



Fuzzy Posets with Fuzzy Order Applied to Fuzzy Ordered Groups

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Abstract. In the framework of lattice valued structures we investigate fuzzy sub-posets of a given poset, in which the underlying set and the order are fuzzy and the reflexivity is specially defined. We introduce and investigate particular fuzzy sub-posets, e.g. fuzzy up-sets and down-sets, fuzzy convex sub-posets, fuzzy intervals etc. We describe the structure of the lattice of all fuzzy orders contained in a given crisp ordering. Then we apply these in defining a fuzzy ordered subgroup of an ordered group. Main features of fuzzy ordered subgroups are introduced and investigated like fuzzy positive and negative cone and fuzzy convex subgroups.

1. Introduction

The investigation elaborated in this paper is situated in a fuzzy framework and it consists of two parts: in the first part we deal with relational structures, and in the second these are applied in an algebraic context. More precisely, we investigate fuzzy ordered sets and apply them to introduce and deal with fuzzy ordered groups.

The structure of membership values is here a complete lattice (approach introduced firstly by Goguen, [17]). So, the co-domain lattice has no additional operations (like residuation or multiplication). The reason is our intention to use the cutworthy approach: under the classical lattice operations (meet and join) many crisp properties fulfilled by cuts are also satisfied in the fuzzy framework by the corresponding fuzzy structure.

Historically, fuzzy orderings have been investigated in the fuzzy context at the very beginning by Zadeh, [33], and then by others. These investigations were motivated not only by theoretical reasons, but also because of their applicability: orderings can model various situations in which different kinds of comparison appear; in rule based systems, in control problems, in solving relational equations. From the earlier period, orderings were introduced by Ovchinnikov (e.g., [23]). In the last decade, there were papers by Bělohlávek (many results are collected in his book [2]). Recent important results concerning fuzzy orders and their representations are related to De Baets, Bodenhofer et al. ([6–8] and the overview about weak fuzzy orders, [9], see also [16]). In most of these investigations, what is fuzzified is the ordering relation,

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while the underlying set is crisp. In addition, the authors use T -norms, or more generally, residuated lattices and corresponding operations. Fuzzy posets, lattices and similar structures were investigated by Demirci, ([13–15], the last one with Eken). In the mentioned papers fuzzy orderings are defined with respect to fuzzy equality instead of the crisp one (this approach was introduced by Höhle, [18]). There are also some recent papers in this field e.g., [10]. In [11], Ćirić, Ignjatović and Bogdanović are dealing with fuzzy equivalence classes, generalizing some known properties (e.g., partitions and semi-partitions). The same authors are investigating algebraic notions in the fuzzy framework, see [19].

Our approach to fuzzy structures is based on papers [26–30] and in particular, concerning fuzzy posets, on [31] and [32]. Fuzzy up and down-sets have been investigated in [20, 21].

Concerning fuzzy groups, we follow the well known approach initiated by Rosenfeld in [24]. Though, our topic are ordered algebraic structures, mainly ordered groups (for the classical approach see the books [4, 5]), which in fuzzy context have not been considerably investigated. Namely, Bhakat and Das (see their paper [3]) have introduced fuzzy ordered subgroups using fuzzy order and the membership values belonging to the unit interval; they have also investigated Archimedean ordered subgroups in fuzzy context. Recently Saibaba in [25] defined fuzzy lattice ordered fuzzy groups. The most extensive recent material in the topic of fuzzy ordered groups is the paper [1] by Bakhshi; in this paper the co-domain of fuzzy structures is the unit interval, while we use the complete lattice. In addition, the corresponding orderings differ.

In the present paper we firstly deal with posets and their fuzzy substructures: fuzzy sub-posets with the crisp order, then fuzzy orderings on a crisp domain, and finally we investigate fuzzy sub-posets with fuzzy ordering. We investigate cut structures, and some special fuzzy sub-posets: fuzzy up-sets and down-sets, fuzzy convex sub-posets, fuzzy intervals. We show their properties and how these are related. We present Theorem of synthesis for fuzzy posets with fuzzy order by crisp orders. Finally, we give the full description of the lattice of all weak fuzzy orders contained in the given crisp order of some poset.

In the second part we use the above mentioned results for the definition of fuzzy (lattice valued) ordered groups as a pair consisting of a fuzzy subgroup and a fuzzy sub-poset. By our relational approach we use so called weak reflexivity, which enable us to have a fuzzy order on a fuzzy structure. Then we introduce the main related features, like positive and negative cones, convexity, and we characterize fuzzy ordered subgroups using these new notions.

Let us mention that our approach to fuzzy ordered groups differs from existing ones by Bhakat and Das, [3], and Saibaba, [25]. Namely, we use particular fuzzy order which is, by reflexivity, essentially different from others, not only in order structures but also in applications. In addition, in [25], the author deals with lattice ordered groups, while our order is not a lattice one.

2. Preliminaries

In this section, some well-known definitions and preliminary results are recalled. These are necessary not only because of notation we use, but also for understanding the new concepts and corresponding properties introduced and investigated in this paper.

2.1. Order

Necessary notions from the classical order theory are listed in the sequel, together with relevant properties. For more comprehensive presentation, see e.g., books [4, 12].

A *poset* is a nonempty set X equipped with an ordering relation \leq . A poset is usually denoted as an ordered pair (X, \leq) , or simply by the underlying set X . A *sub-poset* of (X, \leq) is a poset on a subset Y of X in which the order is the one restricted from X , and usually denoted in the same way (\leq) . An *up-set* (*semi-filter*) on a poset X is any sub-poset U , satisfying the following: for $x \in U, y \in X, x \leq y$ implies $y \in U$. Dually, a *down-set* (*semi-ideal*) on X is any sub-poset D , satisfying: for $x \in D, y \in X, y \leq x$ implies $y \in D$. In particular, for an $x \in X$, a sub-poset $\uparrow x := \{y \in X \mid x \leq y\}$ is a *principal filter* generated by x . The dual notion is a *principal ideal* generated by x , denoted by $\downarrow x$. A *lattice* is a poset L in which for each pair of elements

x, y there is a greatest lower bound (glb, infimum, meet) and a least upper bound (lub, supremum, join), denoted respectively by $x \wedge y$ and $x \vee y$. These are binary operations on L . A non-empty poset L is said to be a *complete* lattice if infima and suprema exist for each subset of L . A complete lattice possesses the *top* (1) and the *bottom* element (0). A lattice L is of a *finite length* if the longest chain in L is finite. An element $a, a \neq 0$ of a lattice L is said to be a *zero divisor* (under \wedge) if there is a non-zero element $b \in L$, such that $a \wedge b = 0$. Consequently, L is a lattice *without zero divisors* if for any $x, y \in L, x \wedge y = 0$ implies $x = 0$ or $y = 0$. In other words, this is a lattice in which 0 is a meet irreducible element. We need the following known way of constructing a complete lattice.

