Filomat 30:4 (2016), 913–920
DOI 10.2298/FIL1604913E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Matrices over Hyperbolic Split Quaternions

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Abstract. In this paper, we present some important properties of matrices over hyperbolic split quaternions. We examine hyperbolic split quaternion matrices by their split quaternion matrix representation.

1. Introduction

The collection of hyperbolic numbers (double, perplex or split complex numbers) is two dimensional commutative algebra over real numbers different from complex and dual numbers. Every hyperbolic number has the form $x + yh$ where x and y are real numbers. The unit h is defined by the property $h^2 = 1$. On the other hand, the split quaternions (or coquaternions) are elements of four dimensional noncommutative and associative algebra. The set of hyperbolic split quaternions is an extension of split quaternions by hyperbolic number coefficients.

The matrices over noncommutative algebras is a new topic. Most well known studies about this topic are related to quaternion matrices such as [10], [11], [4], [5]. Since the rotations in Minkowski 3 space can be stated with timelike split quaternions in [7] such as expressing the Euclidean rotations using quaternions, the set of matrices over split quaternions becomes an interesting area. Firstly, the set of split quaternion matrices is introduced in [1]. Then, eigenvalues of split quaternion matrices are discussed in [2] and the relations between the eigenvalues of a split quaternion matrix and its complex adjoint matrix are presented. On the other hand, the complex split quaternions and their matrices are investigated in [3].

In this paper, we give a brief summary of split quaternions and their matrices. Then we present hyperbolic split quaternions to provide the necessary background. And we introduce the matrices over hyperbolic split quaternions and give some properties of them. Moreover, we define the $2n \times 2n$ split quaternion matrix representation $S(A)$ of a $n \times n$ hyperbolic split quaternion matrix A. We prove that A is invertible if and only if $S(A)$ is invertible. Then, we obtain a method to find the inverse of hyperbolic split quaternion matrices. And, we give the relations between left and right split quaternion eigenvalues of a hyperbolic split quaternion matrix and its split quaternion matrix representation.

2010 *Mathematics Subject Classification*. Primary 15A66; Secondary 51B20
Keywords. Split Quaternion; Hyperbolic Split Quaternion.
Received: 24 June 2015; Accepted: 28 September 2015
Communicated by Gradimir Milovanović and Yilmaz Simsek
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2. Preliminaries

2.1. Split Quaternions and Their Matrices

The set of split quaternions, which was introduced by James Cockle in 1849, can be represented as

$$\widehat{\mathbb{H}} = \{q = q_0 + q_1 i + q_2 j + q_3 k : \ q_0, \ q_1, \ q_2, \ q_3 \in \mathbb{R}\}.$$

Here the imaginary units satisfy the following relations: $i^2 = -1$, $j^2 = k^2 = ijk = 1$, $ij = -ji = k$, $jk = -kj = -i$, $ki = -ik = j$. Unlike quaternion algebra, the set of split quaternions contains zero divisors, nilpotent elements and nontrivial idempotents. For any $q = q_0 + q_1 i + q_2 j + q_3 k \in \widehat{\mathbb{H}}$, we define scalar part of q as $S_q = q_0$, vector part of q as $V_q = q_1 i + q_2 j + q_3 k$ and conjugate of q as $\overline{q} = S_q - V_q = q_0 - q_1 i - q_2 j - q_3 k$. If $S_q = 0$ then q is called pure split quaternion. The set of pure split quaternions are identified with the Minkowski 3 space. The Minkowski 3 space is Euclidean 3 space with the Lorentzian inner product $\langle u, v\rangle_{\mathbb{L}} = -u_1 v_1 + u_2 v_2 + u_3 v_3$, where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{E}^3$ and denoted by $\mathbb{E}^3_1$. The sum and product of two split quaternions $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$ are defined as $p + q = S_p + S_q + V_p + V_q$ and $pq = S_p S_q + \left\langle V_p, V_q \right\rangle_{\mathbb{L}} + S_p V_q + S_q V_p + V_p \times_{\mathbb{L}} V_q$, respectively. Here $\times_{\mathbb{L}}$ denotes Lorentzian vector product and is defined as

