) > 0 \tag{36}$$

for all $n = 0, 1, 2, \ldots$. We define $t_n = \phi(G(x_n, x_{n+1}, x_{n+1}))$. By (34), we observe that for all $n = 0, 1, 2, \ldots,$

$$\begin{aligned} t_{n+1} &= \phi(G(x_{n+1}, x_{n+2}, x_{n+2})) = \phi(G(Tx_n, T^2x_n, Tx_{n+1})) \\ &< \phi(G(x_n, Tx_n, x_{n+1})) = \phi(G(x_n, x_{n+1}, x_{n+1})) = t_n. \end{aligned} \tag{37}$$

By (37), the sequence $\{t_n\}$ is (strictly) decreasing sequence in $\mathbb{R}^+$ and thus it is convergent, say $t \in \mathbb{R}^+$. We claim that $t = 0$. Suppose, to the contrary, that $t > 0$. Hence, we have

$$0 < t = \inf_{n\ge1} \phi(G(x_n, Tx_n, x_{n+1})). \tag{38}$$

Take $\varepsilon = t > 0$. For any $\delta = \delta(\varepsilon) > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\varepsilon < t_n = \phi(G(x_n, Tx_n, x_{n+1})) < \varepsilon + \delta \text{ for all } n \ge n_0.$$

In particular,

$$\varepsilon < t_{n_0} = \phi(G(x_{n_0}, Tx_{n_0}, x_{n_0+1})) < \varepsilon + \delta \text{ for all } n \ge n_0. \tag{39}$$

Taking the condition (34) into account, the expression (39) yields that

$$t_{n_0+1} = \phi(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})) = \phi(G(Tx_{n_0}, T^2x_{n_0}, Tx_{n_0+1})) < \varepsilon = t. \tag{40}$$

which contradicts (38). Hence $t = 0$, that is,

$$t = \lim_{n\to\infty} t_n = 0. \tag{41}$$

As $\phi \in \Psi$, we derive that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{42}$$

On the other hand, by assumption ($C$), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \to z$ as $k \to \infty$, that is, $\lim_{k\to\infty} G(x_{n(k)}, z, z) = 0$. Since $T$ is orbitally continuous, we have

$$\lim_{k\to\infty} G(Tx_{n(k)}, Tz, Tz) = \lim_{k\to\infty} G(x_{n(k)+1}, Tz, Tz) = 0.$$

By the modified triangle inequality (G5) together with Proposition 1.7, we have

$$G(z, Tz, Tz) \le G(z, x_{n(k)}, x_{n(k)}) + G(z, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, Tz, Tz). \tag{43}$$

Letting $k \to \infty$, the inequality (43) yields that $G(z, Tz, Tz) = 0$. Analogously, again by (G5) and Proposition 1.7, we get that $G(Tz, z, z) = 0$. Thus we have,

$$d_G(z, Tz) = G(z, Tz, Tz) + G(Tz, z, z) = 0, \tag{44}$$

and hence $Tz = z$. □

**Corollary 4.4.** *Let $(X, G)$ be a G-metric space and $T : X \to X$ be a self-mapping. Assume that for given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \le G(x, Tx, y) < \varepsilon + \delta \Rightarrow G(Tx, T^2x, Ty) < \varepsilon$$

*for all distinct $x, y \in \overline{O_T(x)}$ with $Tx = y$. Suppose also that*

(C) *for some $x_0 \in X$, the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$.*

*Then, $z$ is a fixed point of $T$ in $\overline{O_T(x_0)}$ provided that $T$ is orbitally continuous at $z$.*

## 5. Cone Asymmetric Meir-Keeler Contractive Mappings

In this section, we recall the notion of cone $G$-metric space [5], and introduce the concept of cone asymmetric Meir-Keeler contractive mapping and establish certain fixed point results for such class of mappings(see also [11]).

**Definition 5.1.** *[10] Let $E$ be a real Banach space with $\theta$ as the zero element and with norm $\|\cdot\|$. A subset $P$ of $E$ is called a cone if and only if the following conditions are satisfied:*

*(i) $P$ is closed, nonempty and $P \neq \{\theta\}$,*

*(ii) $a, b \ge 0$ and $x \in P$ implies $ax + by \in P$,*

*(iii) $x \in P$ and $-x \in P$ implies $x = \theta$.*

*Let $P \subset E$ be a cone, we define a partial ordering $\preceq$ on $E$ with respect to $P$ by $x \preceq y$ if and only if $y - x \in P$; we write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$(the interior of $P$). The cone $P \subset E$ is called normal if there is a positive real number $K$ such that for all $x, y \in E$, $\theta \preceq x \preceq y \Rightarrow \|x\| \le K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of $P$. If $K = 1$ then the cone $P$ is called monotone.*