Proposition 2.1. *Let C be a collection of subsets of a nonempty set X , containing X . If this collection is closed under set intersections and contains the greatest element, then (C, \subseteq) is a complete lattice.*

2.2. Ordered algebraic structures

Here we use the notion of an ordered group. A structure (G, \cdot) with a binary operation is a *groupoid*, which is a *group* if the operation is associative, there is a *neutral element* e ($e \cdot x = x \cdot e = x$ for every $x \in G$) and for each $x \in G$ there is an *inverse* $x^{-1} \in G$ ($x \cdot x^{-1} = x^{-1} \cdot x = e$). For groups we use the language with a binary operation, unary and a constant, denoting it by $(G, \cdot, ^{-1}, e)$.

A groupoid (G, \cdot) may be equipped with an ordering relation \leq which is *compatible* with the binary operation in the following sense: for all $x, y, z \in G$

$$x \leq y \text{ implies } z \cdot x \leq z \cdot y \text{ and } x \cdot z \leq y \cdot z. \tag{1}$$

If there is such an order on G , then the structure (G, \cdot, \leq) is an *ordered groupoid*. An *ordered group* is a structure $(G, \cdot, ^{-1}, e, \leq)$ where $(G, \cdot, ^{-1}, e)$ is a group which is ordered by \leq in the sense of (1). An example of an ordered group is the set of integers under the addition and the classical order \leq . If $(G, \cdot, ^{-1}, e, \leq)$ is an ordered group and the poset (G, \leq) is a lattice, then this structure is said to be a *lattice ordered group*.

If x is an element of an ordered group G , then x is said to be *positive* if $x \geq e$, and *negative* if $x \leq e$. The set of positive elements in G is its positive cone, denoted by P_G , and the set of negative elements, N_G , is the *negative cone* in G .

In dealing with properties of ordered structures, we mostly refer to the books [4, 5].

2.3. Fuzzy (lattice-valued) sets and structures

We present some notions from the theory of fuzzy structures with the membership values belonging to a complete lattice. More details about the relevant properties can be found e.g., in [28, 29].

Fuzzy or *lattice-valued* sets are here considered to be mappings from a non-empty set X (domain) into a complete lattice L (co-domain) with the top and bottom elements 1 and 0, respectively. This concept was introduced by Goguen (see [17]), as a generalization of the concept of a fuzzy set with the membership values belonging to the unit interval of real numbers.

If $\alpha : X \rightarrow L$ is a fuzzy set on X then, for $p \in L$, the set

$$\alpha_p := \{x \in X \mid \alpha(x) \geq p\}$$

is called the *p-cut*, a *cut set* or simply a *cut* of α .

Obviously, a *p-cut* of α is the inverse image (under α) of the principal filter in L generated by p :

$$\alpha_p = \alpha^{-1}(\uparrow p).$$

In particular, the set

$$\text{supp } \alpha := \{x \in X \mid \alpha(x) > 0\}$$

is the *support* of α .

A synthesis of a fuzzy set by its cuts is formulated in the sequel. The proof is known, see e.g., [28].

Proposition 2.2. *If $\mu : X \rightarrow L$ is a fuzzy set on X , then for every $x \in X$*

$$\mu(x) = \bigvee \{p \in L \mid x \in \mu_p\}.$$

□

A mapping $\rho : X^2 \rightarrow L$ is a *fuzzy (binary) relation* on X .

Observe that for $\mu, \nu : X \rightarrow L$, we use notation $\mu \subseteq \nu$ if for every $x \in X$, $\mu(x) \leq \nu(x)$, and the same for fuzzy relations. It is not the classical inclusion, but the notation is common in the literature. Similarly, $\mu = \bigcap_i \mu_i$ means that for every $x \in X$, we have $\mu(x) = \bigwedge_i \mu_i(x)$.

As it is known, if $(G, \cdot, ^{-1}, e)$ is a group and (L, \wedge, \vee, \leq) a complete lattice, then the mapping $\mu : G \rightarrow L$ is a *fuzzy subgroup* of G if the following holds: for all $x, y \in G$

(i) $\mu(x) \wedge \mu(y) \leq \mu(x \cdot y)$

(ii) $\mu(x) \leq \mu(x^{-1})$

(iii) $\mu(e) = 1$.

Observe that by the property of inverse elements, it is true that

$$\mu(x) = \mu(x^{-1}).$$

3. Fuzzy Poset, Fuzzy Orderings

We start with a poset and investigate its fuzzy sub-posets, keeping the crisp order. Then we consider a crisp set together with the fuzzy ordering and investigate its properties. Finally, we deal with fuzzy subsets of a poset domain equipped with the fuzzy order on it. The membership values of all fuzzy structures belong to a fixed complete lattice (L, \wedge, \vee, \leq) .

3.1. Fuzzy sub-poset

Let (P, \leq) be a poset and (L, \wedge, \vee, \leq) , as indicated, a complete lattice (observe the difference between the denotation of orderings in P and L). By k_{\leq} we denote the characteristic function of the order in P : for any $x, y \in P$

$$k_{\leq}(x, y) := \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

A mapping $\alpha : P \rightarrow L$ is a *fuzzy sub-poset* of (P, \leq) . Observe that this definition is analogous to the crisp one: every subset of a crisp poset is its sub-poset, keeping the same order restricted to its domain. Here we simply assume that the (crisp) order \leq remains the same. The following is obvious.

Lemma 3.1. *Every cut of a fuzzy sub-poset α of a poset P is a crisp sub-poset of P .* □

Some special fuzzy sub-posets are given in the sequel. The first two are defined in the paper [20].

