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Generalized Strongly Vector Equilibrium Problem for Set-Valued Mappings

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Abstract. In this paper, we introduce and study a generalized strongly vector equilibrium problem for set-valued mappings in real Banach spaces. Using Minty type lemma and KKM-Fan theorem as basic tools, we prove existence theorems for generalized strongly vector equilibrium problem with and without monotonicity, respectively. Some examples are given.

1. Introduction

Ky Fan inequality has become a versatile tool in nonlinear and convex analysis, for instance, optimization problems, variational inequalities, problems of Nash equilibria etc.

Ky Fan [13] obtained the following famous Ky Fan inequality.

Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $\phi : X\times X \longrightarrow \mathbb{R}$ be a mapping such that

(i) for each fixed $y \in X, \phi(\cdot, y)$ is lower semicontinuous;
(ii) for each fixed $x \in X, \phi(x, \cdot)$ is quasi-concave;
(iii) for all $x \in X, \phi(x, x) \leq 0$.

Then, there exists $x^* \in X$ such that $\phi(x^*, y) \leq 0$, for all $y \in X$.

Since 1961, *Ky Fan* showed that the KKM theorem provides the foundation for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone in the history of KKM theory was erected by *Fan* in 1961[14]. His 1961 KKM lemma (or the Fan-KKM theorem) extended the KKM theorem to arbitrary topological vector spaces and had been applied to various problems in subsequent papers [15–19].

The classical Fan's minimax inequality given by *Fan*[13] typically assumes lower semicontinuity and quasi-concavity for the functions, in addition to convexity and compactness in Hausdorff topological vector spaces. However, in many situations, these assumptions may not be satisfied. The function under consideration may be non-convex and/or non-compact.

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Inspired by the applications of Fan's work and other work done in this direction which has been followed by large number of generalizations, see for example [1–8, 10–12, 19, 22, 23, 26–33] and references therein. Since vector Ky Fan inequality problem is commonly known as vector equilibrium problem, in this paper, we introduce and study a generalized strongly vector equilibrium problem for set-valued mappings. We prove existence theorems for generalized strongly vector equilibrium problem with and without monotonicity, respectively.

2. Preliminaries

Throughout this paper, unless otherwise specified, we take X and Y to be real Banach spaces; let $K \subset X$ be a nonempty and convex set and let C be a pointed, closed and convex cone in Y with apex at the origin. Let $F : K \times K \longrightarrow 2^Y$ be a set-valued function such that

$$F(\lambda x + (1-\lambda)z, x) \supseteq \{0\}, \ \ \forall x, z \in K, \lambda \in (0, 1].$$

We consider the following problem of finding $x \in K$ such that

$$F(\lambda x + (1-\lambda)z, y) \nsubseteq -C \setminus \{0\}, \ \ \forall y, z \in K, \lambda \in (0, 1]. \tag{1}$$

The system (1) is called generalized strongly vector equilibrium problem.

If $\lambda = 1$, then problem (1) reduces to the problem of finding $x \in K$ such that

$$F(x, y) \nsubseteq -C \setminus \{0\}, \ \ \forall y \in K. \tag{2}$$

The system (2) is called vector equilibrium problem.

If $F(\lambda x + (1-\lambda)z, y) = \langle s, x - x_0 \rangle$, for all $x, x_0 \in K$ and $s \in T(x_0)$, where $T : K \longrightarrow 2^{L(X,Y)}$ is a set-valued mapping, $L(X, Y)$ is the Banach space of all continuous linear mappings from X into Y, then the system (1) reduces to the problem of finding $x_0 \in K$ such that

$$s \in T(x_0) \text{ and } \langle s, x - x_0 \rangle \nsubseteq -C \setminus \{0\}. \tag{3}$$

Problem (3) is called set-valued strong vector variational inequality. For detailed study of vector variational inequalities, we refer to [9, 20, 21, 25].

Let us take a brief look at some definitions and results.

Definition 2.1 ([14]). *Let K be a subset of a topological vector space X. A set-valued mapping $T : K \longrightarrow 2^X$ is called a KKM mapping if, for each nonempty finite subset $\{x_1, x_2, \cdots, x_n\} \subset K$, we have*

$$Co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i),$$

where Co denotes the convex hull.

