



On Evaluations of Propositional Formulas in Countable Structures

Aleksandar Perović^a, Dragan Doder^b, Zoran Ognjanović^c, Miodrag Rašković^c

^aUniversity of Belgrade, Faculty of Transportation and Traffic Engineering Vojvode Stepe 305, 11000 Belgrade, Serbia

^bUniversity of Luxembourg, Computer Science and Communications 6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg

^cMathematical Institute of Serbian Academy of Sciences and Arts Kneza Mihaila 36, 11000 Belgrade, Serbia

Abstract. Let L be a countable first-order language such that its set of constant symbols $\text{Const}(L)$ is countable. We provide a complete infinitary propositional logic (formulas remain finite sequences of symbols, but we use inference rules with countably many premises) for description of C -valued L -structures, where C is an infinite subset of $\text{Const}(L)$. The purpose of such a formalism is to provide a general propositional framework for reasoning about \mathbb{F} -valued evaluations of propositional formulas, where \mathbb{F} is a C -valued L -structure. The prime examples of \mathbb{F} are the field of rational numbers \mathbb{Q} , its countable elementary extensions, its real and algebraic closures, the field of fractions $\mathbb{Q}(\varepsilon)$, where ε is a positive infinitesimal and so on.

1. Introduction

The present paper is an extended version of our talk presented at the FLINS 2012 conference [8], where we have introduced infinitary propositional logic for reasoning about countable first-order structures of some recursive first-order language L . What we have in mind is to provide a general propositional framework for reasoning about \mathbb{F} -valued evaluations of propositional formulas of various probability logics, fuzzy logics and possibility and necessity logics, where \mathbb{F} is a recursive L -structure. The central technical result is the proof of the strong completeness theorem (every consistent set of formulas is satisfiable) for the introduced system.

1.1. Motivation

Our main motive was to try to identify semantical essence behind various formal systems that capture numerous aspects of so called weighted logics (probability logics, possibilistic logics, fuzzy logics etc). It turns out that any concept of weighted logic can semantically be reduced to evaluations of certain syntactical objects in some type of a first-order structure. In propositional setting, one of the important tasks is to study effective methods for extending evaluations of propositional letters to the set of all propositional formulas.

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Email addresses: pera@sf.bg.ac.rs (Aleksandar Perović), dragan.doder@uni.lu (Dragan Doder), zorano@mi.sanu.ac.rs (Zoran Ognjanović), miodragr@mi.sanu.ac.rs (Miodrag Rašković)

1.2. Axiomatization issues

Propositional logic L_{ω_1} (countable conjunctions and disjunctions and inference rules with countably many premises are allowed) is expressive enough for description of any countable structure of a recursive first-order language L : we can literally write entire diagram of the underlying structure. However, expressiveness of L_{ω_1} has its price – non-recursive set of formulas and failure of the completeness and compactness theorems.

Completeness issue can be partially solved by restrictions on admissible fragments $L_{\mathcal{A}}$ of L_{ω_1} . Since many important mathematical concepts can be coded by Σ_1 -theories, this approach leads to various interesting applications of Barwise compactness theorem (see for instance [53]).

Even more tamed fragments of L_{ω_1} can be obtained by restricting to finite formulas and application of infinitary inference rules. We have used such fragments to obtain various recursive strongly complete axiomatizations of probability and temporal logics.

The main challenge was to construct the adequate infinitary inference rule that will allow us to adapt Henkin's completion technique for the first-order logic. In order to achieve this, our language must have names for all elements of a model, so we consider recursive languages with countably many symbols for constants. The set of all names is denoted by C .

Similarly as in the Henkin's construction, the universe of the canonical model of a complete L -theory T is the set C/\sim , where $c \sim d$ iff $T \vdash c = d$. So, we need to provide that each term f has a value in C . In other words, we need to satisfy the following condition:

$$\bigvee_{c \in C} f = c.$$

This L_{ω_1} formula can be represented by the infinitary inference rule

$$\frac{\{\phi \rightarrow f \neq c \mid c \in C\}}{\phi \rightarrow \perp}.$$

The implicative form of premises and conclusion is the standard technical requirement for the proof of the deduction theorem. In particular, introduced infinitary rule allows us to prove that for every consistent theory T and any term f exists $c \in C$ such that $T, f = c$ is consistent. Note that this property is a propositional counterpart of the elimination of $\exists x\phi(x)$ in the Henkin's construction.

1.3. Related work

There are many books and papers that are closely related to the work presented here. Though the list of references seems extensive (61 entry), it is actually quite narrow both in size and scope.

Our infinitary techniques have root in the work of Jon Barwise, Jerome Keisler and Douglas Hoover in admissible set theory, infinitary predicate and propositional logics and infinitary logics with probability quantifiers [1, 26, 28–32].