A fuzzy sub-poset $\Upsilon : P \rightarrow L$ of (P, \leq) is a *fuzzy down-set* or a *fuzzy semi-ideal* of P if for all $x, y \in P$

$$x \leq y \text{ implies } \Upsilon(y) \leq \Upsilon(x). \tag{2}$$

Dually fuzzy sub-poset $F : P \rightarrow L$ of (P, \leq) is a *fuzzy up-set* or a *fuzzy semi-filter* of P if for all $x, y \in P$

$$x \leq y \text{ implies } F(x) \leq F(y). \tag{3}$$

As above, we have the analogous properties of cuts. We give it together with its converse, which follows by Proposition 2.2.

Proposition 3.2. *Let $\alpha : P \rightarrow L$ be a fuzzy sub-poset of (P, \leq) . Then α is a fuzzy up-set (down-set) of P if and only if every cut of α is an up-set (down-set) in P .* □

Next we present characterizations of fuzzy down-sets and up-sets, which are used in the subsequent applications.

Lemma 3.3. A fuzzy set $\mu : P \rightarrow L$ is a fuzzy down-set in P if and only if for all $x, y \in P$ the following holds:

$$\mu(x) \wedge k_{\leq}(y, x) \leq \mu(y). \tag{4}$$

Dually, μ is a fuzzy up-set on P if and only if for all $x, y \in P$

$$\mu(x) \wedge k_{\leq}(x, y) \leq \mu(y). \tag{5}$$

Proof. If μ is a fuzzy down-set on P and $y \leq x$ in P , then $k_{\leq}(y, x) = 1$ and (2) implies (4); if $y \not\leq x$, then $k_{\leq}(y, x) = 0$ and (4) holds again. Conversely, if (4) holds and $y \leq x$, then $k_{\leq}(y, x) = 1$ and (2) holds.

The second part is proved dually. □

As for the classical order sets, we introduce special fuzzy down-sets (up-sets).

For $a \in P$, the mapping $f_{\downarrow a} : P \rightarrow L$ is a *fuzzy principal ideal* generated by a , if it is a fuzzy down-set of P satisfying: for every $x \in P$

$$k_{\leq}(x, a) \wedge f_{\downarrow a}(a) \leq f_{\downarrow a}(x) \leq k_{\leq}(x, a). \tag{6}$$

Dually, for $a \in P$, the mapping $f_{\uparrow a} : P \rightarrow L$ is a *fuzzy principal filter* generated by a , if it is a fuzzy up-set of P satisfying: for every $x \in P$

$$k_{\leq}(a, x) \wedge f_{\uparrow a}(a) \leq f_{\uparrow a}(x) \leq k_{\leq}(a, x). \tag{7}$$

It is easy to see that a fuzzy principal ideal (filter) generated by $a \in P$ is any fuzzy down-set (up-set) in P whose value is 0 for all elements in P not belonging to $\downarrow a$ ($\uparrow a$).

Consequently, for $a, b \in P$, we define a *fuzzy interval* $f_{[a,b]}$ on P as a fuzzy set on P , such that for every $x \in P$

$$f_{[a,b]}(x) := (f_{\uparrow a} \cap f_{\downarrow b})(x),$$

for some $f_{\uparrow a}$ and $f_{\downarrow b}$ on P .

In the crisp case, a principal ideal or a filter generated by an element of a poset is unique, while the fuzzy analogue object is not. The same holds for a fuzzy interval.

Example 3.4. Let (P, \leq) be a poset and (L, \wedge, \vee, \leq) a lattice presented by diagrams in Figure 1. The functions

$$f_{\downarrow t} = \begin{pmatrix} a & b & c & d & t \\ t & t & s & r & p \end{pmatrix} \text{ and } f_{\uparrow a} = \begin{pmatrix} a & b & c & d & t \\ p & 0 & s & r & t \end{pmatrix},$$

are a fuzzy principal ideal and a fuzzy principal filter on P , respectively. In addition,

$$f_{[a,t]} = \begin{pmatrix} a & b & c & d & t \\ p & 0 & s & r & p \end{pmatrix}$$

is a fuzzy interval on P .

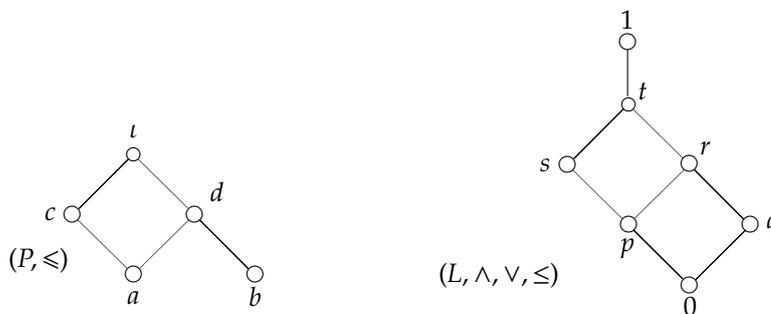


Figure 1

Further, the functions

$$g_{\downarrow t} = \begin{pmatrix} a & b & c & d & t \\ t & 1 & q & s & 0 \end{pmatrix} \quad \text{and} \quad g_{\uparrow a} = \begin{pmatrix} a & b & c & d & t \\ 0 & 0 & r & s & 1 \end{pmatrix},$$

are also a fuzzy principal ideal and a filter on P , respectively, generated correspondingly by the same elements as the previous ones. Consequently,

$$g_{[a,t]} = \begin{pmatrix} a & b & c & d & t \\ 0 & 0 & q & s & 0 \end{pmatrix}$$

is also a fuzzy interval on P . □

Next we introduce a notion of convexity for fuzzy sub-*posets*. A fuzzy set $\mu : P \rightarrow L$ is said to be a *fuzzy convex sub-poset* in P if for all $x, y, z \in P$ the following holds:

$$\mu(x) \wedge \mu(z) \wedge k_{\leq}(x, y) \wedge k_{\leq}(y, z) \leq \mu(y). \tag{8}$$

Combining the properties (4) and (5) from Lemma 3.3, we get the following way to obtain fuzzy convex sub-*posets*.

Proposition 3.5. *A fuzzy subset $\zeta : P \rightarrow L$ of P is fuzzy convex sub-poset of P if*

$$\zeta = F \cap \Upsilon,$$

for some fuzzy up-set F and fuzzy down-set Υ on P .