$$u \times_{\mathbb{L}} v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

for vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ of Minkowski 3 space. The term $I_q = q\overline{q} = \overline{q}q = q_0^2 + q_1^2 - q_2^2 - q_3^2$ characterizes any split quaternion q. The split quaternion q is called spacelike, timelike or null, if $I_q < 0$, $I_q > 0$ or $I_q = 0$, respectively. This characterization of split quaternions plays an important role on expressing the Lorentzian rotations. Also, kind of rotation angle (spherical or hyperbolic) and rotation axis of a Lorentzian rotation depends on characterization of the split quaternion. And the norm of q is defined as $N_q = \sqrt{\left|q_0^2 + q_1^2 - q_2^2 - q_3^2\right|}$. If $N_q = 1$, then q is called unit split quaternion. If $I_q \neq 0$, then $q^{-1} = \dfrac{\overline{q}}{I_q}$. For details about split quaternions, see [8], [9].

The set of $m \times n$ matrices over split quaternions, which is denoted by $M_{m\times n}(\widehat{\mathbb{H}})$, with ordinary matrix addition and multiplication is a ring with unity. For any $A = (a_{ij}) \in M_{m\times n}(\widehat{\mathbb{H}})$, the conjugate of A is defined as $\overline{A} = \left(\overline{a_{ij}}\right) \in M_{m\times n}(\widehat{\mathbb{H}})$, the transpose of A is defined as $A^T = \left(a_{ji}\right) \in M_{n\times m}(\widehat{\mathbb{H}})$ and conjugate transpose of A is defined as $A^* = (\overline{a_{ji}}) \in M_{n\times m}(\widehat{\mathbb{H}})$. For any $n \times n$ split quaternionic matrix A, if there exists any matrix $B \in M_{n\times n}(\widehat{\mathbb{H}})$ such that $AB = BA = I$ then A is called invertible matrix and B is called the inverse of A. In the study [1], it was proved that $AB = I$ implies that $BA = I$ for any $A, B \in M_{n\times n}(\widehat{\mathbb{H}})$. So, the right inverse and left inverse of any square split quaternion matrix are always equal. Since there exists a unique representation of any split quaternion q as $q = c_1 + c_2 j$ where c_1 and c_2 are complex numbers, we may write any $A \in M_{n\times n}(\widehat{\mathbb{H}})$ as $A = A_1 + A_2 j$ where $A_1, A_2 \in M_{n\times n}(\mathbb{C})$ are uniquely determined. Using this representation, we may define the $2n \times 2n$ complex matrix

$$\chi_A = \begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix}$$

which is called *the complex adjoint matrix of A*. Since the set of split quaternions are noncommutative, defining a determinant function for matrices over split quaternion is a special problem. A determinant function is defined by using complex adjoint matrix of A as $|A|_q = \det(\chi_A)$ and is called q determinant of A in [1]. For some nonzero split quaternionic vector x, if $\lambda \in \widehat{\mathbb{H}}$ satisfies the equation $Ax = \lambda x$, (or $Ax = x\lambda$) then λ is called left (or right) eigenvalue of A. The sets of left and right eigenvalues are denoted by $\sigma_l(A)$ and $\sigma_r(A)$, respectively. The eigenvalues of split quaternion matrices are deeply discussed in [2] and [1]. The relation between the right eigenvalue of a split quaternion matrix A and its complex adjoint matrix is found as $\sigma_r(A) \cap \mathbb{C} = \sigma(\chi_A)$ in [2].