Zadeh's seminal work on fuzzy sets [61] and Nilsson's work on probability logics and their application in expert systems [43] have launched $[0, 1]$ -valued logics as important and useful scientific tools in engineering community. Soon the rapid theoretical development of so called weighted logics has followed. The most notable are operator probability logics, fuzzy logics and possibilistic logics. We have tried to credit some of this huge amount of work throughout the paper.

The turning point in mathematical representation of non-monotonic consequence relations and default reasoning was the work of Sarit Kraus, Daniel Lehmann and Menachem Magidor [33, 35]. In particular, characterization theorem of Lehmann and Magidor has provided an essential connection between rational relations and hyperreal-valued conditional probabilities, which we have used in the formalization presented in [57, 58].

1.4. Organization of the paper

The rest of the paper is organized as follows: in Section 2 we introduce syntax and semantics of our system, which can be briefly described as the logic of Σ_0 -sentences of L ; in Section 3 we prove the strong completeness theorem; in Section 4 we show how to code \mathbb{Q} -valued probabilities, $\mathbb{Q}(\varepsilon)$ -valued conditional probabilities and possibility distributions in our formalism; concluding remarks are in the final section.

2. Syntax and Semantics

Recall that a first-order language L is a (possibly empty) set of pairwise distinct symbols that are categorized in three groups: symbols for constants, which are the elements of $\text{Const}(L)$; symbols for relations, which are the elements of $\text{Rel}(L)$; symbols for functions, which are the elements of $\text{Fun}(L)$. Note that each of these three sets can be empty. In addition, for each relation and function symbol is provided its arity (the number of arguments). The arity of a symbol S will be denoted by $\text{ar}(S)$. From now on, we will consider only countable languages with countably many symbols for constants.

Let L be such a language. The shortest description of our syntax is the following one: our formulas are Σ_0 L -sentences. However, it seems prudent to present the extended version of definition as well.

Definition 2.1. *The set Term of terms is the smallest superset of $\text{Const}(L)$ that is closed for all $F \in \text{Fun}(L)$, i.e. if $F \in \text{Fun}(L)$ and $f_1, \dots, f_{\text{ar}(F)} \in \text{Term}$, then $F(f_1, \dots, f_{\text{ar}(F)}) \in \text{Term}$.*

Terms will be denoted by f, g and h , indexed if necessary.

Definition 2.2. *Atomic formulas are formulas of the form $f = g$ and $R(f_1, \dots, f_{\text{ar}(R)})$, where $f, g, f_i \in \text{Term}$ and $R \in \text{Rel}(L)$.*

Definition 2.3. *The set For of formulas is the smallest set containing all atomic formulas that is closed under negation and conjunction.*

Formulas will be denoted by ϕ, ψ and θ , indexed if necessary. The other connectives (disjunction, implication, equivalence) are defined as usual. We conclude the introduction of our syntax by the convention that C would from now on represent some fixed infinite subset of $\text{Const}(L)$.

Concerning semantics, our models are so called C -valued L -structures, i.e. L -structures of the form $\mathbb{A} = (A, \dots)$, where each $a \in A$ is an interpretation of some $c \in C$. Our satisfiability relation coincides with the restriction of the first-order satisfiability relation on Σ_0 -sentences. Precise definitions can be found in any graduate textbook of Mathematical logic or some more specialized branches of it such is Model theory, see for instance [41].

3. Complete Axiomatization

The basic axiom system \mathcal{AS} that we are going to introduce here is fairly simple - it contains only two groups of axioms and two types of inference rules.

Propositional axioms

A1 Substitutional instances of classical tautologies;

Equality axioms

A2 $f = f$;

A3 $f = g \rightarrow (\phi(\dots, f, \dots) \rightarrow \phi(\dots, g, \dots))$;

Inference rules

R1 Modus ponens: from ϕ and $\phi \rightarrow \psi$ infer ψ ;

R2 f -rule: from the set of premises $\{\phi \rightarrow f \neq c \mid c \in C\}$ infer $\phi \rightarrow f \neq f$.

Note that R2 is a schemata rule - one instance for each $f \in Term$. The purpose of R2 is to ensure that value of each term lies in the set of interpretations of C . The implicative form of premises and conclusion in R2 is the standard technical detail that provides an easy proof of the deduction theorem.

As it is usual, sets of formulas will be called theories. Due to the presence of infinitary inference rules, the notion of an inference is somewhat different.

Definition 3.1. Suppose that T is a theory and that ϕ is a formula. We say that ϕ is syntactical consequence of T , or that ϕ can be deduced or inferred from T , if there exists a countable ordinal ξ and a $\xi + 1$ -sequence of formulas $\phi_0, \phi_1, \dots, \phi_\xi$ such that $\phi_\xi = \phi$ and, for all $i \leq \xi$, ϕ_i is an instance of an axiom, or $\phi_i \in T$, or ϕ_i can be derived from some previous members of the sequence by some inference rule.