Proof. Let $\zeta = F \cap \Upsilon$, for a fuzzy up-set F and a fuzzy down-set Υ on P . Then

$$\begin{aligned} &\zeta(x) \wedge \zeta(z) \wedge k_{\leq}(x, y) \wedge k_{\leq}(y, z) = \\ &F(x) \wedge \Upsilon(x) \wedge F(z) \wedge \Upsilon(z) \wedge k_{\leq}(x, y) \wedge k_{\leq}(y, z) \leq \\ &F(y) \wedge \Upsilon(y) \wedge F(z) \wedge \Upsilon(x) \leq \\ &F(y) \wedge \Upsilon(y) = \zeta(y), \end{aligned}$$

and ζ is fuzzy convex. □

Again it is straightforward to check that *all the cuts of a fuzzy convex sub-poset of P are crisp convex sub-posets of P .*

Obviously, any fuzzy interval on P is a fuzzy convex sub-*poset* of P .

3.2. Fuzzy ordering on a crisp domain

As it is well known, a fuzzy relation $\rho : X^2 \rightarrow L$ on a set X is

- *reflexive* if $\rho(x, x) = 1$, for every $x \in X$; (r)

- *antisymmetric* if for all $x, y \in X$,
 $x \neq y$ implies $\rho(x, y) \wedge \rho(y, x) = 0$; (a)

- *transitive* if for all $x, y, z \in X$, $\rho(x, y) \wedge \rho(y, z) \leq \rho(x, z)$. (t)

A fuzzy relation ρ on X is a *fuzzy ordering relation* (*fuzzy order*) on X if it is reflexive, antisymmetric and transitive.

The following property is also connected to fuzzy orderings.

A fuzzy relation $\rho : X^2 \rightarrow L$ on a set X is

- *weakly reflexive* if
 $\rho(x, x) \geq \rho(x, y)$ and $\rho(x, x) \geq \rho(y, x)$, for all $x, y \in X$. (wr)

A fuzzy relation ρ on X is a *weak fuzzy ordering relation* (*weak fuzzy order*) on X if it is weakly reflexive, antisymmetric and transitive.

The following is known, see e.g., [31].

Proposition 3.6. *A relation $\rho : X^2 \rightarrow L$ is a fuzzy ordering relation on X if and only if all cuts except 0-cut are ordering relations on the same set. \square*

Proposition 3.7. *Let $\rho : X^2 \rightarrow L$ be a fuzzy ordering relation, such that L is a complete lattice without zero divisors under \wedge . Then, $\text{supp } \rho$ is a crisp ordering relation on set X .*

Observe that Proposition 3.7 essentially depends on the condition that L has no zero divisors.

If ρ is a fuzzy ordering on X , and $k_{\text{supp } \rho}$ the characteristic function of its support considered as a fuzzy set, then obviously, by the definition of the support, $\rho \subseteq k_{\text{supp } \rho}$ (due to reflexivity).

The following shows reasons for considering weak fuzzy orders.

Proposition 3.8. *If $\rho : X^2 \rightarrow L$ is a weak fuzzy ordering relation on X , and $\delta(\rho) : X \rightarrow L$, defined by $\delta(\rho)(x) := \rho(x, x)$. Then for each non-zero $p \in L$, the cut-relation ρ_p is a crisp order on the cut-subset $\delta(\rho)_p$ of X .*

Let us mention that contrary to the case of fuzzy orders, the zero fuzzy relation ($\rho(x, y) = 0$ for all $x, y \in X$) is a weak fuzzy order on X .

3.3. Fuzzy poset with fuzzy order

Here we combine the two introduced fuzzy notions related to orderings. After some preliminary notions, we start with a crisp poset and consider its fuzzy sub-poset equipped with a fuzzy order. Our approach is new, so we explain it in details.

The domain of all fuzzy structures here is a nonempty set P , equipped if indicated, with a crisp order \leq .

As it is known, a fuzzy relation $\rho : P \rightarrow L$ on a set P is a *fuzzy relation on a fuzzy subset* $\mu : P \rightarrow L$ of P for all $x, y \in P$

$$\rho(x, y) \leq \mu(x) \wedge \mu(y). \tag{9}$$

A fuzzy relation ρ on a fuzzy subset μ is *reflexive*, if for all $x \in P$

$$\rho(x, x) = \mu(x). \tag{10}$$

Observe that the notion of reflexivity here formally differs from the same notion (r) defined for a fuzzy relation on a crisp set. However, if a crisp domain is represented by its characteristic function (with all values constantly equal 1), then the notion (9) coincides with (r). Finally, to connect this with the notion of weak reflexivity (wr), we have the following obvious fact.

Every fuzzy relation $\rho : P \rightarrow L$ which is reflexive on a fuzzy set $\mu : P \rightarrow L$ is weakly reflexive on P .

We say that a fuzzy relation ρ on a fuzzy subset μ is a *fuzzy ordering on μ* , if it is reflexive (in the sense of (9)), antisymmetric as defined by (a) and transitive in the sense of (t).

Now we introduce a fuzzy poset with fuzzy ordering as follows. Let (P, \leq) be a crisp poset and $\mu : P \rightarrow L$ its fuzzy sub-poset. Further, let $\rho : P \rightarrow L$ be a fuzzy relation on P defined as follows:

$$\rho(x, y) := \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y), \tag{11}$$

where k_{\leq} is the characteristic function of the crisp order \leq on P .

The above definition is new, though similar to a fuzzy relation introduced in papers [31] and [32].

Theorem 3.9. *The function ρ defined by (11) is a fuzzy order on fuzzy set μ on P .*

Now, if (P, \leq) is a poset, then a pair (μ, ρ) is a *fuzzy poset with fuzzy ordering* if $\mu : P \rightarrow L$ is a fuzzy subset of P and $\rho : P^2 \rightarrow L$ is the fuzzy ordering on P defined by (11):

$$\rho(x, y) = \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y).$$

Theorem 3.10. Let (P, \leq) be a poset, $\mu : P \rightarrow L$ a fuzzy subset of P , and $\rho : P^2 \rightarrow L$ a fuzzy relation on μ . Then (μ, ρ) is a fuzzy poset with fuzzy order on (P, \leq) , if and only if for every $p \in L, p \neq 0$, pair (μ_p, ρ_p) is a crisp sub-poset of (P, \leq) .