Theorem 2.2 (KKM-Fan Theorem[14]). *Let K be a subset of a topological vector space X and let $T : K \longrightarrow 2^X$ be a KKM mapping. If, for each $x \in K, T(x)$ is closed and for at least one $x \in K, T(x)$ is compact, then*

$$\bigcap_{x \in K} T(x) \neq \emptyset.$$

Definition 2.3. *Let $F : K \times K \longrightarrow 2^Y$ be a set-valued mapping. Then* *(i)* *F is said to be C-strong pseudomonotone with respect to* $[\cdot, z)$ *if*

$$F(\lambda x + (1-\lambda)z, y) \nsubseteq -C \setminus \{0\}$$

implies

$$F(\lambda y + (1-\lambda)z, x) \subseteq -C, \quad \forall x, y, z \in K, \lambda \in (0, 1],$$

where $[\cdot, z)$ *denotes the line-segment excluding the point* z;

(ii) *F is said to be C-function in the first argument if*

$$F(\alpha x + (1-\alpha)y, z) \subseteq \alpha F(x, z) + (1-\alpha)F(y, z) - C, \quad \forall x, y, z \in K, \alpha \in [0, 1];$$

Similarly, one can define F to be C-function in the second argument.

(iii) *F is said to be generalized hemicontinuous if for all* $x, y \in K, \alpha \in [0, 1]$, *the mapping* $\alpha \to F(x + \alpha(y - x))$ *is upper semicontinuous at* 0^+;

(iv) *F is said to be C-quasiconvex-like if for all* $x, z, y_1, y_2 \in K, \lambda \in (0, 1], \alpha \in [0, 1]$, *we have either*

$$F(\lambda x + (1-\lambda)z, \alpha y_1 + (1-\alpha)y_2) \subseteq F(\lambda x + (1-\lambda)z, y_1) - C$$

or

$$F(\lambda x + (1-\lambda)z, \alpha y_1 + (1-\alpha)y_2) \subseteq F(\lambda x + (1-\lambda)z, y_2) - C.$$

Remark 2.4. *When C contains (or equal to, or is contained in) the nonnegative orthant or hyperoctant, then the definition of C-function converges to C-convex (or convex, or strictly C-convex). Similarly, when C contains (or equal to, or is contained in) the nonpositive orthant or hyperoctant, then the definition of C-function converges to C-concave (or concave, or strictly C-concave).*

Example 2.5. *Let* $X = Y = \mathbb{R}$, *and* $K = C = \mathbb{R}^+$. *Let* $F : K \times K \longrightarrow 2^Y$ *be a mapping such that*

$$F(\lambda x + (1-\lambda)z, y) = \langle T(\lambda x + (1-\lambda)z), \sqrt{y} - \sqrt{x} \rangle, \quad \forall x, y, z \in K,$$

where $T : K \longrightarrow L(X, Y)$ *is given by*

$$T(x) = \begin{pmatrix} \sin x + 1 \\ \cos x + 1 \end{pmatrix}, \quad \forall x \in K.$$

Now,

$$\begin{aligned} F(\lambda x + (1-\lambda)z, y) &= \begin{pmatrix} \sin(\lambda x + (1-\lambda)z) + 1 \\ \cos(\lambda x + (1-\lambda)z) + 1 \end{pmatrix} \left(\sqrt{y} - \sqrt{x} \right) \\ &= \begin{pmatrix} \{\sin(\lambda x + (1-\lambda)z) + 1\}(\sqrt{y} - \sqrt{x}) \\ \{\cos(\lambda x + (1-\lambda)z) + 1\}(\sqrt{y} - \sqrt{x}) \end{pmatrix} \nsubseteq -C \setminus \{0\}. \end{aligned}$$

The above inequality implies that $y \geq x$. *Therefore, it follows that*

$$F(\lambda y + (1-\lambda)z, x) = \begin{pmatrix} \{\sin(\lambda y + (1-\lambda)z) + 1\}(\sqrt{x} - \sqrt{y}) \\ \{\cos(\lambda y + (1-\lambda)z) + 1\}(\sqrt{x} - \sqrt{y}) \end{pmatrix} \subseteq -C.$$

Therefore, F is C-strong pseudomonotone with respect to $[\cdot, z)$.