Note that the length of an inference is always at most countable successor ordinal. The notions of satisfiability, finite satisfiability, validity, consistency etc. are defined as usual. As it is usual, $T \vdash \phi$ reads “ ϕ is syntactical consequence of T ”, while $T \models \phi$ reads “ ϕ is semantical consequence of T ”.

Theorem 3.2. The compactness theorem fails for \mathcal{AS} .

Proof. Suppose that $d \in Const(L) \setminus C$. Let $T = T_1 \cup T_2$, where $T_1 = \{d \neq c \mid c \in C\}$ and $T_2 = \{c_1 \neq c_2 \mid c_1, c_2 \in C \text{ and } c_1 \neq c_2 \text{ (syntactically)}\}$. Furthermore, let $\mathbb{A} = (A, \dots)$ be any L -structure that satisfies T_2 . Since all elements of A are interpretations of some elements of C , it is easy to see that T_1 is not satisfiable in \mathbb{A} . On the other hand, $\mathbb{A} \models T_2$ implies that A is countable, so we can use \mathbb{A} to construct a model for any finite subset of T_1 . Hence, T is both unsatisfiable and finitely satisfiable, so compactness theorem fails for \mathcal{AS} . \square

Deduction theorem for \mathcal{AS} can be shown by induction on the length of the inference similarly as in [58], where the variant of the R2 rule is present. In any case we have

Theorem 3.3 (Deduction theorem). Let T be a theory and $\phi, \psi \in For$. Then, $T, \phi \vdash \psi$ iff $T \vdash \phi \rightarrow \psi$.

The completion technique that we will present here follows the guidelines from [58]. Roughly speaking, we will start with an adaptation of Lindenbaum’s lemma (every consistent theory can be maximized), then present a modification of the Henkin’s completeness proof.

Lemma 3.4. Suppose that T is a consistent theory and that f is an arbitrary term. Then, there exists $c \in C$ such that $T, f = c$ is consistent.

Proof. Contrary to the statement of the lemma, suppose that $T, f = c$ is inconsistent for all $c \in C$. Let ϕ be any syntactical consequence of T . Then, for all $c \in C$, $T, f = c \vdash \neg\phi$, so by Theorem 3.3 we obtain $T \vdash f = c \rightarrow \neg\phi$, which is equivalent to $T \vdash \phi \rightarrow f \neq c$. This is true for all $c \in C$, so by R2, $T \vdash \phi \rightarrow \perp$, i.e. $T \vdash \neg\phi$. Hence, T is inconsistent - a contradiction. By the excluding middle, the statement of the lemma must be true. \square

Lemma 3.5. (Lindenbaum’s lemma)

Every consistent theory can be maximized, i.e. extended to a complete consistent theory.

Proof. Suppose that T is a consistent theory. Since L is countable, sets For , $Term$ and C are all countable. Let $For = \{\phi_n \mid n \in \omega\}$, $C = \{c_n \mid n \in \omega\}$ and $Term = \{f_n \mid n \in \omega\}$. We define an ω -sequence $T_0, T_1, T_2 \dots$ of theories as follows:

1. $T_0 = T$;
2. $T_{2n+1} = \begin{cases} T_{2n} \cup \{\phi_n\} & , \quad T_{2n} \cup \{\phi_n\} \text{ is consistent} \\ T_{2n} \cup \{\neg\phi_n\} & , \quad T_{2n} \cup \{\phi_n\} \text{ is inconsistent} \end{cases}$;
3. $T_{2n+2} = T_{2n+1} \cup \{f_n = c_m\}$, where m is the least natural number such that $T_{2n+1} \cup \{f_n = c_m\}$ is consistent (the existence of m is provided by Lemma 3.4).

Let $\bar{T} = \bigcup_{n \in \omega} T_n$. By (2), \bar{T} is complete, i.e. $\bar{T} \vdash \phi$ or $\bar{T} \vdash \neg\phi$ for all $\phi \in \text{For}$. In order to see that \bar{T} is consistent, it is sufficient to show that \bar{T} is deductively closed. This fact can be shown by induction on the length of inference similarly as in [58]. As an illustration of the mentioned argument, we will show the following step: if $\phi \rightarrow f \neq c \in \bar{T}$ for all $c \in C$, then $\neg\phi \in \bar{T}$.