2.2. Hyperbolic Numbers and Hyperbolic Split Quaternions

The set of hyperbolic numbers is defined as follows:

$$H = \{A = a + ha^* : a, a^* \in \mathbb{R}\}$$

where the unit h satisfies $h^2 = 1$. For any hyperbolic numbers $A = a + ha^*$ and $B = b + hb^*$, the sum and product of A and B are defined as $A + B = a + b + h(a^* + b^*)$ and $AB = ab + a^*b^* + h(ab^* + ba^*)$, respectively [6]. The set of hyperbolic split quaternions, which can be considered as an extension of split quaternions by hyperbolic numbers, is represented as

$$\widehat{\mathbb{H}}_H = \{Q = Q_0 + Q_1 i + Q_2 j + Q_3 k : Q_0,\ Q_1,\ Q_2,\ Q_3 \in H\}$$

where $i^2 = -1$, $j^2 = k^2 = 1$, $ij = -ji = k$, $jk = -kj = -i$, $ki = -ik = j$, $hi = ih$, $hj = jh$, $hk = kh$, $h^2 = 1$. As a consequence of this representation, any hyperbolic split quaternion Q can also be written uniquely as $Q = q + hq^* = q + q^*h$ where $q, q^* \in \widehat{\mathbb{H}}$. For any $Q = Q_0 + Q_1 i + Q_2 j + Q_3 k = q + hq^* \in \widehat{\mathbb{H}}_H$, we define the hyperbolic number part of Q as $H_Q = Q_0$, vector part of Q as $V_Q = Q_1 i + Q_2 j + Q_3 k$ and Hamiltonian conjugate of Q as $\overline{Q} = Q_0 - Q_1 i - Q_2 j - Q_3 k = H_Q - V_Q$. If $H_Q = 0$, then Q is called pure hyperbolic split quaternion. The sum and product of two hyperbolic split quaternions $Q = Q_0 + Q_1 i + Q_2 j + Q_3 k = q + hq^*$ and $P = P_0 + P_1 i + P_2 j + P_3 k = p + hp^*$ are defined as $Q + P = q + p + h(q^* + p^*)$, $QP = qp + q^*p^* + h(qp^* + q^*p)$, respectively.

3. Hyperbolic Split Quaternion Matrices

We denote the $m \times n$ matrices with hyperbolic split quaternion entries by $M_{m\times n}(\widehat{\mathbb{H}}_H)$. The set of $n \times n$ hyperbolic split quaternion matrices with standard matrix summation and multiplication is ring with unity. For any $A = (A_{ij}) \in M_{m\times n}(\widehat{\mathbb{H}}_H)$ and $Q \in \widehat{\mathbb{H}}_H$, right and left scalar multiplications are defined as $AQ = (A_{ij}Q)$ and $QA = (QA_{ij})$, respectively. So, $M_{m\times n}(\widehat{\mathbb{H}}_H)$ is both a left and a right module over $\widehat{\mathbb{H}}_H$. For any $A = (A_{ij}) \in M_{m\times n}(\widehat{\mathbb{H}}_H)$, the Hamiltonian conjugate of A is defined as $\overline{A} = \left(\overline{A_{ij}}\right) \in M_{m\times n}(\widehat{\mathbb{H}}_H)$, the transpose of A is defined as $A^T = \left(A_{ji}\right) \in M_{n\times m}(\widehat{\mathbb{H}}_H)$ and the conjugate transpose of A is defined as $A^* = (\overline{A})^T \in M_{n\times m}(\widehat{\mathbb{H}}_H)$. Since any hyperbolic split quaternion Q has a unique representation as $Q = q + hq^*$ where $q, q^* \in \widehat{\mathbb{H}}$, then we may write any $A \in M_{n\times n}(\widehat{\mathbb{H}}_H)$ as $A = A_1 + hA_2$ where $A_1, A_2 \in M_{n\times n}(\widehat{\mathbb{H}})$.

Theorem 3.1. *For any $A, B \in M_{n\times n}(\widehat{\mathbb{H}}_H)$, if $AB = I$, then $BA = I$.*

Proof. Let $A = A_1 + hA_2$ and $B = B_1 + hB_2$ be any elements of $M_{n\times n}(\widehat{\mathbb{H}}_H)$ where $A_1, A_2, B_1, B_2 \in M_{n\times n}(\widehat{\mathbb{H}})$. Suppose that $AB = I$, then we may write

$$AB = A_1B_1 + A_2B_2 + h(A_1B_2 + A_2B_1) = I + h0$$

where I denotes $n \times n$ identity matrix and 0 denotes $n \times n$ zero matrix. This implies that $A_1B_1 + A_2B_2 = I$ and $A_1B_2 + A_2B_1 = 0$. Using these two equalities, we can write the following matrix equality

