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An Optimal Control Problem with Final Observation for Systems Governed by Nonlinear Schrödinger Equation

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Abstract. In this paper, an optimal control problem with final observation for systems governed by nonlinear time-dependent Schrödinger equation is studied. The existence and uniqueness of the solution of considered optimal control problem are proved. The first variation of objective functional is obtained and a necessary optimality condition in the variational form is given.

1. Introduction

The researches for the optimal control of systems governed by partial differential equations have a long history. Especially, the optimal control problems for systems described by Schrödinger equation have drawn a lot of attention in the last year. The optimal control problems for systems governed by Schrödinger equation arise in quantum mechanics, nuclear physics, nonlinear optics, and the various field of modern physics and engineering [5, 22].

Consider the partial differential equation in the form:

$$\varepsilon\frac{\partial\psi}{\partial t}+r_2(x,t,\psi)\frac{\partial^2\psi}{\partial x^2}+r_1(x,t,\psi)\frac{\partial\psi}{\partial x}+r_0(x,t,\psi)\psi=0. \tag{1}$$

Equation (1) describes the slow variation of the function $\psi(x,t)$ in a nonlinear medium with quadratic dispersion, where $\varepsilon = const.$, the function $\psi(x,t)$ is the wave's complex amplitute, x and t are variables of space and time, respectively. The coefficients $r_j(x,t,\psi)$ for $j = 0,1,2$ considered as functions of x and t describes the variation of the medium, while their dependence on the function $\psi(x,t)$ describes the nonlinear properties of the medium [29]. From equation (1) according to the properties of the coefficients $r_j(x,t,\psi)$ for $j = 0,1,2$ is obtained the different variants of Schrödinger equation such as linear and nonlinear Schrödinger equations.

In the case of $r_2(x,t,\psi) = r_2(x,t)$, $r_1(x,t,\psi) = 0$, $r_0(x,t,\psi) = r_0(x,t)$ in the equation (1), a linear Schrödinger equation is obtained from equation (1). The optimal control problems for the different variants of systems described by linear Schrödinger equations were examined in the papers [1-5], [7, 9, 13, 19, 27]. Similarly, in

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the case of $r_2(x,t,\psi) = r_2(x,t), r_1(x,t,\psi) = 0, r_0(x,t,\psi) = r_0(x,t,\psi)$ in the equation (1), a nonlinear Schrödinger equation is obtained from equation (1). The optimal control problems for the different variants of systems described by nonlinear Schrödinger equations were studied in the papers [8], [10-12], [16-17], [23, 26]. Also, in the case of $r_2(x,t,\psi) = r_2(x,t)$, $r_1(x,t,\psi) = r_1(x,t)$, $r_0(x,t,\psi) = r_0(x,t)$ in the equation (1), the optimal control problems for the different variants of systems described by linear Schrödinger equations obtained from equation (1) were examined in the papers [24-25].

As the different from the above mentioned studies, in this paper, we consider an optimal control problem for systems governed by nonlinear Schrödinger equations obtained from (1) in the case of $r_2(x,t,\psi) = r_2(x,t)$, $r_1(x,t,\psi) = r_1(x,t)$, $r_0(x,t,\psi) = r_0(x,t,\psi)$ in the equation (1). In Section 2, the formulation of considered optimal control problem is given. In Section 3, the existence and uniqueness of the solutions of optimal control problem are proved. Finally, the first variation of objective functional is obtained and a necessary optimality condition in the variational form is given in Section 4.

2. The Formulation of Optimal Control Problem

Let l, T be given numbers, $0 \le x \le l, 0 \le t \le T, \Omega_t = (0,l) \times (0,t), \Omega = \Omega_T$.

Let's consider the system described by nonlinear Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} + a_0\frac{\partial^2\psi}{\partial x^2} + ia_1\frac{\partial\psi}{\partial x} - a(x)\psi + v(t)\psi + ia_2|\psi|^2\psi = f(x,t) \tag{2}$$

$$\psi(x,0) = \varphi(x),\ x \in (0,l) \tag{3}$$

$$\psi(0,t) = \psi(l,t) = 0,\ \ t \in (0,T), \tag{4}$$

where $\psi = \psi(x,t)$ is a wave function, $i = \sqrt{-1}$ imaginary unit, $a_0, a_1, a_2 > 0$ are given real numbers, $a(x)$ is a measurable real-valued function which satisfies the condition

$$0 \le a(x) \le \mu_{0,} \text{ for almost all } x \in (0,l)\,,\ \mu_0 = \text{const.} > 0, \tag{5}$$

and the functions $\varphi \in \mathring{W}_2^2(0,l)$, $f \in W_2^{0,1}(\Omega)$ are given complex-valued functions. Here, $\psi = \psi(x,t)$ is the state and $v = v(t)$ is the control.

We can roughly express the control problem as follows: to which extent can the solution $\psi = \psi(x,t)$ of (2)-(4) be perturbed by action of the control v at a given final time $t = T$ in order to reach a given final target?

If we want to formulate this problem as an optimal control problem, we can write it as the problem of finding the minimum of the objective functional

$$J_\alpha(v) = \|\psi(.,T) - y\|^2_{L_2(0,l)} + \alpha\|v - w\|^2_{W_2^1(0,T)} \tag{6}$$

from the conditions (2)-(4), where $y = y(x) \in L_2(0,l)$ is the given final target, $\alpha \ge 0$ is a given number, $w \in W_2^1(0,T)$ is a given element.

The control function $v = v(t)$ is investigated on the set

$$V \equiv \left\{ v : v(t) \in W_2^1(0,T),\ |v(t)| \le b_0,\ \left|\frac{dv(t)}{dt}\right| \le b_1 \text{ for almost all } t \in (0,T)\,,\ b_0, b_1 = \text{const.} > 0 \right\}$$

which is called as the set of admissible controls. Here the Sobolev space $W_2^1(0,T)$ is a Hilbert space consisting of all the elements $L_2(0,T)$ having square summable generalized derivatives of the first order on $(0,T)$. The Sobolev spaces $\mathring{W}_2^2(0,l)$, $W_2^{0,1}(\Omega)$ are widely defined in [15].

In this paper, we will denote the problem of finding the minimum of the objective functional $J_\alpha(v)$ on the set V under the conditions (2)-(4) as the optimal control problem (2)-(4), (6).

When an optimal control problem is examined, the some questions emerge such as the existence and uniqueness of solutions of the optimal control problem, the differentiability of the objective functional, the necessary and sufficient conditions for the solution of the optimal control problem. So, throughout this paper, we investigate the answers of mentioned questions. For this purpose, firstly, we must prove the existence of the solution of the boundary value problem (2)-(4).

Now, let's define the solution of the boundary value problem (2)-(4). Under given conditions, by a solution of the problem (2)-(4), we mean a function $\psi(x,t)$ in the space $B_0 \equiv C^0([0,T], \overset{\circ}{W}{}_2^2(0,l)) \cap C^1([0,T], L_2(0,l))$, which satisfies the equation (2) *for almost all* $x \in (0,l)$ and *any* $t \in [0,T]$, the initial condition (3) *for almost all* $x \in (0,l)$ and the boundary condition (4) *for almost all* $t \in (0,T)$, where $C^k([0,T],B)$ is a Banach space of all $B-$valued, $k \geq 0$ times continuously differentiable functions on $[0,T]$ with the norm

$$\|\psi\|_{C^k([0,T],B)} = \sum_{m=0}^{k} \max_{0\leq t\leq T} \|\frac{d^m\psi(t)}{dt^m}\|_B$$

for $\psi \in C^k([0,T],B)$.