Contrary to the assumption, suppose that $\phi \in \bar{T}$ (\bar{T} is complete so $\phi \in \bar{T}$ or $\neg\phi \in \bar{T}$). By the construction of \bar{T} , there exist $m \in \omega$ and $c \in C$ such that $\phi, f = c, \phi \rightarrow f \neq c \in T_m$. By contraposition, $T_m \vdash f = c \rightarrow \neg\phi$, so by modus ponens $T_m \vdash \neg\phi$, hence T_m is inconsistent. However, by the construction of \bar{T} , each T_k is consistent, so the assumption $\phi \in \bar{T}$ fails. \square

Theorem 3.6 (Completeness theorem). *Every consistent theory is satisfiable.*

Proof. Suppose that T is a consistent theory. By Lemma 3.5, there exists a complete consistent theory \bar{T} that extends T . Similarly as in the Henkin's construction, let $c \sim d$ iff $\bar{T} \vdash c = d$, where $c, d \in C$. The domain A of the canonical model \mathbb{A} is defined as the quotient set $C_{/\sim}$. The corresponding interpretation I of L is defined as follows:

1. If $c \in C$, then $I(c) =_{\text{def}} c_{/\sim}$;
2. If $d \in \text{Const}(L) \setminus C$, then $I(d) =_{\text{def}} c_{/\sim}$, where $c \in C$ satisfies condition $\bar{T} \vdash d = c$. The existence of such c is provided by the construction of \bar{T} , see the proof of Lemma 3.5;
3. If f is a functional symbol of arity n and $c_1, \dots, c_n \in C$, then let

$$I(f)(I(c_1), \dots, I(c_n)) =_{\text{def}} I(c),$$

where $c \in C$ satisfies condition $\bar{T} \vdash f(c_1, \dots, c_n) = c$;

4. If R is a relational symbol of arity n and $c_1, \dots, c_n \in C$, then let

$$I(f)(I(c_1), \dots, I(c_n)) \text{ iff } \bar{T} \vdash R(c_1, \dots, c_n).$$

Firstly we are going to show the correctness of the above definition. Let $c_i \sim d_i$, $i = 1, \dots, k$, f is a k -ary function symbol and R is a k -ary relation symbol. Then, $\bar{T} \vdash c_i = d_i$ for all $i = 1, \dots, k$, so by equality axioms

$$\bar{T} \vdash f(c_1, \dots, c_n) = f(d_1, \dots, d_n) \text{ and } \bar{T} \vdash R(c_1, \dots, c_n) \leftrightarrow R(d_1, \dots, d_n),$$

which establishes correctness of $I(f)$ and $I(R)$.

It remains to show that \mathbb{A} is a model of \bar{T} , i.e. that $\mathbb{A} \models \phi$ for all $\phi \in \bar{T}$. Since we only have Σ_0 -formulas, the only nontrivial step is the verification of validity in \mathbb{A} for atomic formulas in \bar{T} .

Let $\bar{T} \vdash f(u_1, \dots, u_k) = g(v_1, \dots, v_k)$, where u_i, v_j are arbitrary L -terms and f, g are k -ary function symbols. By the construction of \bar{T} (see the proof of Lemma 3.5), there are $c, d \in C$ so that

$$\bar{T} \vdash f(u_1, \dots, u_k) = c \text{ and } \bar{T} \vdash g(v_1, \dots, v_k) = d.$$

By equality axioms, $\bar{T} \vdash c = d$, so the following holds:

- $I(f)(I(u_1), \dots, I(u_k)) = I(c)$ ¹⁾;
- $I(g)(I(v_1), \dots, I(v_k)) = I(d)$;
- $I(c) = I(d)$.

As a consequence we have that

$$\mathbb{A} \models f(u_1, \dots, u_k) = g(v_1, \dots, v_k).$$

Let $\bar{T} \vdash R(v_1, \dots, v_k)$, where v_1, \dots, v_k are arbitrary terms and R is a k -ary relation symbol. By the construction of \bar{T} , there are $c_1, \dots, c_k \in C$ so that $\bar{T} \vdash v_i = c_i$ for all $i = 1, \dots, k$. By equality axioms, $\bar{T} \vdash R(c_1, \dots, c_k)$. Furthermore, $I(v_i) = I(c_i)$ for all $i = 1, \dots, k$, so $I(R)(I(v_1), \dots, I(v_k))$ is satisfied in \mathbb{A} , i.e. $\mathbb{A} \models R(v_1, \dots, v_k)$. \square

¹⁾Interpretation $I(t)$ of any L -term t (in other words the value of t in \mathbb{A}) can be naturally defined as follows:

- For any constant symbol c let $I(c) = c_{/\sim}$;
- For any k -ary function symbol f and any terms t_1, \dots, t_k let $I(f(t_1, \dots, t_k)) = I(f)(I(t_1), \dots, I(t_k))$.

4. Some Examples

In this section we will show how probabilistic evaluations, possibilistic evaluations and evaluations in Gödel's logic can be represented as \mathcal{AS} theories.