Example 2.6. *Let* $Y = \mathbb{R}$ *and* $K = C = \mathbb{R}^+$. *Let* $F : K \times K \longrightarrow 2^Y$ *be a mapping such that*

$$F(\lambda x + (1-\lambda)z, y) = [\lambda x + (1-\lambda)z, y + n], \quad \forall x, y, z \in K \text{ and } n \in \mathbb{N}.$$

For all $y_1, y_2 \in K$ *and* $\alpha \in [0, 1]$, *we can see that*

$$\text{if } y_1 \leq y_2, \quad \text{then } \alpha y_1 + (1-\alpha)y_2 \leq y_2$$

and

$$\text{if } y_1 > y_2, \quad \text{then } \alpha y_1 + (1-\alpha)y_2 \leq y_1.$$

Therefore, for all $w \in F(\lambda x + (1-\lambda)z, \alpha y_1 + (1-\alpha)y_2)$, we have

$$w = \begin{cases} (y_2 + n) - \{(y_2 + n) - w\}; & \text{if } y_1 \leq y_2, \\ (y_1 + n) - \{(y_1 + n) - w\}; & \text{if } y_1 > y_2. \end{cases}$$

Hence, we have either

$$F(\lambda x + (1-\lambda)z, \alpha y_1 + (1-\alpha)y_2) \subseteq F(\lambda x + (1-\lambda)z, y_1) - C$$

or

$$F(\lambda x + (1-\lambda)z, \alpha y_1 + (1-\alpha)y_2) \subseteq F(\lambda x + (1-\lambda)z, y_2) - C.$$

Thus, F is C-quasiconvex-like.

3. Existence of Solutions

To prove an existence theorem for generalized strongly vector equilibrium problem with monotonicity, we first prove the following Minty type lemma.

Lemma 3.1. *Let K be a nonempty convex subset of X and let $F : K \times K \longrightarrow 2^Y$ be a set-valued mapping such that F is generalized hemicontinuous in the first argument and C-function in the second argument. Assume that F is C-strongly pseudomonotone with respect to $[\cdot, z)$ and $F(\lambda x + (1-\lambda)z, x) \supseteq \{0\}$, for all $x, z \in K, \lambda \in (0, 1]$. Then the following two statements are equivalent.*

(I) Find $x \in K$ such that $F(\lambda x + (1-\lambda)z, y) \nsubseteq -C \setminus \{0\}, \ \forall y, z \in K, \lambda \in (0, 1]$.
(II) Find $x \in K$ such that $F(\lambda y + (1-\lambda)z, x) \subseteq -C, \ \forall y, z \in K, \lambda \in (0, 1]$.

Proof. (I) $\Rightarrow$ (II). It follows from the definition of C-strong pseudomonotonicity of F with respect to $[\cdot, z)$.

(II) $\Rightarrow$ (I). Let $x \in K$ such that

$$F(\lambda y + (1-\lambda)z, x) \subseteq -C, \ \forall y, z \in K, \lambda \in (0, 1].$$

For each $y \in K, t \in (0, 1)$, set $y_t = ty + (1-t)x$. Then, clearly $y_t \in K$ and hence

$$F(\lambda y_t + (1-\lambda)z, x) \subseteq -C. \tag{4}$$

Since F is C-function in the second argument and

$$\begin{aligned} 0 &\in F(\lambda y_t + (1-\lambda)z, y_t) \\ &\subseteq F(\lambda y_t + (1-\lambda)z, ty + (1-t)x) \\ &\subset tF(\lambda y_t + (1-\lambda)z, y) + (1-t)F(\lambda y_t + (1-\lambda)z, x) - C. \end{aligned} \tag{5}$$

By (4) and using the fact that C is a convex cone, we have

$$tF(\lambda y_t + (1-\lambda)z, y) \cap C \neq \emptyset,$$

and hence

$$F(\lambda y_t + (1-\lambda)z, y) \cap C \neq \emptyset. \tag{6}$$

As F is generalized hemicontinuous in the first argument and $y_t \to x$, from (6) we have

$$F(\lambda x + (1-\lambda)z, y) \cap C \neq \emptyset.$$

If not, then

$$F(\lambda x + (1-\lambda)z, y) \subseteq O,$$

where O is an open set.

By the generalized hemicontinuity of F in first argument, there exists $t_0 \in (0, 1]$ such that for all $t \in [0, t_0)$,

$$F(\lambda y_t + (1-\lambda)z, y) \subseteq O,$$

i.e.,

$$F(\lambda y_t + (1-\lambda)z, y) \cap C = \emptyset,$$

which contradicts (6). As C is a pointed convex cone, we obtain

$$F(\lambda x + (1-\lambda)z, y) \nsubseteq -C \setminus \{0\}, \quad \forall x, y, z \in K, \lambda \in (0, 1].$$