We prove that the following theorem is valid for the solution of the boundary value problem (2)-(4):

Theorem 2.1. *Assume that* $\varphi \in \overset{\circ}{W}{}_2^2(0,l)$, $f \in W_2^{0,1}(\Omega)$ *and the function* $a(x)$ *satisfies the conditions (5). Then, the initial boundary value problem (2)-(4) has a unique solution* $\psi \in B_0$ *for any* $v \in V$ *and the following estimation is valid for this solution:*

$$\begin{aligned}\|\psi(.,t)\|^2_{\overset{\circ}{W}{}_2^2(0,l)} + \left\|\frac{\partial\psi}{\partial t}\right\|^2_{L_2(0,l)} &\leq c_0\Big(\|\varphi\|^2_{\overset{\circ}{W}{}_2^2(0,l)} + \|\varphi\|^6_{\overset{\circ}{W}{}_2^2(0,l)} + \|\varphi\|^6_{\overset{\circ}{W}{}_2^1(0,l)}\\ &\quad +\|\varphi\|^{18}_{\overset{\circ}{W}{}_2^1(0,l)} + \|f\|^2_{W_2^{0,1}(\Omega)} + \|f\|^6_{W_2^{0,1}(\Omega)}\Big) \qquad (7)\end{aligned}$$

for any $t \in [0,T]$, *where the constant* $c_0 > 0$ *is independent from* φ, f *and* t.

3. Existence and Uniqueness Theorems

In this section, two different cases are investigated for the solution of the optimal control problem (2)-(4), (6). Firstly, it is shown that the optimal control problem (2)-(4), (6) has a unique solution for $\alpha > 0$ on a dense subset G of the space $W_2^1(0,T)$ by using theorem 3.1 and secondly, the problem has at least one solution for any $\alpha \geq 0$ on the space $W_2^1(0,T)$.

Theorem 3.1. *(**The corollary of Goebel theorem** [6]) Let* $\widetilde{X}$ *be a uniformly convex space,* U *be a closed bounded set on* $\widetilde{X}$ *, the functional* $I(v)$ *be lower semicontinuous and lower bounded on* U, $\alpha > 0$ *be a given number. Then, there is a dense subset* G *of the space* $\widetilde{X}$ *such that for any* $w \in G$ *the functional*

$$I_\alpha(v) = I(v) + \alpha\|v-w\|^2_{\widetilde{X}}$$

takes its minimum value at a unique point on U.

Lemma 3.2. *The functional* $J_0(v) = \|\psi(.,T) - y\|^2_{L_2(0,l)}$ *is continuous on the set* V.

Proof. Let $\Delta v \in W_\infty^1(0,T)$ be an increment of any element $v \in V$ such that $v+\Delta v \in V$, $\psi = \psi(x,t) \equiv \psi(x,t;v)$ be the solution of the problem (2)-(4) corresponding to $v \in V$ and . $\Delta\psi = \Delta\psi(x,t) \equiv \psi(x,t;v+\Delta v) - \psi(x,t;v) = \psi_\Delta - \psi(x,t)$, where the function $\psi_\Delta = \psi(x,t;v+\Delta v)$ is a solution of the problem (2)-(4) for *any* $v+\Delta v$.From the conditions (2)-(4) is obtained that the function $\Delta\psi = \Delta\psi(x,t)$ is a solution of the following boundary value problem:

$$\begin{aligned}& i\frac{\partial\Delta\psi}{\partial t} + a_0\frac{\partial^2\Delta\psi}{\partial x^2} + ia_1\frac{\partial\Delta\psi}{\partial x} - a(x)\Delta\psi + (v+\Delta v)\Delta\psi\\ & +ia_2\left[\left(|\psi_\Delta|^2 + |\psi|^2\right)\Delta\psi + \psi_\Delta\psi\Delta\bar{\psi}\right]\\ = \;& -\Delta v(t)\psi, \quad (x,t)\in\Omega \qquad (8)\end{aligned}$$

$$\Delta\psi(x,0)=0,\quad x\in(0,l), \tag{9}$$

$$\Delta\psi(0,t)=\Delta\psi(l,t)=0, t\in(0,T). \tag{10}$$

Now, let's evaluate the solution of the boundary value problem (8)-(10). For this purpose, let us multiply both sides of the equation (8) by $\Delta\bar{\psi}(x,t)$ and integrate over Ω_t. Then, integrating by part we get

$$\begin{aligned}
&\int\limits_{\Omega_t}\left[i\frac{\partial\Delta\psi}{\partial t}\Delta\bar{\psi}-a_0\left|\frac{\partial\Delta\psi}{\partial x}\right|^2+ia_1\frac{\partial\Delta\psi}{\partial x}\Delta\bar{\psi}-a(x)\left|\Delta\psi\right|^2+(v(t)+\Delta v(t))|\Delta\psi|^2\right.\\
&\left.+ia_2\left(|\psi_\Delta|^2+|\psi|^2\right)|\Delta\psi|^2+ia_2\psi_\Delta\psi(\Delta\bar{\psi})^2\right]dxd\tau\\
=\;&-\int\limits_{\Omega_t}\Delta v(t)\psi\Delta\bar{\psi}dxd\tau.
\end{aligned}$$

If we subtract the complex conjugate of the above equality from itself and use the condition (9), we obtain

$$\begin{aligned}
&\left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)}+2a_2\int\limits_{\Omega_t}(|\psi_\Delta|^2+|\psi|^2)|\Delta\psi|^2\,dxd\tau\\
=\;&-2\int\limits_{\Omega_t}\operatorname{Im}\left(\Delta v\psi\Delta\overline{\psi}\right)dxd\tau-2a_2\int\limits_{\Omega_t}\operatorname{Re}\left(\psi_\Delta\psi\left(\Delta\overline{\psi}\right)^2\right)dxd\tau
\end{aligned}$$

and by Young's inequality

$$\begin{aligned}
&\left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)}+a_2\int\limits_{\Omega_t}(|\psi_\Delta|^2+|\psi|^2)|\Delta\psi|^2\,dxd\tau\\
\leq\;&2\int\limits_{\Omega_t}|\Delta v|\left|\psi\right|\left|\Delta\psi\right|dxd\tau\\
\leq\;&\int_0^t\int_0^l|\Delta v|^2\left|\psi\right|^2dxd\tau+\int\limits_{\Omega_t}\left|\Delta\psi\right|^2dxd\tau\\
=\;&\int_0^t|\Delta v|^2\left\|\psi(.,t)\right\|^2_{L_2(0,l)}d\tau+\int_0^t\left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)}d\tau\\
\leq\;&\max_{0\leq t\leq T}\left\|\psi(.,t)\right\|^2_{L_2(0,l)}\int_0^T|\Delta v|^2\,d\tau+\int_0^t\left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)}d\tau
\end{aligned}$$

for any $t\in[0,T]$. In the above inequality, if we use the estimation (7), we get

$$\|\Delta\psi(.,t)\|^2_{L_2(0,l)}+a_2\int\limits_{\Omega_t}(|\psi_\Delta|^2+|\psi|^2)|\Delta\psi|^2\,dxd\tau\leq c_1\|\Delta v\|^2_{L_2(0,T)}+\int_0^t\|\Delta\psi(.,t)\|^2_{L_2(0,l)}d\tau. \tag{11}$$

Thus, ıf we take into account $a_2>0$,we have the inequality

$$\|\Delta\psi(.,t)\|^2_{L_2(0,l)}\leq c_1\|\Delta v\|^2_{L_2(0,T)}+\int_0^t\|\Delta\psi(.,t)\|^2_{L_2(0,l)}d\tau, \tag{12}$$

for any $t \in [0,T]$, where the constant $c_1 > 0$ is independent from $\Delta\psi$, Δv and t. Applying the Gronwall's inequality to (12), we get

$$\left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)} \leq c_2\|\Delta v\|^2_{L_2(0,T)} \text{ for any } t \in [0,T] \tag{13}$$

and thus

$$\left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)} \leq c_2\|\Delta v\|^2_{L_2(0,T)} \leq c_3\|\Delta v\|^2_{W_2^1(0,T)} \leq c_4\|\Delta v\|^2_{W_\infty^1(0,T)} \text{ for any } t \in [0,T] \tag{14}$$

where the constants $c_2, c_3, c_4 > 0$ are independent from Δv and t.