4.1. Simple probabilities

Modal variants of probability logics are logics with modal like operators that formally capture the measure of the truthfulness of formulas in some underlying formalism. Usually, the basic syntactical construct involves a formula of the underlying formalism (say classical propositional logic) and the subinterval of the real unit interval $[0, 1]$ which refers to the estimation of the probability of the given formula. For example, formulas

$$w(p) \geq 0.5, [0.5, 1]p, P_{\geq 0.5}p \text{ and } 0.5 \rightarrow_L p$$

have the same meaning: the probability of the propositional letter p is between 0.5 and 1. Some of books and papers that are relevant for our previous and current work are [13, 14, 21–23, 25, 27, 34, 37–39, 42–50, 53–56, 59, 60].

In the single agent case we will assume that $L = \{+, -, \cdot, \leq\} \cup \text{Const}_L$, where $-$ is a unary function symbol, $+$ and \cdot are binary function symbols, \leq is a binary relation symbol and

$$\text{Const}_L = \{P(\alpha) \mid \alpha \in \text{For}_{Cl}\} \cup \mathbb{Q}.$$

Here For_{Cl} is the set of classical propositional formulas and $C = \mathbb{Q}$. Now we can assure that P actually behaves like probability by the following \mathcal{AS} -theory:

- Diagram of the ordered field of rational numbers;
- $f + g = g + f$;
- $f + (g + h) = (f + g) + h$
- $f + 0 = f$;
- $f - f = 0$ ($f - g$ is an abbreviation for $f + (-g)$);
- $f \cdot g = g \cdot f$;
- $f \cdot (g \cdot h) = (f \cdot g) \cdot h$;
- $f \cdot 1 = f$;
- $f \cdot (g + h) = (f \cdot g) + (f \cdot h)$;
- $f \leq f$;
- $f \leq g \wedge g \leq f \rightarrow f = g$;
- $f \leq g \wedge g \leq h \rightarrow f \leq h$;
- $f \leq g \vee g \leq f$;
- $f \leq g \leftrightarrow f + h \leq g + h$;
- $f \leq g \leftrightarrow (h > 0 \rightarrow f \cdot h \leq g \cdot h)$;
- $0 \leq P(\alpha) \leq 1$;
- $P(\alpha) = 1$, whenever α is a tautology;
- $P(\alpha) = P(\beta)$, whenever α and β are equivalent;

- $P(\alpha \vee \beta) = P(\alpha) + P(\beta) - P(\alpha \wedge \beta)$;
- $P(\neg\alpha) = 1 - P(\alpha)$.

In the multi agent case the main idea is to use indices for designation of probability estimations for each agent.

Though probabilities are not truth-functional, they can be effectively applied on certain classification problems (such as Grabisch problem of student classification [20]). Namely, it is easy to see that value of each probability μ on formulas is uniquely determined by its values on conjunctions of pairwise distinct propositional letters. As a consequence, any effective method of evaluation of finite conjunction of pairwise distinct propositional letters (say product of values of letters, or its minimum) would yield an effective procedure for computation of probability of any formula. As a consequence, we have gained an effective method for extending valuations of propositional letters to valuation of all formulas. We will illustrate this in the formalization of the well known Grabisch's classification example treated in [20, 51].

Originally Grabisch has considered the classification of students according to their academic scores in three courses. As a preprocessing, academic scores should be normalized in the usual way: if we have n grades displayed increasingly as a_1, a_2, \dots, a_n , then their normalization is defined by $\|a_i\| = \frac{i}{n}$. Courses C_1 , C_2 and C_3 can be represented by propositional letters p_1 , p_2 and p_3 respectively. So, normalized academic scores of the each student generate evaluation of propositional letters p_1 , p_2 and p_3 . As we have shown in [51], the obvious classification can be derived if we extend those evaluations on entire For_C by means of probability. Namely, if A is a nonempty finite set of pairwise disjoint propositional letters, then let $e(\bigwedge_{p \in A} p) = \prod_{p \in A} e(p)$. Here $e(p_i)$ coincides with normalized academic score in i -th course of some student and $e(p) = 1$ for $p \notin \{p_1, p_2, p_3\}$. What follows is a slight generalization of the original Grabisch's example.

Objects A , B , C and D are described by quality attributes p_1 , p_2 and p_3 , whose values are given in the following table:

| object | p_1 | p_2 | p_3 |
|--------|-------|-------|-------|
| A | 0.75 | 0.9 | 0.3 |
| B | 0.75 | 0.8 | 0.4 |
| C | 0.3 | 0.65 | 0.1 |
| D | 0.3 | 0.55 | 0.2 |

Objects A , B , C and D should be classified according to the following criteria:

- ϕ_1 : The average value of quality attributes;
- ϕ_2 : If the analyzed object is good with respect to p_1 , then p_3 is more important than p_2 . Otherwise, p_2 is more important than p_3 .