Proof. Suppose that (μ, ρ) is a fuzzy poset with a fuzzy order on (P, \leq) . We prove that for any $p \in L, p \neq 0$, the relation ρ_p is an ordering relation on $\mu_p \subseteq P$. Similarly as in the proof of Proposition 3.8, one can show that ρ_p is a relation on μ_p . Next we prove its ordering properties.

Reflexivity: for any $x \in \mu_p, (x, x) \in \rho_p$ if and only if $\rho(x, x) \geq p$ if and only if $\mu(x) \wedge k_{\leq}(x, x) \geq p$ if and only if $\mu(x) \geq p$ if and only if $x \in \mu_p$.

Antisymmetry: If $(x, y), (y, x) \in \rho_p$ for $x, y \in \mu_p$, then $\rho(x, y) \geq p$ and $\rho(y, x) \geq p$. Hence $\rho(x, y) \wedge \rho(y, x) \geq p$. By antisymmetry of ρ , if $x \neq y$ then $\rho(x, y) \wedge \rho(y, x) = 0$, therefore $x = y$.

Transitivity: If $(x, y), (y, z) \in \rho_p$ for $x, y, z \in \mu_p$, then $\rho(x, y) \geq p$ and $\rho(y, z) \geq p$, and by transitivity of ρ , $\rho(x, z) \geq p$, finally $(x, z) \in \rho_p$.

Conversely, suppose that for every $p \in L, p \neq 0$, pair (μ_p, ρ_p) is a crisp sub-poset of (P, \leq) . Since ρ is by assumption of the theorem a relation on μ , we have $\rho(x, y) \leq \mu(x) \wedge \mu(y)$. If $x \leq y$, we have also

$$\rho(x, y) \leq \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y).$$

Let $\mu(x) \wedge \mu(y) = p$, hence $\mu(x) \geq p$ and $\mu(y) \geq p$. Therefore $\mu_p(x) = \mu_p(y) = 1$ and by $x \leq y$ we get $\rho_p(x, y) = 1$, implying $\rho(x, y) \geq p$. Therefore

$$\rho(x, y) \geq \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y).$$

The opposite inequality as shown above holds, hence we have the equality.

If $x \not\leq y$, then for every $p \in L, p \neq 0$ we have $\rho_p(x, y) = 0$. Hence by Proposition 2.2 applied to fuzzy relation, $\rho(x, y) = 0$. From $x \not\leq y$ it also follows that $k_{\leq}(x, y) = 0$ and the equality

$$\rho(x, y) = \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y)$$

holds also in this case. □

The synthesis of a fuzzy poset by crisp ones being its cuts is straightforward, as follows. Observe that a sub-poset of a given poset P is its nonempty subset Q , ordered by the restriction to Q of the order on P . The construction presented in the following proof is similar to other analogue constructions (see e.g., [28, 29]), though the proof itself has some specific parts.

Theorem 3.11. Let \mathcal{F} be a collection of sub-posets of a poset (P, \leq) , closed under set intersections and containing P as a member. Then there is a lattice L and a fuzzy ordered (L -valued) fuzzy sub-poset (M, ρ) of P so that the collection of its cuts coincides with \mathcal{F} . Moreover, the crisp order on each cut is the corresponding cut of ρ .

Proof. Consider the pair (\mathcal{F}, \supseteq) , i.e., the given collection of sub-posets of P ordered by the dual of set inclusion. By Proposition 2.1 this is a complete lattice and we take it to be the membership values structure, denoted from now on by (L, \leq) . Next, if $M : P \rightarrow L$ and $\rho : P^2 \rightarrow L$ are defined respectively by

$$M(x) := \bigcap \{ \phi \in \mathcal{F} \mid x \in \phi \}, \text{ and } \rho(x, y) := M(x) \wedge M(y) \wedge k_{\leq}(x, y),$$

then it is straightforward to check that (M, ρ) is a fuzzy poset with fuzzy order on P . In addition, the cuts of M coincide with the sub-posets of the collection \mathcal{F} , namely for every $\phi \in \mathcal{F}$, we have $M_\phi = \phi$. Still we have to prove that for every $\phi \in \mathcal{F}$, such that ϕ is not the bottom (0) of L , we have that the pair $(M_\phi, \rho_\phi) = (\phi, \rho_\phi)$ is a crisp sub-poset of P . Indeed, for $x, y \in \phi$,

$(x, y) \in \rho_\phi$ if and only if $\rho(x, y) \geq \phi$ if and only if $M(x) \wedge M(y) \wedge k_{\leq}(x, y) \geq \phi \neq 0$ if and only if $M(x) \geq \phi$, $M(y) \geq \phi$, and $k_{\leq}(x, y) = 1$ if and only if $x, y \in \phi$ and $x \leq y$. □

3.4. Structure of all weak fuzzy suborders of (P, \leq)

As defined above, we are investigating all fuzzy sub–posets with fuzzy ordering, denoted with (μ, ρ) , of a given poset (P, \leq) . The relation ρ is contained in the crisp order \leq , (represented by its characteristic function), and it is a fuzzy order on fuzzy set μ . Considered as a fuzzy relation on P it is a weak fuzzy order on this crisp domain. Clearly, there are other weak fuzzy orders on P also contained in this crisp order.

Here we present a full description of the structure of all these weak fuzzy orderings.

So, let (P, \leq) be a poset and \mathcal{FP} the collection of all weak fuzzy orders on P , contained in \leq :

$$\mathcal{FP} := \{\rho : P^2 \rightarrow L \mid \rho \subseteq k_{\leq} \text{ and } \rho \text{ is a weak fuzzy order on } P\}.$$

The above inclusion is componentwise defined, and the whole collection can be ordered by the same relation: for $\rho, \sigma \in \mathcal{FP}$,

$$\rho \subseteq \sigma \text{ if for all } x, y \in P, \rho(x, y) \leq \sigma(x, y).$$

In the following, Δ is the diagonal relation of P : $\Delta = \{(x, x) \mid x \in P\}$, and by $\uparrow\Delta, \downarrow\Delta$ we denote respectively the filter and the ideal in the poset $(\mathcal{FP}, \subseteq)$, generated by Δ .