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Since this theorem holds for split quaternion matrices [1], we may write

$$\begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

This implies that $B_1A_1 + B_2A_2 = I$ and $B_2A_1 + B_1A_2 = 0$. By using these equalities, we have $I = I + h0 = B_1A_1 + B_2A_2 + h(B_2A_1 + B_1A_2) = BA$. □

Definition 3.2. *Let $A = A_1 + hA_2 \in M_{n\times n}(\widehat{\mathbb{H}}_H)$ where $A_1, A_2 \in M_{n\times n}(\widehat{\mathbb{H}})$. We define the $2n \times 2n$ split quaternionic matrix*

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$$

as split quaternion matrix representation of A and denote by $S(A)$.

Theorem 3.3. *For any $A, B \in M_{n\times n}(\widehat{\mathbb{H}}_H)$, the followings are satisfied;*
i. *$S(I_n) = I_{2n}$;*
ii. *$S(A + B) = S(A) + S(B)$;*
iii. *$S(AB) = S(A)S(B)$;*
iv. *If A is invertible then $S(A)^{-1} = S(A^{-1})$.*

Proof. Let $A = A_1 + hA_2$ and $B = B_1 + hB_1$ be any elements of $M_{n\times n}(\widehat{\mathbb{H}}_H)$ where $A_1, A_2, B_1, B_2 \in M_{n\times n}(\widehat{\mathbb{H}})$.
i.

$$S(I_n) = S(I_n + h0_n) = \begin{bmatrix} I_n & 0_n \\ 0_n & I_n \end{bmatrix} = I_{2n}.$$

ii. We have $A + B = (A_1 + B_1) + h(A_2 + B_2)$. By definition, we get

$$S(A + B) = \begin{bmatrix} A_1 + B_1 & A_2 + B_2 \\ A_2 + B_2 & A_1 + B_1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix} = S(A) + S(B).$$

iii. We have $AB = (A_1B_1 + A_2B_2) + h(A_1B_2 + A_2B_1)$. We obtain

$$S(AB) = \begin{bmatrix} A_1B_1 + A_2B_2 & A_1B_2 + A_2B_1 \\ A_1B_2 + A_2B_1 & A_1B_1 + A_2B_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix} = S(A)S(B).$$

iv. Suppose A is invertible that is $AA^{-1} = A^{-1}A = I$. By using the properties i. and iii., we may write

$$I_{2n} = S(I_n) = S(AA^{-1}) = S(A)S(A^{-1}) \text{ and } I_{2n} = S(I_n) = S(A^{-1}A) = S(A^{-1})S(A).$$

This implies $S(A)$ is also invertible and $S(A)^{-1} = S(A^{-1})$. □

Theorem 3.4. *For any $A \in M_{n\times n}(\widehat{\mathbb{H}}_H)$, if $S(A)$ is invertible then A is invertible.*

Proof. Let $A = A_1 + hA_2 \in M_{n\times n}(\widehat{\mathbb{H}}_H)$ where $A_1, A_2 \in M_{n\times n}(\widehat{\mathbb{H}})$. Suppose that $S(A)$ is invertible and

$$S(A)^{-1} = \begin{bmatrix} B_1 & B_3 \\ B_2 & B_4 \end{bmatrix}$$

where $B_1, B_2, B_3, B_4 \in M_{n\times n}(\widehat{\mathbb{H}})$. Then we have

$$S(A)S(A)^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} B_1 & B_3 \\ B_2 & B_4 \end{bmatrix} = \begin{bmatrix} I_n & 0_n \\ 0_n & I_n \end{bmatrix}.$$

By this relation, we get $A_1B_1 + A_2B_2 = I_n$ and $A_1B_2 + A_2B_1 = 0_n$. If we chose $B = B_1 + hB_2$, then we get $AB = I_n$. By Theorem 3.1, we have $BA = I_n$ that is A is also invertible and $A^{-1} = B_1 + hB_2$. □

Corollary 3.5. *For any $A \in M_{n\times n}(\widehat{\mathbb{H}}_H)$, A is invertible if and only if $S(A)$ is invertible.*