According to the first criterion, A and B are incomparable, C and D are incomparable, and both A and B are better than C and D . According to the second criterion, B is slightly better than A and C is slightly better than D . Hence, we get $D < C < A < B$.

In order to formally express described problem, we will assume the multi-agent formalization of simple probabilities. The second classification criterion can be propositionally coded by the formula $(p_1 \wedge p_3) \vee (\neg p_1 \wedge p_2)$ (of course, this is not the only way). We will assume that p_1 , p_2 and p_3 are independent. In terms of probability this transcribes as follows:

- $P_i(p_1 \wedge p_2) = P_i(p_1) \cdot P_i(p_2)$;
- $P_i(p_1 \wedge p_3) = P_i(p_1) \cdot P_i(p_3)$;
- $P_i(p_2 \wedge p_3) = P_i(p_2) \cdot P_i(p_3)$;
- $P_i(p_1 \wedge p_2 \wedge p_3) = P_i(p_1) \cdot P_i(p_2) \cdot P_i(p_3)$.

Moreover, we will formalize the given table in the obvious way:

- $P_i(p_j) = q_{i,j}$ (for instance $q_{1,1} = 0.75$).

Now for each i we can formally compute

$$1/6 \cdot (P_i(p_1) + P_i(p_2) + P_i(p_3)) + 1/2P_i((p_1 \wedge p_3) \vee (\neg p_1 \wedge p_2)),$$

compare calculated values and deduce the intended classification.

4.2. Rational relations

In [35] Lehmann and Magidor have shown that for each rational consequence relation ρ on For_{CI} there exists a hyper-real probability measure $\mu : For_{CI} \rightarrow [0, 1]^*$ such that $\alpha \rho \beta$ iff $\mu(\beta|\alpha) \approx 1$ (i.e. $1 - \mu(\beta|\alpha)$ is an infinitesimal. More on non-monotonic reasoning the reader can found in [3, 5–7, 17–19, 33, 62, 63].

In [58] we have presented an infinitary propositional logic for reasoning about $\mathbb{Q}(\varepsilon)$ -valued conditional probabilities (ε is a positive infinitesimal), which provides a complete and decidable formal system for reasoning about rational relations. Here we will rewrite this system as an \mathcal{AS} -theory \mathfrak{a} .

Let $L = \{+, -, \cdot, \leq, \approx\} \cup \mathbb{Q}(\varepsilon) \cup \{P(\alpha), CP(\alpha, \beta) \mid \alpha, \beta \in For_{CI}\}$. Here $P(\alpha)$ reads “the probability of α ”, while $CP(\alpha, \beta)$ reads “the conditional probability of α given β ”.

- Diagram of the ordered field $\mathbb{Q}(\varepsilon)$;
- $f + g = g + f$;
- $f + (g + h) = (f + g) + h$
- $f + 0 = f$;
- $f - f = 0$ ($f - g$ is an abbreviation for $f + (-g)$);
- $f \cdot g = g \cdot f$;
- $f \cdot (g \cdot h) = (f \cdot g) \cdot h$;
- $f \cdot 1 = f$;
- $f \cdot (g + h) = (f \cdot g) + (f \cdot h)$;
- $f \leq f$;
- $f \leq g \wedge g \leq f \rightarrow f = g$;
- $f \leq g \wedge g \leq h \rightarrow f \leq h$;
- $f \leq g \vee g \leq f$;
- $f \leq g \leftrightarrow f + h \leq g + h$;
- $f \leq g \leftrightarrow (h > 0 \rightarrow f \cdot h \leq g \cdot h)$;
- $0 \leq CP(\alpha, \beta) \leq 1$;
- $P(\alpha) = CP(\alpha, \top)$;
- $CP(\alpha, \beta) = CP(\alpha', \beta')$, where α is equivalent to α' and β is equivalent to β' ;
- $P(\neg\alpha) = 1 - P(\alpha)$;
- $P(\alpha \vee \beta) = P(\alpha) + P(\beta) - P(\alpha \wedge \beta)$;
- $P(\beta) = 0 \rightarrow CP(\alpha, \beta) = 1$;

- $P(\beta) = b \wedge P(\alpha \wedge \beta) = a \wedge b > 0 \rightarrow CP(\alpha, \beta) = \frac{a}{b}$;
- $CP(\alpha, \beta) = a \rightarrow CP(\alpha, \beta) \approx \text{st}(a)$, where $\text{st}(a)$ is the standard part of a ;
- $CP(\alpha, \beta) \approx 0 \rightarrow CP(\alpha, \beta) \leq \frac{1}{n+1}$;
- $CP(\alpha, \beta) \approx 1 \rightarrow CP(\alpha, \beta) \geq \frac{n}{n+1}$;
- $CP(\alpha, \beta) \approx r \rightarrow s \leq CP(\alpha, \beta) \leq t$, for all $r, s, t \in (0, 1)_{\mathbb{Q}}$ so that $s < r < t$.