Theorem 3.12. For a given poset (P, \leq) , the following holds:

- (i) The structure $(\mathcal{FP}, \subseteq)$ is a complete lattice;
- (ii) $\uparrow\Delta$ consists of all fuzzy orders on P ;
- (iii) $\downarrow\Delta$ is isomorphic to the lattice of all fuzzy subsets of P ;
- (iv) If μ is a fuzzy set on P and $\underline{\rho(\mu)}, \overline{\rho(\mu)} \in \mathcal{FP}$ such that

$$\underline{\rho(\mu)}(x, y) = \begin{cases} \mu(x) & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases} \quad \overline{\rho(\mu)}(x, y) = \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y),$$

then the interval $[\underline{\rho(\mu)}, \overline{\rho(\mu)}]$ consists of all $\sigma \in \mathcal{FP}$ with $\sigma(x, x) = \mu(x)$.

- (v) $\mathcal{FP} = \bigcup\{[\underline{\rho(\mu)}, \overline{\rho(\mu)}] \mid \mu : P \rightarrow L\}$.

Proof. (i) The poset $(\mathcal{FP}, \subseteq)$ is closed under arbitrary intersections. Indeed, let ρ be the (componentwise defined) intersection of a family $\{\rho_i \mid i \in I\} \subseteq \mathcal{FP}$:

$$\rho = \bigcap\{\rho_i \mid i \in I\}.$$

It is straightforward to check that ρ is weakly reflexive, antisymmetric and transitive relation on P , i.e., that $\rho \in \mathcal{FP}$ (ρ may be the zero relation, which is also a weak fuzzy order). All fuzzy relations in \mathcal{FP} are contained (under \subseteq) in k_{\leq} which is the greatest element in \mathcal{FP} . Hence, the collection \mathcal{FP} is closed under intersections and contains the greatest element. By Proposition 2.1 it is a complete lattice.

(ii) Since for every $x \in P$, we have $\Delta(x, x) = 1$, the same holds for every fuzzy relation $\rho \in \uparrow\Delta$, i.e., $\rho(x, x) = 1$ and ρ is a fuzzy order which is reflexive.

(iii) $\downarrow\Delta$ consists of all diagonal fuzzy relations on P . If $\mathcal{F}(P)$ is the lattice of all fuzzy sub–sets of P , then the mapping $f : \downarrow\Delta \rightarrow \mathcal{F}(P)$ such that for $\rho \in \downarrow\Delta$, we have $f(\rho) = \mu$, with $\mu(x) = \rho(x, x)$, then it is easy to check that f is an order isomorphism.

(iv) Let $\mu : P \rightarrow L$ be an arbitrary fuzzy subset of P . Then $\underline{\rho(\mu)}$, as defined above, is a diagonal fuzzy relation belonging to $\downarrow\Delta$ i.e., to \mathcal{FP} . In addition, $\overline{\rho(\mu)}$ is by Theorem 3.9 a fuzzy order on μ , hence it is a weak fuzzy order on P . Clearly, for every fuzzy relation $\sigma \in [\underline{\rho(\mu)}, \overline{\rho(\mu)}]$ the diagonal is fixed: $\sigma(x, x) = \mu(x)$. On the other hand, if $\rho \in \mathcal{FP}$ and $\rho(x, x) = \mu(x)$, then $\underline{\rho(\mu)} \subseteq \rho$. By weak reflexivity $\rho(x, y) \leq \rho(x, x) = \mu(x)$ and analogously $\rho(x, y) \leq \mu(y)$. Since also $\rho(x, y) \leq k_{\leq}(x, y)$, it follows that, by the definition of $\overline{\rho(\mu)}$, $\rho \leq \overline{\rho(\mu)}$. Therefore, $\rho \in [\underline{\rho(\mu)}, \overline{\rho(\mu)}]$.

- (v) By (iii), if $\rho \in \mathcal{FP}$ and $\mu : P \rightarrow L$ such that $\mu(x) = \rho(x, x)$, then $\rho \in [\underline{\rho(\mu)}, \overline{\rho(\mu)}]$ hence

$$\mathcal{FP} \subseteq \bigcup\{[\underline{\rho(\mu)}, \overline{\rho(\mu)}] \mid \mu : P \rightarrow L\}.$$

The opposite inclusion is obvious, which proves the equality. □

4. Fuzzy-Ordered Structures

Here we connect the above introduced fuzzy ordering notions with fuzzy algebraic structures.

Namely, we take a crisp ordered group and consider both, a subgroup and a sub-poset to be its fuzzy substructures. For this purpose we essentially use our approach to fuzzy posets, developed above.

4.1. Compatibility

First we introduce the compatibility of a fuzzy relation with a binary operation, both, on a crisp as well as on a fuzzy algebraic structure.

Let (G, \cdot) be a groupoid and $\rho : G^2 \rightarrow L$ a fuzzy relation on G . We say that ρ is *compatible with operation " \cdot "* on G , if for all $x, y, z \in G$ the following holds:

$$\rho(x, y) \leq \rho(x \cdot z, y \cdot z) \wedge \rho(z \cdot x, z \cdot y). \tag{12}$$

Let (G, \cdot) be a groupoid and $\mu : G \rightarrow L$ its fuzzy subgroupoid. We say that a fuzzy relation $\rho : G^2 \rightarrow L$ on μ is *compatible with operation " \cdot "* on μ , if for all $x, y, z \in G$ the following holds:

$$\mu(z) \wedge \rho(x, y) \leq \rho(x \cdot z, y \cdot z) \wedge \rho(z \cdot x, z \cdot y). \tag{13}$$

In the crisp case, i.e., if L is a two-element chain, then (12) reduces to the classical compatibility of order on G , and (13) gives the compatibility of the order restricted to the subgroupoid of G .

Finally, let us mention that a fuzzy relation ρ on a crisp groupoid G which is compatible with the operation in the sense of (12), is for every fuzzy subgroupoid μ of G also compatible in the sense of (13). The converse does not hold, i.e., compatibility over a fuzzy subgroupoid does not imply compatibility over G .

4.2. Fuzzy-ordered subgroup

Here we introduce the notion of a fuzzy ordered subgroup. Like before, (L, \wedge, \vee, \leq) is a complete lattice of membership values.

Proposition 4.1. *Let $(G, \cdot, ^{-1}, e, \leq)$ be an ordered group and $\mu : G \rightarrow L$ a fuzzy subgroup of G . The fuzzy relation $\rho : G^2 \rightarrow L$ on μ defined by (11), i.e., by*

$$\rho(x, y) = \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y),$$

is a fuzzy order on μ which is compatible with the group operation in the sense of (13).