Now, let us find the increment of the functional $J_0(v)$ for *any* $v \in V$. If we use the definition of the functional $J_0(v)$, we obtain

$$\begin{aligned}
\Delta J_0(v) &= J_0(v+\Delta v) - J_0(v) \\
&= \int_0^l \left|\psi(x,T;v+\Delta v) - y(x)\right|^2 dx - \int_0^l \left|\psi(x,T;v) - y(x)\right|^2 dx \\
&= \int_0^l (\psi_\Delta(x,T) - y(x))\left(\overline{\psi}_\Delta(x,T) - \overline{y}(x)\right) dx - \int_0^l (\psi(x,T) - y(x))\left(\overline{\psi}(x,T) - \overline{y}(x)\right) dx \\
&= 2\int_0^l \mathrm{Re}[(\psi(x,T) - y(x))(\Delta\bar{\psi}(x,T)]dx + \left\|\Delta\psi(.,T)\right\|^2_{L_2(0,l)}. \qquad (15)
\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the equality (15) and later using the estimations (7), (14), we get the following inequality for the increment of functional $J_0(v)$:

$$|J_0(v+\Delta v) - J_0(v)| \leq c_5\left(\|\Delta v\|_{W_2^1(0,T)} + \|\Delta v\|^2_{W_2^1(0,T)}\right)$$

for *any* $v \in V$, where the constant $c_5 > 0$ is independent from Δv.Thus, since $\Delta J_0(v) \to 0$ for $\|\Delta v\|_{W_\infty^1(0,T)} \to 0$ and *any* $v \in V$, we can easily say that the functional $J_0(v)$ is continuous on the set V. □

Now, let's give the following theorem stated the uniqueness of the solution of the optimal control problem (2)-(4), (6) on a dense subset G of the space $W_2^1(0,T)$.

Theorem 3.3. *Assume that the conditions of Theorem (2.1) are fulfilled and $w \in W_2^1(0,T)$, $y \in L_2(0,l)$ are given functions. Then, there is a dense subset G of the space $W_2^1(0,T)$ such that the optimal control problem (2)-(4), (6) has a unique solution for any $w \in G$ and $\alpha > 0$.*

Proof. Since the functional $J_0(v)$ is continuous on the set V, it is a lower semicontinuous functional. Also, since $J_0(v) \geq 0$ for *any* $v \in V$, the functional $J_0(v)$ is lower bounded. Additionally, the set V is a closed, bounded and convex set of the uniformly convex space $W_2^1(0,T)$ [28]. Thus, the conditions of Theorem 3.1 hold. So, we can say that there is a dense subset G of the space $W_2^1(0,T)$ such that the optimal control problem (2)-(4), (6) has a unique solution for *any* $w \in G$ and $\alpha > 0$. □

The next theorem state that the optimal control problem (2)-(4), (6) has at least one solution for any $\alpha \geq 0$ on the space $W_2^1(0,T)$.

Theorem 3.4. *Suppose that $\alpha \geq 0$, the conditions of Theorem (2.1) hold and $w \in W_2^1(0,T)$ is a given function. Then, the optimal control problem (2)-(4), (6) has at least one solution.*

Proof. Let $\{v^m\} \subset V$ be a minimizing sequence such that $\lim_{m\longrightarrow\infty} J_\alpha(v^m) = J_{\alpha^*} = \inf_{v\in V} J_\alpha(v)$ for the functional $J_\alpha(v)$. Let $\psi_m = \psi(x,t) \equiv \psi(x,t;v^m)$ be a solution of the boundary value problem (2)-(4) for any $v^m \in V$. Since $v^m \in V$, $m = 1,2,\ldots$, from theorem 2.1, we can say that the problem (2)-(4) has a unique solution $\psi_m = \psi(x,t) \equiv \psi(x,t;v^m) \in B_0$ for each $m = 1,2,\cdots$. Moreover, the following estimation is valid:

$$\|\psi_m(.,t)\|^2_{\overset{0}{W}{}^2_2(0,l)} + \left\|\frac{\partial\psi_m(.,t)}{\partial t}\right\|^2_{L_2(0,l)} \le c_6, \text{ for any } t \in [0,T] \text{ and } m = 1,2,... \tag{16}$$

where the constant $c_6 > 0$ is independent from m and indicates the right side of the estimation (7).

Since the set V is a closed, bounded and convex set of the Hilbert space $W^1_\infty(0,T)$,we can choose a subsequence $\left\{v^m_p\right\}$ of the sequence $\{v^m\}$ such that

$$v^m_p \xrightarrow{*\text{-weakly}} v \text{ as } m \to \infty \text{ in } L_\infty(0,T)$$

$$\frac{dv^m_p}{dt} \xrightarrow{*\text{-weakly}} \frac{dv}{dt} \text{ as } m \to \infty \text{ in } L_\infty(0,T).$$

For simplicity, let's denote the subsequence $\left\{v^m_p\right\}$ by $\{v^m\}$. Namely, the following limit relations for the subsequence $\{v^m\}$ are written:

$$v^m \xrightarrow{*\text{-weakly}} v \text{ as } m \to \infty \text{ in } L_\infty(0,T) \tag{17}$$

$$\frac{dv^m}{dt} \xrightarrow{*\text{-weakly}} \frac{dv}{dt} \text{ as } m \to \infty \text{ in } L_\infty(0,T) \tag{18}$$

Since the set V is a closed, bounded and convex set of the Hilbert space $W^1_\infty(0,T)$, it is a *-weakly closed set on the set $W^1_\infty(0,T)$ according to the known theorem from [14]. Namely, $v \in V$. So, from (17) and (18) we can write the following limit relations:

$$\int_0^T v^m(t)q(t)dt \longrightarrow \int_0^T v(t)q(t)dt \text{ for any } q \in L_1(0,T) \text{ as } m \to \infty, \tag{19}$$

$$\int_0^T \frac{dv^m(t)}{dt}q_1(t)dt \longrightarrow \int_0^T \frac{dv(t)}{dt}q_1(t)dt \text{ for } any\ q_1 \in L_1(0,T) \text{ as } m \to \infty. \tag{20}$$

From the estimation (16), it is obtained the sequence $\{\psi_m\}$ is uniformly bounded in B_0. Hence, we can choose a subsequence $\{\psi^p_m\}$ of the sequence $\{\psi_m\}$ such that the subsequences $\{\psi^p_{km}(x,t)\}$, $\{\frac{\partial\psi^p_{km}(x,t)}{\partial x}\}$, $\{\frac{\partial^2\psi^p_{km}(x,t)}{\partial x^2}\}$, $\{\frac{\partial\psi^p_{km}(x,t)}{\partial t}\}$ weakly converge to the functions ψ_k, $\frac{\partial\psi_k}{\partial x}$, $\frac{\partial^2\psi_k}{\partial x^2}$, $\frac{\partial\psi_k}{\partial t}$ in $L_2(0,l)$ for *each* $t \in [0,T]$ and $m \to \infty$, respectively. For simplicity, let's denote the subsequence $\{\psi^p_m\}$ by $\{\psi_m\}$. Thus, we can write the limit relations:

$$\begin{aligned}
\psi_m &\xrightarrow{weakly} \psi \text{ in } L_2(0,l) \text{ for } each\ t \in [0,T] \text{ as } m \to \infty\\
\frac{\partial\psi_m}{\partial t} &\xrightarrow{weakly} \frac{\partial\psi}{\partial t} \text{ in } L_2(0,l) \text{ for } each\ t \in [0,T] \text{ as } m \to \infty\\
\frac{\partial\psi_m}{\partial x} &\xrightarrow{weakly} \frac{\partial\psi}{\partial x} \text{ in } L_2(0,l) \text{ for } each\ t \in [0,T] \text{ as } m \to \infty\\
\frac{\partial^2\psi_m}{\partial x^2} &\xrightarrow{weakly} \frac{\partial^2\psi}{\partial x^2} \text{ in } L_2(0,l) \text{ for } each\ t \in [0,T] \text{ as } m \to \infty
\end{aligned} \tag{21}$$