Let us show that $T = \{s \leq CP(\alpha, \beta) \leq t, \mid s, t \in (0, 1)_{\mathbb{Q}} \text{ and } s < r < t\} \vdash CP(\alpha, \beta) \approx r, r \in (0, 1)_{\mathbb{Q}}$.

Indeed, suppose that $T \not\vdash CP(\alpha, \beta) \approx r$. Then, $T' = T \cup \{\neg CP(\alpha, \beta) \approx r\}$ is a consistent theory. Let T^* be any completion of T' . There is unique $a \in [0, 1]_{\mathbb{Q}(\varepsilon)}$ such that $CP(\alpha, \beta) = a$. It is easy to see that $|a - r|$ must be an infinitesimal (this is a straightforward consequence of T). Consequently, $\text{st}(a) = r$, so $T^* \vdash CP(\alpha, \beta) \approx r$; contradiction.

4.3. Possibility and necessity functions

Possibility theory is one of the prominent branches of research in the field of nonmonotonic reasoning and reasoning under uncertainty, see [2, 4, 9–11, 15, 52]. Qualitative possibilities were introduced independently by Lewis in [36] and Dubois in [9], while qualitative necessities were introduced by Dubois in [9]. Though possibility and necessity relations are originally defined on algebras of sets, it is quite natural to treat them as relations on propositional formulas since all Boolean algebras are, up to isomorphism, Lindenbaum–Tarski propositional algebras.

A binary relation \leq_{Π} on the set For_C of classical propositional formulas is a qualitative possibility if it satisfies the following conditions:

- \leq_{Π} is a nontrivial weak order, i.e. it is linear (each pair of formulas is comparable), transitive and $\perp <_{\Pi} \top$, where \perp is any contradiction, \top is any tautology and $<_{\Pi}$ has the obvious meaning $\alpha <_{\Pi} \beta$ and $\beta \not\leq_{\Pi} \alpha$;
- \leq_{Π} is compatible with the equivalence of classical propositional formulas. More precisely, if α is equivalent to α' and β is equivalent to β' , then $\alpha \leq_{\Pi} \beta$ iff $\alpha' \leq_{\Pi} \beta'$;
- For all $\alpha \in For_C$, $\perp \leq_{\Pi} \alpha$;
- \leq_{Π} satisfies so called disjunctive stability: if $\beta \leq_{\Pi} \gamma$, then, for all α , $\alpha \vee \beta \leq_{\Pi} \alpha \vee \gamma$.

A binary relation \leq_N on For_C is a qualitative possibility if it has the following properties:

- \leq_N is a nontrivial weak order;
- \leq_N is compatible with the equivalence of classical propositional formulas;
- For all $\alpha \in For_C$, $\alpha \leq_N \top$;
- \leq_N satisfies so called conjunctive stability: if $\beta \leq_{\Pi} \gamma$, then, for all α , $\alpha \wedge \beta \leq_{\Pi} \alpha \wedge \gamma$.

It is quite natural to treat qualitative possibilities and necessities as dual notions, since any qualitative possibility relation \leq_{Π} generates the unique qualitative necessity relation $\leq_{N_{\Pi}}$ by

$$\alpha \leq_{N_{\Pi}} \beta \text{ iff } \neg\beta \leq_{\Pi} \neg\alpha,$$

and vice versa. Moreover, for any possibility (necessity) relation \leq_R , the quotient structure $\langle For_C / \sim, \leq_R / \sim \rangle$ ($\alpha \sim \beta$ iff $\alpha \leq \beta$ and $\beta \leq \alpha$) is at most countable (For_C is countable) linear ordering with endpoints, so it can be embedded into $\langle [0, 1], \leq \rangle$. If F is one such embedding, then by $f : \alpha \mapsto F(\alpha / \sim)$ is defined a distribution f of \leq_R , i.e. $\alpha \leq_R \beta$ iff $f(\alpha) \leq f(\beta)$. Such functions are also called possibility (necessity) functions.

If \leq_{Π} is a qualitative possibility relation and π its arbitrary distribution, then

$$\pi(\alpha \vee \beta) = \max(\pi(\alpha), \pi(\beta)) \quad (\text{maxitivity condition}).$$

Moreover, if $f : For_C \rightarrow [0, 1]$ satisfies maxitivity condition, then the relation \leq_f defined by

$$\alpha \leq_f \beta \quad \text{iff} \quad f(\alpha) \leq f(\beta)$$

is a qualitative possibility relation. Dually, minitivity condition

$$v(\alpha \wedge \beta) = \min(v(\alpha), v(\beta))$$

fully characterizes qualitative necessity relations.