**Definition 5.2.** *Let $(E, \|\cdot\|)$ be a real Banach space with a monotone solid cone $P$. A cone G-metric on $X$ is a mapping $G_c : X \times X \times X \longrightarrow E$ satisfying the following conditions:*

*(F1) If $x = y = z$, then $G_c(x, y, z) = \theta$,*

*(F2) $\theta \ll G_c(x, y, y)$, for any $x, y \in X$ with $x \neq y$,*

*(F3) $G_c(x, x, y) \leq G_c(x, y, z)$ for any points $x, y, z \in X$, with $y \neq z$,*

*(F4) $G_c(x, y, z) = G_c(x, z, y) = G_c(y, z, x) = \cdots$, symmetry in all three variables,*

*(F5) $G_c(x, y, z) \leq G_c(x, a, a) + G_c(a, y, z)$ for any $x, y, z, a \in X$.*

*The pair $(X, G_c)$ is called a cone G-metric space.*

**Lemma 5.3.** *([9, 23]) Let $(E, \|\cdot\|)$ be a real Banach space with a monotone solid cone $P$. Then*

$$\theta \leq x \ll y \Rightarrow \|x\| < \|y\|.$$

**Proposition 5.4.** *([23]) Let $(E, \|\cdot\|)$ be a real Banach space with a monotone solid cone $P$. If $G_c : X \times X \times X \longrightarrow E$ is a G-cone metric on $X$, then the function $G : X \times X \times X \longrightarrow [0, +\infty)$ defined by $G(x, y, z) = \|G_c(x, y, z)\|$ is a G-metric on $X$ and $(X, G)$ a G-metric space.*

**Definition 5.5.** *Let $(E, \|\cdot\|)$ be a real Banach space with a monotone solid cone $P$ and $(X, G_c)$ be a cone G-metric space. Suppose that $T : X \to X$ is a self-mapping satisfying the following condition:*
*for each $\Upsilon \in intP$ there exists $\Delta \in intP$ such that for all $x, y \in X$*

$$\begin{cases} \Upsilon - G_c(x, Tx, y) \notin intP, \\ G_c(x, Tx, y) - (\Upsilon + \Delta) \notin P, \end{cases} \Rightarrow G_c(Tx, T^2x, Ty) \ll \Upsilon. \tag{45}$$

*Then $T$ is called cone asymmetric Meir-Keeler contractive mapping.*

**Theorem 5.6.** *Let $(E, \|\cdot\|)$ be a real Banach space with a monotone solid cone $P$ and $(X, G_c)$ be a G-complete G-cone metric space and $T$ be a cone asymmetric Meir-Keeler contractive mapping on $X$. Then $T$ has a fixed point.*

*Proof.* For a given $\varepsilon > 0$, let $\varepsilon \leq G(x, Tx, y)$ where $G = \|G_c\|$. This implies

$$\frac{\varepsilon H}{\|H\|} - G_c(x, Tx, y) \notin intP \tag{46}$$

for given $H \in intP$. In fact, $\frac{\varepsilon H}{\|H\|} - G_c(x, Tx, y) \in intP$ implies,

$$G_c(x, Tx, y) \ll \frac{\varepsilon H}{\|H\|}$$

and so by Lemma 5.3 we have, $G(x, Tx, y) < \varepsilon$ which is a contradiction. Therefore (46) holds. Similarly, $G(x, Tx, y) < \varepsilon + \delta$ implies,

$$G_c(x, Tx, y) - (\frac{\varepsilon H}{\|H\|} + \frac{\delta H}{\|H\|}) \notin P. \tag{47}$$

Now, by (45), (46) and (47), we have

$$G_c(Tx, T^2x, Ty) \ll \frac{\varepsilon H}{\|H\|}.$$

Again by Lemma 5.3 we get

$$G(Tx, T^2x, Ty) < \varepsilon.$$

That is, conditions of Corollary 2.8 hold and $T$ has a fixed point. □

Similarly we, can deduce the following result.

**Theorem 5.7.** *Let $(E, \|\cdot\|)$ be a real Banach space with a monotone solid cone $P$ and $(X, G_c)$ be a G-complete G-cone metric space. Assume that for each $\Upsilon \in intP$ there exists $\Delta \in intP$ such that*

$$\begin{cases} \Upsilon - G_c(x, Tx, y) \notin intP, \\ G_c(x, Tx, y) - (\Upsilon + \Delta) \notin P, \end{cases} \Rightarrow G_c(Tx, T^2x, Ty) \ll \Upsilon.$$

*for all distinct $x, y \in \overline{O_T(x)}$ with $Tx = y$. Suppose also that*

(C) *for some $x_0 \in X$, the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$.*