This completes the proof. □

Theorem 3.2. *Let K be a nonempty, closed, bounded and convex subset of a reflexive Banach space X and let $F : K \times K \longrightarrow 2^Y$ be generalized hemicontinuous in first argument, continuous in second argument, C-function in second argument and also affine in second argument. Assume that F is C-strong pseudomonotone with respect to $[\cdot, z)$ and C-quasiconvex-like with respect to $[\cdot, z)$ and $F(\lambda x + (1-\lambda)z, x) \supseteq \{0\}$, for all $x, z \in K, \lambda \in (0, 1]$. Then generalized strongly vector equilibrium problem* (1) *admits a solution.*

Proof. We define two set-valued mappings $A, B : K \longrightarrow 2^K$ by

$$A(y) = \Big\{x \in K : F(\lambda x + (1-\lambda)z, y) \nsubseteq -C \setminus \{0\}\Big\},$$
$$B(y) = \{x \in K : F(\lambda y + (1-\lambda)z, x) \subseteq -C\}, \quad \forall y, z \in K, \lambda \in (0, 1].$$

Since $y \in A(y) \cap B(y)$, for all $y \in K$, it follows that $A(y)$ and $B(y)$ both are nonempty.

We claim that A is KKM mapping. Suppose that A is not a KKM mapping, then there exists some $x \in Co\{y_1, y_2, \cdots, y_n\}$ such that for all $t_i \in [0, 1], i = 1, 2, \cdots, n$ with $\sum_{i=1}^{n} t_i = 1$, we have

$$x = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} A(y_i).$$

Then, we have

$$F(\lambda x + (1-\lambda)z, y_i) \subseteq -C \setminus \{0\},$$

and

$$\sum_{i=1}^{n} t_i F(\lambda x + (1-\lambda)z, y_i) \subseteq -C \setminus \{0\}. \tag{7}$$

Since F is affine in the second argument, it follows from (7) that

$$F(\lambda x + (1-\lambda)z, \sum_{i=1}^{n} t_i y_i) \subseteq -C \setminus \{0\},$$

or,

$$F(\lambda x + (1-\lambda)z, x) \subseteq -C \setminus \{0\},$$

which implies that $0 \in -C \setminus \{0\}$, which is not possible. Hence A is a KKM mapping. By Lemma 3.1, we see that

$$\bigcap_{y \in K} A(y) = \bigcap_{y \in K} B(y).$$

Next, we show that $B(y)$ is bounded, convex and closed in K. As $B(y) \subset K, B(y)$ is bounded. Also for each $y \in K, B(y)$ is convex. To see this, let $x_1, x_2 \in B(y)$ and since F is C-quasiconvex-like with respect to $[\cdot, z)$, then for all $\alpha \in [0,1]$ we have

either,

$$\begin{aligned} F(\lambda y + (1-\lambda)z, \alpha x_1 + (1-\alpha)x_2) &\subseteq F(\lambda y + (1-\lambda)z, x_1) - C \\ &\subseteq -C - C \\ &\subseteq -C, \end{aligned}$$

or,

$$\begin{aligned} F(\lambda y + (1-\lambda)z, \alpha x_1 + (1-\alpha)x_2) &\subseteq F(\lambda y + (1-\lambda)z, x_2) - C \\ &\subseteq -C - C \\ &\subseteq -C. \end{aligned}$$

In both cases, we get

$$F(\lambda y + (1-\lambda)z, \alpha x_1 + (1-\alpha)x_2) \subseteq -C, \tag{8}$$

which implies that $B(y)$ is convex.

Next, we claim that $B(y)$ is closed. Let $\{x_n\}$ be a net in $B(y)$ such that $x_n \to x$. As F is continuous in the second argument, we have

$$F(\lambda y + (1-\lambda)z, x_n) \to F(\lambda y + (1-\lambda)z, x) \subseteq -C$$

and by closedness of $-C$, it follows that $B(y)$ is closed.

Now, we equip X with the weak topology and as $B(y)$ is closed, bounded and convex subset of a reflexive Banach space X, it turns out to be weakly compact for all $y \in K$. Thus, by KKM theorem 2.2, we have

$$\bigcap_{y \in K} A(y) = \bigcap_{y \in K} B(y) \neq \emptyset.$$

Hence

$$F(\lambda x + (1-\lambda)z, y) \nsubseteq -C \setminus \{0\},$$

i.e., generalized strongly vector equilibrium problem (1) is solvable. This completes the proof. □

Now, we prove another existence theorem for generalized strongly vector equilibrium problem (1) without monotonicity, to do this we state the following theorem.