Hence, in order to formally capture evaluations that behaves like possibility functions and necessity functions, we just need to provide fulfillment of the corresponding characterizing condition (maxitivity for possibility functions, minitivity for necessity functions). Again we will only discuss the single agent case.

Let $L = \{\leq\} \cup [0, 1]_{\mathbb{Q}} \cup \{\Pi(\alpha), N(\alpha) \mid \alpha \in For_{Cl}\}$, where $C = [0, 1]_{\mathbb{Q}}$. Here $\Pi(\alpha)$ stands for the possibility measure of α , while $N(\alpha)$ stands for the necessity measure of α . We can obtain desired properties of Π and N by the following \mathcal{AS} -theory:

- Diagram of $\langle [0, 1]_{\mathbb{Q}}, \leq \rangle$;
- $f \leq g$;
- $f \leq g \wedge g \leq f \rightarrow f = g$;
- $f \leq g \wedge g \leq h \rightarrow f \leq h$;
- $f \leq g \vee g \leq f$;
- $\Pi(\alpha) = \Pi(\beta)$, if α and β are equivalent;
- $\Pi(\perp) = 0$; $\Pi(\top) = 1$;
- $\Pi(\alpha) \leq \Pi(\beta) \rightarrow \Pi(\alpha \vee \beta) = \Pi(\beta)$;
- $N(\alpha) = N(\beta)$, if α and β are equivalent;
- $N(\perp) = 0$; $N(\top) = 1$;
- $N(\alpha) \leq N(\beta) \rightarrow N(\alpha \wedge \beta) = N(\alpha)$.

4.4. Evaluations in Gödel's fuzzy logic

Roughly speaking, Gödel's fuzzy logic is an extension of the basic logic (BL) with axioms that ensures that conjunction and disjunction behaves like min and max respectively. Furthermore, Gödel's implication (residuum of min) and negation are defined by

$$x \Rightarrow y = \begin{cases} 1 & , \quad x \leq y \\ y & , \quad y < x \end{cases}$$

and

$$\neg x = \begin{cases} 0 & , \quad x > 0 \\ 1 & , \quad x = 0 \end{cases} .$$

More information on fuzzy logic the reader can found in [12, 16, 21, 23, 24, 39, 40, 61].

Let $L = \{\leq\} \cup [0, 1]_{\mathbb{Q}} \cup \{G(\alpha) \mid \alpha \in For_{Cl}\}$. The following \mathcal{AS} -theory formally captures semantics of Gödel's logic in the single agent case:

- Diagram of $\langle [0, 1]_{\mathbb{Q}}, \leq \rangle$;
- $f \leq f$;
- $f \leq g \wedge g \leq f \rightarrow f = g$;
- $f \leq g \wedge g \leq h \rightarrow f \leq h$;
- $f \leq g \vee g \leq f$;
- $0 < G(\alpha) \rightarrow G(\neg\alpha) = 0$;
- $G(\alpha) = 0 \rightarrow G(\neg\alpha) = 1$;
- $G(\alpha) \leq G(\beta) \rightarrow G(\alpha \rightarrow \beta) = 1$;
- $G(\alpha) > G(\beta) \rightarrow G(\alpha \rightarrow \beta) = G(\beta)$;
- $G(\alpha) \leq G(\beta) \rightarrow G(\alpha \wedge \beta) = G(\alpha)$;
- $G(\alpha) \leq G(\beta) \rightarrow G(\alpha \vee \beta) = G(\beta)$.

It is easy to see that in any model M of the introduced theory, by $e(\alpha) = G^M(\alpha)$ is correctly defined an evaluation in Gödel's logic.

5. Conclusion

Motivation for our research was prior work on uncertainty reasoning based on numerical-valued evaluation of propositional formulas. Our goal was to develop a general formal environment for modelling main numerical-valued concepts such as probability and possibility logics, and different types of fuzzy logics, which were investigated in various papers [9, 11, 12, 21, 24, 37, 38, 48, 53, 58].

This paper deals with \mathbb{F} -valued evaluations, where \mathbb{F} is a recursive structure of a given first-order recursive language. We investigated the logical language appropriate for description of many-valued logics. We axiomatized this language and we proved that the axiomatization is sound and complete with respect to corresponding semantics. Then we showed how the logics mentioned above may be represented as theories in our formalism. The examples include simple and conditional probabilities, possibility and necessity functions and evaluations in Gödel's fuzzy logics.

We showed that logic presented here is not compact. Hence, any finitary axiomatization would be incomplete, so we achieved completeness using infinitary inference rules.

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