Remark 3.6. *For any invertible split quaternion matrix $A \in M_{n\times n}(\widehat{\mathbb{H}})$, we know that the complex adjoint matrix χ_A of A is also invertible by the study [1]. If we have*

$$(\chi_A)^{-1} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

where $C_1, C_2, C_3, C_4 \in M_{n\times n}(\mathbb{C})$, then $A^{-1} = C_1 + C_2 j$. By using the proof of above theorem, we can also find inverse of any hyperbolic split quaternion matrix with its split quaternion matrix representation. The following example is given to show how to find the inverse of any hyperbolic split quaternion matrix.

Example 3.7. *Consider the 2×2 hyperbolic split quaternion matrix*

$$A = \begin{bmatrix} i & 1+i \\ 0 & k \end{bmatrix} + h \begin{bmatrix} 1 & 0 \\ i & k \end{bmatrix}.$$

Here the split quaternion matrix representation of A is found as

$$S(A) = \begin{bmatrix} i & 1+i & 1 & 0 \\ 0 & k & i & k \\ 1 & 0 & i & 1+i \\ i & k & 0 & k \end{bmatrix}.$$

Since $|S(A)|_q = 5 \neq 0$, then $S(A)$ is invertible. So that A is also invertible. By using complex adjoint matrix of $S(A)$ and the method given in above remark, we find the inverse of $S(A)$ as

$$S(A)^{-1} = \frac{1}{5} \begin{bmatrix} 0 & 5i & 0 & -5i \\ 2+i & 4+2i+3j+4k & -1+2i & -4-2i \\ 3-i & 1-2i+2j-4k & 1-2i & -1+2i \\ -2-i & -4-2i-3j+k & 1-2i & 4+2i \end{bmatrix}.$$

Here we get

$$B_1 = \frac{1}{5} \begin{bmatrix} 0 & 5i \\ 2+i & 4+2i+3j+4k \end{bmatrix} \text{ and } B_2 = \frac{1}{5} \begin{bmatrix} 3-i & 1-2i+2j-4k \\ -2-i & -4-2i-3j+k \end{bmatrix}.$$

Thus, we find

$$A^{-1} = B_1 + B_2 h = \frac{1}{5} \begin{bmatrix} 0 & 5i \\ 2+i & 4+2i+3j+4k \end{bmatrix} + \frac{h}{5} \begin{bmatrix} 3-i & 1-2i+2j-4k \\ -2-i & -4-2i-3j+k \end{bmatrix}.$$

Definition 3.8. *Let $A \in M_{n\times n}(\widehat{\mathbb{H}}_H)$ and $\lambda \in \widehat{\mathbb{H}}_H$. If λ satisfies the equation $Ax = \lambda x$ for some nonzero hyperbolic split quaternion vector x, then λ is called a left eigenvalue of A. The set of left eigenvalues of A is called left spectrum of A and denoted by*

$$\sigma_l(A) = \{\lambda \in \widehat{\mathbb{H}}_H : Ax = \lambda x \text{ and } x \neq 0\}.$$

If λ satisfies the equation $Ax = x\lambda$ for some nonzero hyperbolic split quaternion vector x, then λ is called a right eigenvalue of A. The set of right eigenvalues of A is called right spectrum of A and denoted by

$$\sigma_r(A) = \{\lambda \in \widehat{\mathbb{H}}_H : Ax = x\lambda \text{ and } x \neq 0\}.$$

Theorem 3.9. *For any $A \in M_{n\times n}(\widehat{\mathbb{H}}_H)$, the following relations are satisfied:*

$$\sigma_l(A) \cap \widehat{\mathbb{H}} = \sigma_l(S(A)) \text{ and } \sigma_r(A) \cap \widehat{\mathbb{H}} = \sigma_r(S(A)).$$