Now, let us show that the limit function $\psi(x,t)$ satisfies the equation (2) for *almost all* $x \in (0,l)$ and for *any* $t \in [0,T]$. Since the function $\psi_m(x,t)$ for *each* $m = 1,2,...$ is the solution of the problem (2)-(4), we can write the integral identity

$$\int_0^l \left[i\frac{\partial \psi_m}{\partial t} + a_0 \frac{\partial^2 \psi_m}{\partial x^2} + ia_1 \frac{\partial \psi_m}{\partial x} - a(x)\psi_m + v^m(t)\psi_m + ia_2|\psi_m|^2\psi_m - f \right] \bar{g}(x)dx = 0 \tag{22}$$

for each $t \in [0,T]$ and *any* $g \in L_2(0,l)$.

According to the embedding theorems in the studies [1, 20], the space B_0 is compactly embedded into the space $C^0([0,T], L_\infty(0,l))$ which implies that

$$\|\psi_m(.,t) - \psi(.,t)\|_{L_\infty(0,l)} \xrightarrow{\text{uniformly according to } t} 0 \quad \text{as } m \to \infty. \tag{23}$$

It is clear that

$$\begin{aligned} \int_0^l v^m(t)\psi_m(x,t)\bar{g}(x)dx &= \int_0^l v^m(t)\left(\psi_m(x,t) - \psi(x,t)\right)\bar{g}(x)dx + \int_0^l \left(v^m(t) - v(t)\right)\psi(x,t)\bar{g}(x)dx \\ &+ \int_0^l v(t)\psi(x,t)\bar{g}(x)dx. \end{aligned} \tag{24}$$

for *any* $g \in L_2(0,l)$. Also, it is written that

$$\begin{aligned} \left| \int_0^l v^m(t)\left(\psi_m(x,t) - \psi(x,t)\right)\bar{g}(x)dx \right| &\le |v^m(t)| \int_0^l \left|\psi_m(x,t) - \psi(x,t)\right| \left|g(x)\right| dx \\ &\le b_0 \sqrt{l} \|\psi_m(.,t) - \psi(.,t)\|_{L_\infty(0,l)} \|g\|_{L_2(0,l)} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \left| \int_0^l \left(v^m(t) - v(t)\right)\psi(x,t)\bar{g}(x)dx \right| &\le |v^m(t) - v(t)| \int_0^l \left|\psi(x,t)\right| \left|g(x)\right| dx \\ &\le \|v^m - v\|_{C^0[0,T]} \|\psi(.,t)\|_{L_2(0,l)} \|g\|_{L_2(0,l)} \end{aligned} \tag{26}$$

Since $\{v^m\} \in W^1_\infty(0,T)$ and the space $W^1_\infty(0,T)$ is compactly embedded into $C^0[0,T]$, the limit relation

$$\|v^m - v\|_{C^0[0,T]} \to 0 \text{ for } any\ t \in [0,T] \text{ as } m \to \infty \tag{27}$$

is written. Thus, taking into account the limit relations (23) and (27), the inequalities (25) and (26) in (24) if we get to the limit in (24) as $m \to \infty$, we obtain the limit relation

$$\int_0^l v^m(t)\psi_m(x,t)\bar{g}(x)dx \to \int_0^l v(t)\psi(x,t)\bar{g}(x)dx \tag{28}$$

for each $t \in [0,T]$ and *any* $g \in L_2(0,l)$. Now, let's prove that the limit relation

$$\lim_{m\to\infty} \int_0^l a_2|\psi_m(x,t)|^2\psi_m(x,t)\bar{g}(x)dx = \int_0^l a_2|\psi(x,t)|^2\psi(x,t)\bar{g}(x)dx$$

for each $t \in [0, T]$ and *any* $g \in L_2(0, l)$ is valid. It is clear that

$$\||\psi_m(.,t)|^2\psi_m(.,t)\|_{L_2(0,l)} = \left(\int_0^l \left||\psi_m(x,t)|^2\psi_m(x,t)\right|^2 dx\right)^{\frac{1}{2}} = \|\psi_m(.,t)\|^3_{L_6(0,l)}. \tag{29}$$

In (29), if we use the following known inequality in [15]

$$\|\psi_m(.,t)\|^3_{L_6(0,l)} \leq \beta_1\|\frac{\partial\psi_m(.,t)}{\partial x}\|_{L_2(0,l)}\|\psi_m(.,t)\|^2_{L_2(0,l)} \ \ for\ any\ t \in [0,T]\,,\ \beta_1 = const. > 0$$

and the estimation (16), we obtain

$$\||\psi_m(.,t)|^2\psi_m(.,t)\|_{L_2(0,l)} \leq c_7,\ m = 1, 2, 3, \cdots \tag{30}$$

where the constant $c_7 > 0$ is independent from m and t. Also, from the limit relation (23) we can easily say that the sequence $\{\psi_m(.,t)\}$ converges to the function $\psi(x,t)$ for *each* $t \in [0,T]$ at almost everywhere on $(0,l)$ .Thus, taking into account the inequality (30) and using the known Lemma in [15], we can write the limit relation

$$|\psi_m(x,t)|^2\psi_m(x,t) \xrightarrow{\ weakly\ } |\psi(x,t)|^2\psi(x,t) \text{ in } L_2(0,l) \ \ for\ each\ t \in [0,T] \text{ as } m \to \infty. \tag{31}$$

Thus, considering the limit relations (21), (28), (31) and taking the limit of (22) as $m \to \infty$ then we get the integral identity

$$\int_0^l \left[i\frac{\partial\psi}{\partial t} + a_0\frac{\partial^2\psi}{\partial x^2} + ia_1\frac{\partial\psi}{\partial x} - a(x)\psi + v(t)\psi + ia_2|\psi|^2\psi - f(x,t)\right]\bar{g}(x)dx = 0 \tag{32}$$

for *each* $t \in [0,T]$ and *any* $g \in L_2(0,l)$. From (32), we deduce that the function $\psi(x,t)$ satisfies the equation (2) for *almost all* $x \in (0,l)$ and *any* $t \in [0,T]$.

Now let us prove that the limit function $\psi(x,t)$ satisfies the initial condition (3). Since the space B_0 is compactly embedded into the space $C^0([0,T], L_2(0,l))$, it is written that

$$\|\psi_m(.,t) - \psi(.,t)\|_{L_2(0,l)} \ \to \ 0 \ \ for\ any\ t \in [0,T] \text{ as } m \to \infty. \tag{33}$$

It is clear that

$$0 \leq \int_0^l |\psi(x,0) - \varphi(x)|^2 dx \leq 2\int_0^l |\psi(x,0) - \psi_m(x,0)|^2 dx + 2\int_0^l |\psi_m(x,0) - \varphi(x)|^2 dx. \tag{34}$$

If we take the limit of (34) as $m \to \infty$ and use the initial condition $\psi_m(x,0) = \varphi(x)$ for $x \in (0,l)$ and the limit relation (33) for $t = 0$, we obtain

$$\int_0^l |\psi(x,0) - \varphi(x)|^2 dx = 0$$

which implies that

$$\psi(x,0) = \varphi(x) \ for\ almost\ all\ x \in (0,l)$$

it is follows that the limit function $\psi(x,t)$ satisfies the initial condition (3).