Proof. By Theorem 3.9, ρ is a fuzzy ordering relation on μ . We prove that it is compatible with \cdot , i.e., that it fulfills (13). Indeed, for $x, y, z \in G$ we have

$$\begin{aligned} \mu(z) \wedge \rho(x, y) &= \\ &= \mu(z) \wedge \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y) \leq \mu(x \cdot z) \wedge \mu(y \cdot z) \wedge k_{\leq}(x \cdot z, y \cdot z) = \rho(x \cdot z, y \cdot z), \end{aligned}$$

since μ is a fuzzy subgroup of G , and the order \leq on G is compatible with the group operation.

Similarly one can prove the formula

$$\mu(z) \wedge \rho(x, y) \leq \rho(z \cdot x, z \cdot y),$$

hence (13) holds. □

Now we are ready for the definition.

Let $(G, \cdot, ^{-1}, e, \leq)$ be an ordered group. Let also $\mu : G \rightarrow L$ and $\rho : G^2 \rightarrow L$ be a fuzzy set on G and a fuzzy relation on μ , respectively. The pair (μ, ρ) is a *fuzzy ordered subgroup of G* if the following hold:

1. μ is a fuzzy subgroup of G ;

2. ρ is the fuzzy relation on μ defined by

$$\rho(x, y) = \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y).$$

Obviously, by Proposition 4.1, ρ is a fuzzy ordering relation on μ , compatible with the group operation.

Example 4.2. Let $(\mathbb{Z}, +, -, 0, \leq)$ be the additive group of integers under the usual order. Let (L, \leq) be a four element chain: $0 < \mathbf{a} < \mathbf{b} < \mathbf{1}$. Denoting as usual x divides y by $x|y$, we define a fuzzy set $\mu : \mathbb{Z} \rightarrow L$ as follows: for $x \in \mathbb{Z}$

$$\mu(x) = \begin{cases} \mathbf{1} & \text{if } 12|x \\ \mathbf{b} & \text{if } 6|x \text{ and } 4 \nmid x \\ \mathbf{a} & \text{if } 3|x \text{ and } 2 \nmid x \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

It is straightforward to check that μ is a fuzzy subgroup of $(\mathbb{Z}, +, -, 0)$.

Further, let $\rho : \mathbb{Z}^2 \rightarrow L$ be the fuzzy relation on \mathbb{Z} defined by (11):

$$\rho(x, y) := \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y).$$

Then we have for all $x, y \in \mathbb{Z}$

$$\rho(x, y) = \begin{cases} \mathbf{1} & \text{if } 12|x, 12|y \text{ and } x \leq y \\ \mathbf{b} & \text{if } 6|x, 6|y, (4 \nmid x \text{ or } 4 \nmid y) \text{ and } x \leq y \\ \mathbf{a} & \text{if } 3|x, 3|y, (2 \nmid x \text{ or } 2 \nmid y) \text{ and } x \leq y \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The pair (μ, ρ) is a fuzzy ordered subgroup of \mathbb{Z} , according to our definition. □

Theorem 4.3. Let G be an ordered group, $\mu : G \rightarrow L$ a fuzzy subset of G and $\rho : G^2 \rightarrow L$ a fuzzy relation on μ . Then (μ, ρ) is a fuzzy-ordered subgroup of G if and only if for every $p \in L$, the cut μ_p is an ordered subgroup of G .

Proof. Let (μ, ρ) be a fuzzy-ordered subgroup of G . It is well known that for lattice valued structures being a subalgebra (subgroup) is a cutworthy property. This means that for every $p \in L$, the cut μ_p of μ is a subgroup of G . By Theorem 3.10, (μ_p, ρ_p) is a poset with the corresponding ordering, hence μ_p is an ordered subgroup of G .

The converse is well known for groups, and for the ordering part it follows again by Theorem 3.10. □

An illustration of Theorem 4.3 can be found in Example 4.2. E.g., the b -cut of μ is the naturally ordered subgroup of integers divisible by 6.

The next theorem is known as a *Theorem of synthesis*. In the case of fuzzy ordered subgroups it consists of two parts, algebraic and relational. For both, the construction is similar to the one given in Theorem 3.11, which also proves the order–relational part. The algebraic aspect is known (see e.g., [28, 29]).

Theorem 4.4. Let \mathcal{F} be a collection of subgroups of an ordered subgroup $(G, \cdot, ^{-1}, e, \leq)$ which is closed under set intersections and contains G . Then there is a complete lattice L and an ordered fuzzy subgroup (μ, ρ) of G , such that for every subgroup $H \in \mathcal{F}$, the cut μ_H coincides with H and it is ordered by ρ_H . □

4.3. Fuzzy cones

In order to investigate the structure of fuzzy ordered subgroups, we look at the so called positive and negative elements, the sets of which are in the classical theory of ordered groups known as *cones* (positive and negative). We introduce the analogue objects in the fuzzy framework and investigate their properties.

If (μ, ρ) is a fuzzy-ordered subgroup of G , then we define the *fuzzy positive cone* on μ , as a fuzzy set $\pi_\mu : G \rightarrow L$, in the following way:

$$\pi_\mu(x) := \rho(e, x),$$

where e is the neutral element of G .

Obviously,

$$\pi_\mu(x) = \begin{cases} \mu(x) & \text{if } x \geq e, \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, if (μ, ρ) is a fuzzy-ordered subgroup of G , we define the *fuzzy negative cone* as a function (fuzzy set) $\nu_\mu : G \rightarrow L$, in the following way:

$$\nu_\mu(x) := \rho(x, e)$$

It follows that

$$\nu_\mu(x) = \begin{cases} \mu(x) & \text{if } x \leq e, \\ 0 & \text{otherwise.} \end{cases}$$

A connection between the two cones is straightforward, as in the crisp case:

$$\pi_\mu(x) \wedge \nu_\mu(x) = \begin{cases} 1, & \text{for } x = e \\ 0, & \text{otherwise.} \end{cases}$$

There is a connection between a fuzzy cone and the corresponding fuzzy order, as follows.