Proof. We will prove only first relation, the second one can be done by similar way. Let $A = A_1 + hA_2 \in M_{n\times n}(\widehat{\mathbb{H}}_H)$ where $A_1, A_2 \in M_{n\times n}(\widehat{\mathbb{H}})$ and $\lambda \in \widehat{\mathbb{H}}$ be a left eigenvalue of A. So there exists at least one nonzero hyperbolic split quaternion vector $x = x_1 + hx_2$ such that $Ax = \lambda x$. Here x_1 and x_2 are split quaternionic vectors. Thus we may write

$$(A_1 + hA_2)(x_1 + hx_2) = \lambda(x_1 + hx_2) \Rightarrow (A_1x_1 + A_2x_2) + h(A_1x_2 + A_2x_1) = (\lambda x_1) + h(\lambda x_2).$$

We get $A_1x_1 + A_2x_2 = \lambda x_1$ and $A_1x_2 + A_2x_1 = \lambda x_2$. Using these two equalities, we get

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This means $\lambda \in \sigma_l(S(A))$. Now, suppose that $\lambda \in \sigma_l(S(A))$. That is $S(A)x = \lambda x$ for any nonzero split quaternionic vector $x = (x_1, x_2)$ where x_1, x_2 are n dimensional split quaternionic vectors. So we may write

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This matrix equation implies that $A_1x_1 + A_2x_2 = \lambda x_1$ and $A_1x_2 + A_2x_1 = \lambda x_2$. Using these equations, we obtain

$$(A_1x_1 + A_2x_2) + h(A_1x_2 + A_2x_1) = (\lambda x_1) + h(\lambda x_2) \Rightarrow (A_1 + hA_2)(x_1 + hx_2) = \lambda(x_1 + hx_2).$$

Thus, we get $\lambda \in \sigma_l(A) \cap \widehat{\mathbb{H}}$. □

Theorem 3.10. *For any $A \in M_{n\times n}(\widehat{\mathbb{H}}_H)$, the following relation holds*

$$\sigma_r(A) \cap \mathbb{C} = \sigma(\chi_{S(A)}).$$

Proof. Let $A \in M_{n\times n}(\widehat{\mathbb{H}}_H)$ and $\lambda \in \mathbb{C}$ be any right eigenvalue of A. Since $\mathbb{C} \subset \widehat{\mathbb{H}}$, we have $\lambda \in \sigma_r(A) \cap \widehat{\mathbb{H}}$. By previous theorem, we get $\lambda \in \sigma_r(S(A))$. By Theorem 7 in the study [2], we know that $\sigma_r(S_A) \cap \mathbb{C} = \sigma(\chi_{S(A)})$. Thus, we obtain $\lambda \in \sigma(\chi_{S(A)})$. Now suppose that $\lambda \in \sigma(\chi_{S(A)})$. This implies $\lambda \in \sigma_r(S(A)) \cap \mathbb{C}$. By previous theorem, we get $\lambda \in \sigma_r(A) \cap \mathbb{C}$. □

Corollary 3.11. *For any $A \in M_{n\times n}(\widehat{\mathbb{H}}_H)$, A has at most $4n$ different complex right eigenvalues.*

Example 3.12. *Consider the 2×2 hyperbolic split quaternion matrix*

$$A = \begin{bmatrix} i & j \\ 1+k & 0 \end{bmatrix} + h \begin{bmatrix} 1 & i \\ 0 & k \end{bmatrix}.$$

Then complex adjoint matrix of $S(A)$ is obtained as

$$\chi_{S(A)} = \begin{bmatrix} i & 0 & 1 & i & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & i & 0 & 0 & i \\ 1 & i & i & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & i & i & 0 \\ 0 & 1 & 0 & 0 & -i & 0 & 1 & -i \\ -i & 0 & 0 & -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -i & -i & 0 \\ 0 & -i & -i & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

By long and tedious computations, we find that one of the eigenvalues of $\chi_{S(A)}$ as $\lambda = 1 + i$. Since $\sigma_r(A) \cap \mathbb{C} = \sigma(\chi_{S(A)})$, then $\lambda = 1 + i$ is a right eigenvalue of A. Really, for the nonzero hyperbolic split quaternion vector

$$x = \begin{bmatrix} -1 \\ i+j \end{bmatrix} + h \begin{bmatrix} -1 \\ i+j \end{bmatrix}$$

the relation $Ax = \lambda x$ is satisfied.