Now, let us prove that the limit function $\psi(x,t)$ satisfies the the boundary condition (4). Since the space $C^0([0,T],W_2^2(0,l))\cap C^1([0,T],L_2(0,l))$ is compactly embedded into the space $C^0([0,l],L_2(0,T))$, the limit relation

$$\|\psi_m(x,.)-\psi(x,.)\|^2_{L_2[0,T]} \quad\to\quad 0 \;\; for\ any\ x\in[0,l] \text{ as } m\to\infty. \tag{35}$$

is written. If we get to the limit as $m\to\infty$ in the inequalities

$$\begin{aligned} 0 &\leq \int_0^T |\psi(0,t)|^2dt \leq 2\int_0^T |\psi(0,t)-\psi_m(0,t)|^2dt + 2\int_0^T |\psi_m(0,t)|^2dt, \\ 0 &\leq \int_0^T |\psi(l,t)|^2dt \leq 2\int_0^T |\psi(l,t)-\psi_m(l,t)|^2dt + 2\int_0^T |\psi_m(l,t)|^2dt \end{aligned}$$

and use the boundary conditions $\psi_m(0,t)=\psi_m(l,t)=0$ for *any* $t\in(0,T)$, the limit relation (35) for $x=0$ and $x=l$, we obtain

$$\int_0^T |\psi(0,t)|^2dt = 0 \text{ and } \int_0^T |\psi(l,t)|^2dt = 0$$

which implies that

$$\psi(0,t)=\psi(l,t)=0 \;\; for\ almost\ all\ t\in(0,T),$$

it follows that the limit function $\psi(x,t)$ satisfies the boundary condition (4).

Consequently, we have proved that the function $\psi=\psi(x,t)=\psi(x,t;v)$ corresponding to limit function $v(t)$ of the sequence $\{v^m\}\subset V$ is a solution of the boundary value problem (2)-(4), which is the limit function of the sequence $\{\psi_m(x,t)\}$. Since the solution of the problem (2)-(4) is unique we write $\psi\in B_0$.

As known, the norm functions of the spaces $L_2(0,l)$ and $L_2(0,T)$ are lower weakly semicontinuous functionals [21]. If we consider that $\alpha\geq 0$, we can say that the functional $J_\alpha(v)$ is a lower weakly semicontinuous functional at $v\in V$. Therefore, we can write the relation

$$J_{\alpha^*} \leq J_\alpha(v) \leq \varliminf_{m\longrightarrow\infty} J_\alpha(v^m) = J_{\alpha^*}$$

it follows that $v\in V$ is a minimum of the functional $J_\alpha(v)$ on the set V. Namely, $v\in V$ is a solution of the optimal control problem (2)-(4), (6). □

4. Differentiability of the Functional

In this section, we constitute a adjoint problem and obtain the first variation of the objective functional $J_\alpha(v)$ by via of the adjoint problem. Later, a necessary optimality condition is given in variational form.

Firstly, we reformulate the optimization problem (2)-(4), (6) using a Lagrange multiplier function. Consider the minimization problem

$$L(v,\psi,\eta)\to\inf$$

where

$$\begin{aligned} L(v,\psi,\eta) &= J_\alpha(v) + \frac{1}{2}\int_\Omega \left(i\frac{\partial\psi}{\partial t} + a_0\frac{\partial^2\psi}{\partial x^2} + ia_1\frac{\partial\psi}{\partial x} - a(x)\psi + v(t)\psi + ia_2|\psi|^2\psi - f\right)\overline{\eta}dxdt \\ &+\frac{1}{2}\int_\Omega \left(-i\frac{\partial\overline{\psi}}{\partial t} + a_0\frac{\partial^2\overline{\psi}}{\partial x^2} - ia_1\frac{\partial\overline{\psi}}{\partial x} - a(x)\overline{\psi} + v(t)\overline{\psi} - ia_2|\psi|^2\overline{\psi} - \overline{f}\right)\eta dxdt, \end{aligned}$$

the function $\eta = \eta(x,t) = \eta(x,t;v)$ is Lagrange multiplier and the function $\psi = \psi(x,t)$ is a solution of the boundary value problem (2)-(4).Thus, from the stationarity condition of the Lagrange functional $L(v,\psi,\eta)$, we obtain the following adjoint problem:

$$i\frac{\partial\eta}{\partial t} + a_0\frac{\partial^2\eta}{\partial x^2} + ia_1\frac{\partial\eta}{\partial x} - a(x)\eta + v(t)\eta - 2ia_2|\psi|^2\eta + ia_2\psi^2\bar{\eta} = 0, \tag{36}$$

$$\eta(x,T) = -2i(\psi(x,T) - y(x)),\ \ x \in (0,l) \tag{37}$$

$$\eta(0,t) = \eta(l,t) = 0,\ \ t \in (0,T) \tag{38}$$

where the function $\psi = \psi(x,t) \equiv \psi(x,t;v)$ is a solution of problem (2)-(4) for any $v \in V$. If we apply the transform $\tau = T - t$ to adjoint problem (36)-(38) for $y \in \mathring{W}_2^2(0,l)$, it is seen that the adjoint problem (36)-(38) is a initial boundary value problem in the form of problem (2)-(4). Therefore, we can write the following theorem for the solution of adjoint problem (36)-(38):

Theorem 4.1. *Assume that the hypotheses of Theorem 2.1 hold and let $y \in \mathring{W}_2^2(0,l)$ be given function. Then, the adjoint problem (36)-(38) has a unique solution $\eta \in B_0$ for any $v \in V$ and the following estimation is valid for this solution:*

$$\begin{aligned}\|\eta(.,t)\|^2_{\mathring{W}_2^2(0,l)} + \left\|\frac{\partial\eta}{\partial t}\right\|^2_{L_2(0,l)} \le\ & c_8\Big(\|\varphi\|^2_{\mathring{W}_2^2(0,l)} + \|\varphi\|^6_{\mathring{W}_2^2(0,l)} + \|\varphi\|^6_{\mathring{W}_2^1(0,l)} \\ & + \|\varphi\|^{18}_{\mathring{W}_2^1(0,l)} + \|f\|^2_{W_2^{0,1}(\Omega)} + \|f\|^6_{W_2^{0,1}(\Omega)} + \|y\|^2_{\mathring{W}_2^2(0,l)}\Big)\end{aligned} \tag{39}$$

for any $t \in [0,T]$, where $c_8 > 0$ is independent from t.

Using Galerkin's method we can easily prove the theorem 4.1 as the proof of theorem 2.1.