*Then, $z$ is a fixed point of $T$ in $\overline{O_T(x)}$ provided that $T$ is orbitally continuous at $z$.*

## 6. Application

As an application of our results obtained in previous sections, we deduce some fixed point results for mappings satisfying a Meir-Keeler type contraction of an integral type. For this purpose let

$$Y = \left\{ \begin{array}{c} \chi : \mathbb{R}^+ \to \mathbb{R}^+, \chi \text{ is Lebesgue integrable, summable} \\ \text{on each compact subset of } \mathbb{R}^+ \text{and } \int_0^\varepsilon \chi(t)\,dt > 0 \text{ for each } \varepsilon > 0 \end{array} \right\}$$

**Theorem 6.1.** *Let $(X, G)$ be a $G-$complete G-metric space and $\phi \in \Psi$. Suppose that $T : X \to X$ is a triangular $\alpha$-admissible mapping and satisfying the following condition: For each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \int_0^{\phi(G(x,Tx,y))} \chi(t)\,dt < \varepsilon + \delta \Longrightarrow \int_0^{\phi(G(Tx,T^2x,Ty))} \chi(t)\,dt < \varepsilon \tag{48}$$

*for $\chi \in Y$ and for all $x, y, \in X$with $\alpha(x, y) \geq 1$.*

*Now if $T$ is continuous, then $T$ has a fixed point.*

*Proof.* For $\chi \in Y$, consider the function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Lambda(x) = \int_0^x \chi(t)\,dt$. We note that $\Lambda \in \Psi$. The inequality (49) implies that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \Lambda\left(\phi\left(G(x, Tx, y)\right)\right) < \varepsilon + \delta \Longrightarrow \Lambda\left(\phi\left(G\left(Tx, T^2x, Ty\right)\right)\right) < \varepsilon$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Setting $\Lambda \circ \phi = \varphi$, we have $\varphi \in \Psi$ and so $T$ is a $\varphi$-asymmetric Meir-Keeler contractive. Hence, by Theorem 2.3, $T$ has a fixed point. □

**Corollary 6.2.** *Let $(X, G)$ be a $G-$complete G-metric space and let $T : X \to X$ be a triangular $\alpha$-admissible mapping satisfying the following condition: For each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \int_0^{G(x,Tx,y)} \chi(t)\,dt < \varepsilon + \delta \Longrightarrow \int_0^{G(Tx,T^2x,Ty)} \chi(t)\,dt < \varepsilon$$

*for $\chi \in Y$ and for all $x, y, \in X$with $\alpha(x, y) \geq 1$.*

*Now if $T$ is continuous, then $T$ has a fixed point.*

Using the technique of the proof of the above theorem and Theorem 2.4, we obtain following results.

**Theorem 6.3.** *Let $(X, G)$ be a $G-$complete G-metric space and $\phi \in \Psi$. Suppose that $T : X \to X$ is an $\alpha$-admissible mapping satisfying the following condition: For each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \int_0^{\phi(G(x,Tx,y))} \chi(t)\, dt < \varepsilon + \delta \Longrightarrow \int_0^{\phi(G(Tx,T^2x,Ty))} \chi(t)\, dt < \varepsilon$$

*for $\chi \in Y$ and for all $x, y, \in X$with $\alpha(x, y) \geq 1$.*

*Now if $(T3)$ holds, then $T$ has a fixed point.*

**Corollary 6.4.** *Let $(X, G)$ be a $G-$complete G-metric space and let $T : X \to X$ be an $\alpha$-admissible mapping satisfying the following condition: For each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \int_0^{G(x,Tx,y)} \chi(t)\, dt < \varepsilon + \delta \Longrightarrow \int_0^{G(Tx,T^2x,Ty)} \chi(t)\, dt < \varepsilon$$

*for $\chi \in Y$ and for all $x, y, \in X$with $\alpha(x, y) \geq 1$.*

*Now if $(T3)$ holds, then $T$ has a fixed point.*

We also have the following result:

**Theorem 6.5.** *Let $(X, G)$ be a G-complete $G-$metric space and $T : X \to X$ is a self-mapping. Assume that given $\varepsilon > 0$ there exist $\phi \in \Psi$ and $\delta > 0$ such that*

$$\varepsilon \leq \int_0^{\phi(G(x,Tx,y))} \chi(t)\, dt < \varepsilon + \delta \Longrightarrow \int_0^{\phi(G(Tx,T^2x,Ty))} \chi(t)\, dt < \varepsilon$$

*for $\chi \in Y$ and for all $x, y \in X$*

*Suppose also that*

**(C)** for some $x_0 \in X$, the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$.

Then, $z$ is a fixed point of $T$ in $\bar{O_T}(x_0)$ provided that $T$ is orbitally continuous at $z$.

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