4. Applications

Consider the following linear hyperbolic split quaternionic equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n, \end{aligned}$$

where x_i are hyperbolic split quaternionic unknowns for $i = 1, 2, ..., n$. Here $a_{ij} \in \widehat{\mathbb{H}}_H$ for all $i, j = 1, 2, ..., n$ and $b_i \in \widehat{\mathbb{H}}_H$ for $i = 1, 2, ..., n$. This system of equation can be rewritten as $A\mathbf{x} = \mathbf{b}$ where

$$A = (a_{ij}) \in M_{n\times n}(\widehat{\mathbb{H}}_H),\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

We may write $A = A_1 + hA_2$, $\mathbf{x} = y + hz$, $\mathbf{b} = c + hd$. Here $A_1, A_2 \in M_{n\times n}(\widehat{\mathbb{H}})$ and y, z, c and d are split quaternionic column vectors. So, the equation $A\mathbf{x} = \mathbf{b}$ is equivalent to the following split quaternionic system with $2n$ split quaternion unknowns;

$$A_1 y + A_2 z = c \text{ and } A_1 z + A_2 y = d.$$

This system can be written as follows;

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

Let us denote

$$\mathbf{X} = \begin{bmatrix} y \\ z \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

So, the linear equation of the form $A\mathbf{x} = \mathbf{b}$ with n unknowns is equivalent to split quaternionic system $S(A)\mathbf{X} = \mathbf{B}$ with $2n$ split quaternion unknowns.

Example 4.1. *Consider the hyperbolic split quaternion equations*

$$(i + hk)x + (1 + j + h(1 - i))y = 1 + h(1 + i),\ jx + (1 - hk)y = j + h(1 + k)$$

where x and y are hyperbolic split quaternion unknowns. This equation is equivalent the linear system $A\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} i + hk & 1 + j + h(1 - i) \\ j & 1 - hk \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + h(1 + i) \\ j + h(1 + k) \end{bmatrix}.$$

The above hyperbolic split quaternion matrix can be written as follows

$$A = \begin{bmatrix} i + hk & 1 + j + h(1 - i) \\ j & 1 - hk \end{bmatrix} = \begin{bmatrix} i & 1 + j \\ j & 1 \end{bmatrix} + h\begin{bmatrix} k & 1 - i \\ 0 & -k \end{bmatrix}.$$

The given system is equivalent to split quaternionic system $S(A)\mathbf{x} = \mathbf{b}$ that is

$$\begin{bmatrix} i & 1 + j & k & 1 - i \\ j & 1 & 0 & -k \\ k & 1 - i & i & 1 + j \\ 0 & -k & j & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ j \\ 1 + i \\ 1 + k \end{bmatrix}$$

where $x = x_1 + hx_2$, $y = y_1 + hy_2$. Here, we find

$$\chi_{S(A)} = \begin{bmatrix} i & 1 & 0 & 1-i & 0 & 1 & i & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -i \\ 0 & 1-i & i & 1 & i & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -i & 1 & 0 \\ 0 & 1 & -i & 0 & -i & 1 & 0 & 1+i \\ 1 & 0 & 0 & i & 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 1 & 0 & 1+i & -i & 1 \\ 0 & i & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $|S(A)|_q = \det(\chi_{S(A)}) = 48 \neq 0$. This means $S(A)$ is invertible and the system $S(A)\mathbf{X} = \mathbf{B}$ has a unique solution. By long and tedious computations, we find

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 6 - 15i + 9j \\ -6 + 3i + 3j - 6k \\ -3i - 3j - 6k \\ 9 - 3i + 3j + 3k \end{bmatrix}.$$

Therefore, the unique solution of given hyperbolic split quaternion system is obtained as

$$x = \frac{1}{12}(6 - 15i + 9j) + \frac{h}{12}(-6 + 3i + 3j - 6k), \; y = \frac{1}{12}(-3i - 3j - 6k) + \frac{h}{12}(9 - 3i + 3j + 3k).$$

Note that the unique solution case appears when $S(A)$ is invertible.

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