Now, let us find the increment of the functional $J_\alpha(v)$ for $\forall v \in V$. Let the function $\Delta v \in W_\infty^1(0,T)$ be an increment given to any $v \in V$ such that $v + \Delta v \in V$. Then, using the formula (6) and (15) we can write the increment of the functional $J_\alpha(v)$ for any $v \in V$ as the following:

$$\begin{aligned}\Delta J_\alpha(v) &= J_\alpha(v+\Delta v) - J_\alpha(v) \\ &= \int_0^l \left|\psi(x,T;v+\Delta v) - y(x)\right|^2 dx + \alpha\int_0^T |v+\Delta v - w|^2 dt + \alpha\int_0^T \left|\frac{d(v+\Delta v)}{dt} - \frac{dw}{dt}\right|^2 dt \\ &\quad - \int_0^l \left|\psi(x,T;v) - y(x)\right|^2 dx - \alpha\int_0^T |v(t) - w(t)|^2 dt - \alpha\int_0^T \left|\frac{dv}{dt} - \frac{dw}{dt}\right|^2 dt \\ &= \Delta J_0(v) + 2\alpha\int_0^T (v(t) - w(t))\,\Delta v(t)dt + 2\alpha\int_0^T \left(\frac{dv}{dt} - \frac{dw}{dt}\right)\frac{d\Delta v}{dt}dt + \alpha\|\Delta v\|^2_{W_2^1(0,T)} \\ &= 2\int_0^l \mathrm{Re}[(\psi(x,T) - y(x))(\Delta\bar{\psi}(x,T)]dx + \left\|\Delta\psi(.,T)\right\|^2_{L_2(0,l)} \\ &\quad + 2\alpha\int_0^T (v(t) - w(t))\,\Delta v(t)dt + 2\alpha\int_0^T \left(\frac{dv(t)}{dt} - \frac{dw(t)}{dt}\right)\frac{d\Delta v(t)}{dt}dt + \alpha\|\Delta v\|^2_{W_2^1(0,T)}\end{aligned} \tag{40}$$

where $\Delta\psi = \Delta\psi(x,t) \equiv \psi(x,t;v+\Delta v) - \psi(x,t;v)$ is the solution of the problem (8)-(10) for $v \in V$.

Lemma 4.2.

$$
\begin{aligned}
2\int_0^l \operatorname{Re}\left[(\psi(x,T)-y(x))(\Delta\bar{\psi}(x,T)\right]dx = \quad & \int_\Omega \operatorname{Re}(\psi(x,t)\overline{\eta}(x,t))\Delta v(t)dxdt + \int_\Omega \operatorname{Re}(\Delta\psi\bar{\eta})\Delta v(t)dxdt \\
& -a_2\int_\Omega (|\psi_\Delta|^2-|\psi|^2)\operatorname{Im}(\Delta\psi\bar{\eta})dxdt \\
& -a_2\int_\Omega |\Delta\psi|^2\operatorname{Im}\left(\psi\bar{\eta}\right)dxdt.
\end{aligned}
$$

Proof. It is clear that since $\psi \in B_0$, the function $\Delta\psi = \Delta\psi(x,t) \equiv \psi(x,t;v+\Delta v) - \psi(x,t;v)$ satisfies integral identity

$$
\begin{aligned}
& \int_\Omega \left(i\frac{\partial\Delta\psi}{\partial t} + a_0\frac{\partial^2\Delta\psi}{\partial x^2} + ia_1\frac{\partial\Delta\psi}{\partial x} - a(x)\Delta\psi + (v+\Delta v)\Delta\psi\right)\overline{\phi}_1(x,t)dxdt \\
& + \int_\Omega ia_2\left[\left(|\psi_\Delta|^2+|\psi|^2\right)\Delta\psi + \psi_\Delta\psi\Delta\bar{\psi}\right]\overline{\phi}_1(x,t)dxdt \\
= \quad & -\int_\Omega \Delta v(t)\psi\overline{\phi}_1(x,t)dxdt \qquad (41)
\end{aligned}
$$

for any function $\phi_1 = \phi_1(x,t) \in L_2(\Omega)$ and the conditions (9), (10). Taking the function $\eta = \eta(x,t) \in L_2(\Omega)$ instead of the test function $\phi_1(x,t)$ in identity (41) we obtain the identity

$$
\begin{aligned}
& \int_\Omega \left[i\frac{\partial\Delta\psi}{\partial t} + a_0\frac{\partial^2\Delta\psi}{\partial x^2} + ia_1\frac{\partial\Delta\psi}{\partial x} - a(x)\Delta\psi + (v+\Delta v)\Delta\psi\right. \\
& \left. + ia_2\left[\left(|\psi_\Delta|^2+|\psi|^2\right)\Delta\psi + \psi_\Delta\psi\Delta\bar{\psi}\right]\right]\overline{\eta}(x,t)dxdt \\
= \quad & -\int_\Omega \Delta v(t)\psi\overline{\eta}(x,t)dxdt. \qquad (42)
\end{aligned}
$$

Also, since the function $\eta \in B_0$ is a solution of the adjoint problem (36)-(38), it satisfies the following identity for *any* $\phi_2 = \phi_2(x,t) \in L_2(\Omega)$

$$
\int_\Omega \left[i\frac{\partial\eta}{\partial t} + a_0\frac{\partial^2\eta}{\partial x^2} + ia_1\frac{\partial\eta}{\partial x} - a(x)\eta + v(t)\eta - 2ia_2|\psi|^2\eta + ia_2\psi^2\bar{\eta}\right]\overline{\phi}_2dxdt = 0. \qquad (43)
$$

Let's put the function $\Delta\psi(x,t)$ instead of the test function $\phi_2 = \phi_2(x,t)$ in identity (43). Later, if we apply the integration by parts formula to obtained identity, we get

$$
\begin{aligned}
& \int_\Omega \left[\left(-i\frac{\partial\Delta\overline{\psi}}{\partial t} + a_0\frac{\partial^2\Delta\overline{\psi}}{\partial x^2} - ia_1\frac{\partial\Delta\overline{\psi}}{\partial x} - a(x)\Delta\overline{\psi} + v(t)\Delta\overline{\psi} - 2ia_2|\psi|^2\Delta\overline{\psi}\right)\eta + ia_2\psi^2\bar{\eta}\Delta\overline{\psi}\right]dxdt \\
= \quad & -2\int_0^l \left(\psi(x,T)-y(x)\right)\Delta\overline{\psi}(x,T)dx
\end{aligned}
$$

and its complex conjugate

$$\begin{aligned}
&\int_{\Omega}\left(i\frac{\partial\Delta\psi}{\partial t}+a_0\frac{\partial^2\Delta\psi}{\partial x^2}+ia_1\frac{\partial\Delta\psi}{\partial x}-a(x)\Delta\psi+v(t)\Delta\psi+2ia_2|\psi|^2\Delta\psi\right)\eta dxdt\\
&-ia_2\int_{\Omega}\overline{\psi}^2\eta\Delta\psi dxdt\\
=\;&-2\int_0^l\left(\overline{\psi}(x,T)-y(x)\right)\Delta\psi(x,T)dx. \qquad (44)
\end{aligned}$$

Also, if we put the function $\eta(x,t)$ instead of the function $\phi_1(x,t)\in L_2(\Omega)$ in (41), we have

$$\begin{aligned}
&\int_{\Omega}\Bigg[i\frac{\partial\Delta\psi}{\partial t}+a_0\frac{\partial^2\Delta\psi}{\partial x^2}+ia_1\frac{\partial\Delta\psi}{\partial x}-a(x)\Delta\psi+(v+\Delta v)\Delta\psi\\
&+ia_2\left[\left(|\psi_\Delta|^2+|\psi|^2\right)\Delta\psi+\psi_\Delta\psi\Delta\bar{\psi}\right]\Bigg]\overline{\eta}(x,t)dxdt\\
=\;&-\int_{\Omega}\Delta v(t)\psi\overline{\eta}(x,t)dxdt \qquad (45)
\end{aligned}$$

Subtracting the (44) from (45), we get

$$\begin{aligned}
&\int_{\Omega}\left[\Delta v\Delta\psi\overline{\eta}+ia_2\left(\left(|\psi_\Delta|^2-|\psi|^2\right)\Delta\psi+\psi_\Delta\psi\Delta\bar{\psi}\right)\overline{\eta}+ia_2\overline{\psi}^2\eta\Delta\psi\right]dxdt\\
=\;&-\int_{\Omega}\Delta v(t)\psi\overline{\eta}(x,t)dxdt+2\int_0^l\left(\overline{\psi}(x,T)-y(x)\right)\Delta\psi(x,T)dx.
\end{aligned}$$