Proposition 4.5. *Let $(G, \cdot, ^{-1}, e, \leq)$ be an ordered group and (μ, ρ) its fuzzy ordered subgroup. Then for all $x, y \in G$*

$$\pi_\mu(x^{-1} \cdot y) \geq \rho(x, y).$$

Proof.

$$\begin{aligned} \pi_\mu(x^{-1} \cdot y) &= \rho(e, x^{-1} \cdot y) = \mu(e) \wedge \mu(x^{-1} \cdot y) \wedge k_{\leq}(e, x^{-1} \cdot y) = \\ &= \mu(x^{-1} \cdot y) \wedge k_{\leq}(e, x^{-1} \cdot y). \end{aligned}$$

Now, if $x \leq y$, then $x^{-1} \cdot y \geq e$ and we have

$$\begin{aligned} \pi_\mu(x^{-1} \cdot y) &= \mu(x^{-1} \cdot y) \geq \mu(x^{-1}) \wedge \mu(y) = \mu(x) \wedge \mu(y) = \\ &= \mu(x) \wedge \mu(y) \wedge k_{\leq}(x, y) = \rho(x, y). \end{aligned}$$

If $x \not\leq y$, then $k_{\leq}(x, y) = 0$, hence $\rho(x, y) = 0$, and again we have

$$\pi_\mu(x^{-1} \cdot y) \geq \rho(x, y). \quad \square$$

Observe that in a lattice ordered group G , the set of positive elements, positive cone, and the set of negative elements, negative cone, are denoted respectively by P_G and N_G (as in [5]). Here we use the same denotation for their characteristic functions: for every $x \in G$

$$P_G(x) = \begin{cases} 1 & \text{if } x \geq e, \\ 0 & \text{otherwise;} \end{cases}$$

$$N_G(x) = \begin{cases} 1 & \text{if } x \leq e. \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.6. Let $(G, \cdot, ^{-1}, e, \leq)$ be an ordered group and (μ, ρ) a fuzzy-ordered subgroup of G . The following holds:

$$\pi_\mu = \mu \cap P_G.$$

Proof. Obviously, for every $x \in G$,

$$\pi_\mu(x) = \rho(e, x) = \mu(x) \wedge k_{\leq}(e, x) = (\mu \cap P_G)(x). \quad \square$$

Recall that by our definition (formula (8)), a fuzzy subset $\mu : P \rightarrow L$ of a poset (P, \leq) is fuzzy-convex on (P, \leq) if for all $x, y, z \in P$

$$\mu(x) \wedge \mu(z) \wedge k_{\leq}(x, y) \wedge k_{\leq}(y, z) \leq \mu(y).$$

Consequently, we say that a fuzzy-ordered subgroup (μ, ρ) of an ordered group $(G, \cdot, ^{-1}, e, \leq)$ is a *fuzzy-convex subgroup* of G if μ is a fuzzy-convex subset on the poset (G, \leq) .

In the next theorem we use P_G and N_G as ordinary sub-posets of G .

Theorem 4.7. Let (μ, ρ) be a fuzzy ordered subgroup of $(G, \cdot, ^{-1}, e, \leq)$. Then, the following are equivalent:

- (i) (μ, ρ) is a fuzzy convex subgroup of G .
- (ii) The restriction of π_μ to P_G is a fuzzy down-set in P_G .
- (iii) The restriction of ν_μ to N_G is a fuzzy up-set in N_G .

Proof. (i) \Rightarrow (ii). Suppose that (μ, ρ) is a fuzzy-convex subgroup of G and let $x, y \in P_G$. Let $y \leq x$. Then by fuzzy convexity

$$\mu(e) \wedge \mu(x) \wedge k_{\leq}(e, y) \wedge k_{\leq}(y, x) \leq \mu(y). \quad (14)$$

Since $x, y \in P_G$, we have

$$\pi_\mu(x) = \mu(x), \pi_\mu(y) = \mu(y), \text{ and } k_{\leq}(e, y) = 1,$$

and by (14) we get

$$\pi_\mu(x) \wedge k_{\leq}(y, x) = \mu(x) \wedge k_{\leq}(y, x) \leq \mu(y) = \pi_\mu(y).$$

Hence, the restriction of π_μ is a fuzzy down-set in P_G .

(ii) \Rightarrow (i). Suppose that the restriction of π_μ to P_G is a fuzzy down-set in P_G . We prove that μ is fuzzy-convex, i.e., that for all $x, y, z \in G$, we have

$$\mu(x) \wedge \mu(z) \wedge k_{\leq}(x, y) \wedge k_{\leq}(y, z) \leq \mu(y).$$

If it is not true that $x \leq y \leq z$, then the above formula is trivially fulfilled. If $x \leq y \leq z$, we get

$$e \leq x^{-1} \cdot y \leq x^{-1} \cdot z,$$

and hence, since π_μ is a fuzzy down-set on P_G ,

$$\mu(x^{-1} \cdot z) \wedge 1 = \pi_\mu(x^{-1} \cdot z) \wedge k_{\leq}(x^{-1} \cdot y, x^{-1} \cdot z) \leq \pi_\mu(x^{-1} \cdot y) = \mu(x^{-1} \cdot y).$$

Thereby, since $k_{\leq}(x, y) = k_{\leq}(y, z) = 1$, we have

$$\mu(x) \wedge \mu(z) \wedge k_{\leq}(x, y) \wedge k_{\leq}(y, z) \leq \mu(x) \wedge \mu(x^{-1} \cdot z) \leq \mu(x) \wedge \mu(x^{-1} \cdot y) \leq \mu(x \cdot x^{-1} \cdot y) \leq \mu(y).$$

Hence, μ is a fuzzy-convex subgroup of G .

The proofs of (i) \Rightarrow (iii) and (iii) \Rightarrow (i) are fully analogue to the above ones. □

5. Conclusion

We have here tried to present a consistent and detailed description of fuzzy (lattice valued) posets, with fuzzy underlying sets or/and fuzzy ordering relation. To do this, we had to deal with particularly defined fuzzy reflexivity of fuzzy relations on fuzzy domains, which on the crisp domain becomes weak reflexivity. Since the structure of fuzzy weak orders contained in a given crisp order is surprisingly regular (Theorem 3.12), our next task is investigation of this lattice and properties of its special elements (e.g., diagonal relations).

This investigation is used in the second part, for the definition and development of properties for fuzzy ordered subgroups of an ordered group. What we have not analyzed here, are cases when the group is lattice ordered. This important topic deserves some new investigations in the fuzzy framework. In addition, due to definition of fuzzy positive and fuzzy negative cones, one could define and investigate a kind of fuzzy absolute value of an element and a related notion of orthogonality.

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