Summing the above equality with its complex conjugate, we obtain

$$\begin{aligned}
4\int_0^l\mathrm{Re}\left(\psi(x,T)-y(x)\right)\Delta\overline{\psi}(x,T)dx =\;& 2\int_{\Omega}\mathrm{Re}(\psi(x,t)\overline{\eta}(x,t))\Delta v(t)dxdt\\
&+2\int_{\Omega}\mathrm{Re}(\Delta\psi\bar{\eta})\Delta v(t)dxdt\\
&-2a_2\int_{\Omega}(|\psi_\Delta|^2-|\psi|^2)\mathrm{Im}(\Delta\psi\bar{\eta})dxdt\\
&-2a_2\int_{\Omega}|\Delta\psi|^2\mathrm{Im}\left(\psi\bar{\eta}\right)dxdt
\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
2\int_0^l \operatorname{Re}\left(\psi(x,T)-y(x)\right)\Delta\overline{\psi}(x,T)dx = & \int_\Omega \operatorname{Re}(\psi(x,t)\overline{\eta}(x,t))\Delta v(t)dxdt \\
& + \int_\Omega \operatorname{Re}(\Delta\psi\overline{\eta})\Delta v(t)dxdt \\
& - a_2\int_\Omega (|\psi_\Delta|^2-|\psi|^2)\operatorname{Im}(\Delta\psi\overline{\eta})dxdt \\
& - a_2\int_\Omega |\Delta\psi|^2\operatorname{Im}\left(\psi\overline{\eta}\right)dxdt.
\end{aligned}
$$

Thus, lemma (4.2) is proved.

Using lemma (4.2) in (40), we can write the increment of $J_\alpha(v)$ as

$$
\begin{aligned}
\Delta J_\alpha(v) \quad = \quad & \int_\Omega \operatorname{Re}(\psi(x,t)\overline{\eta}(x,t))\Delta v(t)dxdt + 2\alpha\int_0^T \left(v(t)-w(t)\right)\Delta v(t)dt \\
& + 2\alpha\int_0^T \left(\frac{dv(t)}{dt}-\frac{dw(t)}{dt}\right)\frac{d\Delta v(t)}{dt}dt + R
\end{aligned}
$$

where

$$
\begin{aligned}
R \quad = \quad & \int_\Omega \operatorname{Re}(\Delta\psi\overline{\eta})\Delta v(t)dxdt - a_2\int_\Omega (|\psi_\Delta|^2-|\psi|^2)\operatorname{Im}(\Delta\psi\overline{\eta})dxdt \\
& - a_2\int_\Omega |\Delta\psi|^2\operatorname{Im}\left(\psi\overline{\eta}\right)dxdt + \left\|\Delta\psi(.,T)\right\|^2_{L_2(0,l)} + \alpha\|\Delta v\|^2_{W_2^1(0,T)}.
\end{aligned}
\tag{46}
$$

□

Theorem 4.3. *Assume that the conditions of Theorem 4.1 are fulfilled and let $w \in W_2^1(0,T)$, $y \in \overset{\circ}{W}{}_2^2(0,l)$ be given functions. Then, the functional $J_\alpha(v)$ is differentiable on the set V and the following formula is valid for its first variation:*

$$
\delta J_\alpha(v,h) = \int_\Omega \operatorname{Re}(\psi(x,t)\overline{\eta}(x,t))h(t)dxdt + 2\alpha\int_0^T \left(v(t)-w(t)\right)h(t)dt + 2\alpha\int_0^T \left(\frac{dv(t)}{dt}-\frac{dw(t)}{dt}\right)\frac{dh(t)}{dt}dt
$$

for any $h \in W_\infty^1(0,T)$, where $\psi = \psi(x,t)$ is a solution of the initial boundary value problem (2)-(4), $\eta = \eta(x,t)$ is a solution of the adjoint problem (36)-(38) for any $v \in V$.

Proof. Firstly, it is proved that $R = o\left(\|\Delta v\|_{W_\infty^1(0,T)}\right)$, where $o\left(\|\Delta v\|_{W_\infty^1(0,T)}\right)$ represents ″ higher-order terms″ which go to 0 faster than $\|\Delta v\|_{W_\infty^1(0,T)}$ as $\|\Delta v\|_{W_\infty^1(0,T)}$ approaches 0, i.e.

$$
\lim_{\|\Delta v\|_{W_\infty^1(0,T)}\to 0} \frac{o\left(\|\Delta v\|_{W_\infty^1(0,T)}\right)}{\|\Delta v\|_{W_\infty^1(0,T)}} = 0.
$$

From (46), we get

$$
\begin{aligned}
|R| \leq\ & \int_{\Omega} \left|\Delta\psi\right| \left|\eta\right| |\Delta v|\, dxdt + a_2 \int_{\Omega} \left| |\psi_\Delta|^2 - |\psi|^2 \right| \left|\Delta\psi\right| \left|\eta\right| dxdt \\
& + a_2 \int_{\Omega} |\Delta\psi|^2 \left|\psi\right| \left|\eta\right| dxdt + \left\|\Delta\psi(.,T)\right\|^2_{L_2(0,l)} + \alpha \|\Delta v\|^2_{W_2^1(0,T)}.
\end{aligned}
$$

If we apply Young's inequality to the above inequality, we obtain

$$
\begin{aligned}
|R| \leq\ & \frac{1}{2}\int_{\Omega} \left|\Delta\psi\right|^2 dxdt + \frac{1}{2}\int_{\Omega} \left|\eta\right|^2 |\Delta v|^2 dxdt + a_2 \int_{\Omega} (|\psi_\Delta| + |\psi|) \left|\Delta\psi\right|^2 \left|\eta\right| dxdt \\
& + a_2 \int_{\Omega} |\Delta\psi|^2 \left|\psi\right| \left|\eta\right| dxdt + \left\|\Delta\psi(.,T)\right\|^2_{L_2(0,l)} + \alpha \|\Delta v\|^2_{W_2^1(0,T)} \\
\leq\ & \frac{1}{2}\int_{\Omega} \left|\Delta\psi\right|^2 dxdt + \frac{1}{2}\int_{\Omega} \left|\eta\right|^2 |\Delta v|^2 dxdt + \frac{1}{2}a_2 \int_{\Omega} (|\psi_\Delta| + |\psi|)^2 \left|\Delta\psi\right|^2 dxdt \\
& + \frac{1}{2}a_2 \int_{\Omega} \left|\Delta\psi\right|^2 \left|\eta\right|^2 dxdt + \frac{1}{2}a_2 \int_{\Omega} |\Delta\psi|^2 \left|\psi\right|^2 dxdt + \frac{1}{2}a_2 \int_{\Omega} |\Delta\psi|^2 \left|\eta\right|^2 dxdt \\
& + \left\|\Delta\psi(.,T)\right\|^2_{L_2(0,l)} + \alpha \|\Delta v\|^2_{W_2^1(0,T)} \\
\leq\ & \frac{1}{2}\int_{\Omega} \left|\Delta\psi\right|^2 dxdt + \frac{1}{2}\int_{\Omega} \left|\eta\right|^2 |\Delta v|^2 dxdt + a_2 \int_{\Omega} \left(|\psi_\Delta|^2 + |\psi|^2\right) \left|\Delta\psi\right|^2 dxdt \\
& + a_2 \int_{\Omega} \left|\Delta\psi\right|^2 \left|\eta\right|^2 dxdt + \frac{1}{2}a_2 \int_{\Omega} |\Delta\psi|^2 \left|\psi\right|^2 dxdt + \left\|\Delta\psi(.,T)\right\|^2_{L_2(0,l)} + \alpha \|\Delta v\|^2_{W_2^1(0,T)}
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
|R| \leq\ & \frac{T}{2}\max_{0\leq t\leq T} \left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)} + \frac{1}{2}\left(\max_{0\leq t\leq T} \left\|\eta(.,t)\right\|^2_{L_\infty(0,l)}\right) \|\Delta v\|^2_{L_2(0,T)} \\
& + a_2 \int_{\Omega} \left(|\psi_\Delta|^2 + |\psi|^2\right) \left|\Delta\psi\right|^2 dxdt + a_2 \left(\max_{0\leq t\leq T} \left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)}\right) \int_0^T \left\|\eta(.,t)\right\|^2_{L_\infty(0,l)} dt \\
& + \frac{1}{2}a_2 \left(\max_{0\leq t\leq T} \left\|\Delta\psi(.,t)\right\|^2_{L_2(0,l)}\right) \int_0^T \left\|\psi(.,t)\right\|^2_{L_\infty(0,l)} dt + \left\|\Delta\psi(.,T)\right\|^2_{L_2(0,l)} + \alpha \|\Delta v\|^2_{W_2^1(0,T)}.
\end{aligned}
\tag{47}
$$

According to known inequality in [15], we have the inequality

$$
\begin{aligned}
\left\|\psi(.,t)\right\|^2_{L_\infty(0,l)} &\leq \beta_2 \left\|\frac{\partial\psi(.,t)}{\partial x}\right\|_{L_2(0,l)} \left\|\psi(.,t)\right\|_{L_2(0,l)}, \ \beta_2 = const. > 0 \\
\left\|\psi(.,t)\right\|^2_{L_\infty(0,l)} &\leq \frac{\beta_2}{2}\left(\left\|\frac{\partial\psi(.,t)}{\partial x}\right\|^2_{L_2(0,l)} + \left\|\psi(.,t)\right\|^2_{L_2(0,l)}\right).
\end{aligned}
\tag{48}
$$

for any $t \in [0,T]$. Using the inequalities (13), (48) and (11), the estimations (7), (39) in the inequality (47), we obtain

$$\begin{aligned}|R| &\leq c_9\,\|\Delta v\|^2_{L_2(0,T)} + \alpha\|\Delta v\|^2_{W_2^1(0,T)}\\ &\leq c_{10}\|\Delta v\|^2_{W_2^1(0,T)}\\ &\leq c_{11}\|\Delta v\|^2_{W_\infty^1(0,T)}\end{aligned}$$

which shows that $R = o\left(\|\Delta v\|_{W_\infty^1(0,T)}\right)$, where the constants $c_9, c_{10}, c_{11} > 0$ are independent from Δv and t.

Thus, we can write the increment of the functional $J_\alpha(v)$ as

$$\begin{aligned}\Delta J_\alpha(v) &= \int_\Omega \mathrm{Re}(\psi(x,t)\overline{\eta}(x,t))\Delta v(t)dxdt + 2\alpha\int_0^T (v(t)-w(t))\,\Delta v(t)dt\\ &\quad +2\alpha\int_0^T\left(\frac{dv(t)}{dt}-\frac{dw(t)}{dt}\right)\frac{d\Delta v(t)}{dt}dt + o\left(\|\Delta v\|^2_{W_\infty^1(0,T)}\right).\end{aligned}\tag{49}$$

If we consider the function $\theta h \in W_\infty^1(0,T)$ for any $0<\theta<1$ and $h \in W_\infty^1(0,T)$ instead of the function $\Delta v \in W_\infty^1(0,T)$ in (49), we can easily write that

$$\begin{aligned}\Delta J_\alpha(v) &= J_\alpha(v+\theta h) - J_\alpha(v)\\ &= \int_\Omega \mathrm{Re}(\psi(x,t)\overline{\eta}(x,t))\theta h(t)dxdt + 2\alpha\int_0^T (v(t)-w(t))\,\theta h(t)dt\\ &\quad +2\alpha\int_0^T\left(\frac{dv(t)}{dt}-\frac{dw(t)}{dt}\right)\frac{d\theta h(t)}{dt}dt + o(\theta)\end{aligned}$$

which is equivalent to

$$\begin{aligned}J_\alpha(v+\theta h) - J_\alpha(v) &= \theta\left[\int_\Omega \mathrm{Re}(\psi(x,t)\overline{\eta}(x,t))h(t)dxdt + 2\alpha\int_0^T (v(t)-w(t))\,h(t)dt\right.\\ &\quad \left. + 2\alpha\int_0^T\left(\frac{dv(t)}{dt}-\frac{dw(t)}{dt}\right)\frac{dh(t)}{dt}dt\right] + o(\theta)\end{aligned}$$

which shows that for the first variation $\delta J_\alpha(v,h)$ of the functional $J_\alpha(v)$ is valid the formula

$$\begin{aligned}\delta J_\alpha(v,h) &= \lim_{\theta\to 0^+}\frac{J_\alpha(v+\theta h)-J_\alpha(v)}{\theta}\\ &= \int_\Omega \mathrm{Re}(\psi(x,t)\overline{\eta}(x,t))h(t)dxdt\\ &\quad +2\alpha\int_0^T (v(t)-w(t))\,h(t)dt + 2\alpha\int_0^T\left(\frac{dv(t)}{dt}-\frac{dw(t)}{dt}\right)\frac{dh(t)}{dt}dt.\end{aligned}$$

□

Theorem 4.4. *Suppose that the conditions of Theorem 4.3 are fulfilled and let*

$$V_* \equiv \left\{ v^* : v^* \in V,\ J_\alpha(v^*) = \inf_{v\in V} J_\alpha(v) = J_{\alpha^*} \right\}$$

be the set of solutions of the optimal control problem (2)-(4), (6). Then, for any $v^* \in V_*$ the inequality

$$\int_\Omega \mathrm{Re}(\psi^*(x,t)\overline{\eta}^*(x,t))\,(v(t)-v^*(t))\,dxdt + 2\alpha\int_0^T (v^*(t)-w(t))\,(v(t)-v^*(t))\,dt$$
$$+2\alpha\int_0^T \left(\frac{dv^*(t)}{dt}-\frac{dw(t)}{dt}\right)\left(\frac{dv(t)}{dt}-\frac{dv^*(t)}{dt}\right)dt \geq 0,\ \forall v\in V$$

is valid, where the functions $\psi^*(x,t) \equiv \psi(x,t,;v^*)$ and $\eta^*(x,t) \equiv \eta(x,t;v^*)$ are solutions of the boundary value problem (2)-(4) and adjoint problem (36)-(38) for the $v^* \in V$, respectively .

Proof. Let $v \in V$ be any control, $v^* \in V$ be any optimal control. Since the set V is a convex set it is written that $v^* + \theta(v - v^*) \in V$ for $v^* \in V$ and any $v \in V$, $\forall\theta \in (0,1)$. Therefore, according to known theorem from [18] we can write the inequality

$$\left.\frac{d}{d\theta}J_\alpha\left(v^*+\theta(v-v^*)\right)\right|_{\theta=0} = \delta J_\alpha(v^*, v-v^*) \geq 0,\ \ \forall v\in V$$

which is equivalent to

$$\int_\Omega \mathrm{Re}(\psi^*(x,t)\overline{\eta}^*(x,t))\,(v(t)-v^*(t))\,dxdt + 2\alpha\int_0^T (v^*(t)-w(t))\,(v(t)-v^*(t))\,dt$$
$$+2\alpha\int_0^T \left(\frac{dv^*(t)}{dt}-\frac{dw(t)}{dt}\right)\left(\frac{dv(t)}{dt}-\frac{dv^*(t)}{dt}\right)dt \geq 0,\ \ \forall v\in V$$

□

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