Theorem 3.3. *Let K be a nonempty, closed, bounded and convex subset of a reflexive Banach space X. Assume that the set-valued mapping $F : K \times K \longrightarrow 2^Y$ be C-function in second argument. Suppose that for each $y \in K$, the set*

$$\left\{x \in K : F(\lambda x + (1-\lambda)z, y) \subseteq -C \setminus \{0\}\right\}, \ \ \forall z \in K, \lambda \in (0,1]$$

is open. Then the generalized strongly vector equilibrium problem (1) *admits a solution.*

Proof. Using the same arguments in *Kum* and *Wong*[24], one can easily prove this theorem. □

Theorem 3.4. *Let K be a nonempty convex subset of a real Banach space X and let $F : K \times K \longrightarrow 2^Y$ be a set-valued C-function in second argument. Assume that*

(i) for each $y, z \in K, \lambda \in (0,1]$, the set

$$\left\{x \in K : F(\lambda x + (1-\lambda)z, y) \subseteq -C \setminus \{0\}\right\}$$

is open.

(ii) K is locally compact and there is an $\gamma > 0$ and $x_0 \in K$, $\|x_0\| < \gamma$, such that for all $y \in K$, $\|y\| = \gamma$, $z \in K$, $\lambda \in (0,1]$,

$$F(\lambda x + (1-\lambda)z, x_0) \subseteq -C.$$

Then

$$F(\lambda x + (1-\lambda)z, y) \not\subseteq -C \setminus \{0\},$$

i.e., generalized strongly vector equilibrium problem (1) *is solvable.*

Proof. Let

$$K_\gamma = \{x \in K : \|x\| \leq \gamma\}.$$

Since K is locally compact, K_γ is compact and hence it follows from Theorem 3.3 that there exists an $\tilde{x} \in K_\gamma$ such that

$$F(\lambda\tilde{x} + (1-\lambda)z, y) \not\subseteq -C \setminus \{0\}, \quad \forall y, z \in K_\gamma, \lambda \in (0,1]. \tag{9}$$

Now, we will show that $\tilde{x}$ is the solution of the problem (1).

(I) If $\|\tilde{x}\| = \gamma$, by assumption (ii) we have

$$F(\lambda\tilde{x} + (1-\lambda)z, x_0) \subseteq -C. \tag{10}$$

For any $y \in K$, choose $\alpha \in (0,1]$ such that $y_\alpha = \alpha y + (1-\alpha)x_0 \in K_\gamma$. Then from (9), it follows that

$$F(\lambda\tilde{x} + (1-\lambda)z, y_\alpha) \not\subseteq -C \setminus \{0\}.$$

Since F is a C-function in the second argument, we have

$$\begin{aligned} F(\lambda\tilde{x} + (1-\lambda)z, y_\alpha) &\subseteq \alpha F(\lambda\tilde{x} + (1-\lambda)z, y) \\ &\quad + (1-\alpha)F(\lambda\tilde{x} + (1-\lambda)z, x_0) - C. \\ \Rightarrow \alpha F(\lambda\tilde{x} + (1-\lambda)z, y) &\subseteq \Big[Y \setminus \Big(-C \setminus \{0\}\Big)\Big] + (1-\alpha)C + C \\ &\subseteq \Big[Y \setminus \Big(-C \setminus \{0\}\Big)\Big] + C \\ &= \Big[Y \setminus \Big(-C \setminus \{0\}\Big)\Big]. \end{aligned}$$

i.e.,

$$F(\lambda\tilde{x} + (1-\lambda)z, y) \not\subseteq -C \setminus \{0\}, \quad \forall y, z \in K, \lambda \in (0,1].$$

(II) If $\|\tilde{x}\| < \gamma$, for any $y \in K$, choose $\alpha \in (0,1]$ such that $y_\alpha = \alpha y + (1-\alpha)\tilde{x} \in K_\gamma$. Then, it follows from (9) that

$$F(\lambda\tilde{x} + (1-\lambda)z, y_\alpha) \not\subseteq -C \setminus \{0\}.$$

Since F is a C-function in the second variable, we have

$$\begin{aligned} F(\lambda\tilde{x} + (1-\lambda)z, \alpha y + (1-\alpha)\tilde{x}) &\subseteq \alpha F(\lambda\tilde{x} + (1-\lambda)z, y) \\ &\quad + (1-\alpha)F(\lambda\tilde{x} + (1-\lambda)z, \tilde{x}) - C \\ &\subseteq \alpha F(\lambda\tilde{x} + (1-\lambda)z, y) - C, \end{aligned}$$

which implies that

$$F(\lambda\tilde{x} + (1-\lambda)z, y) \not\subseteq -C \setminus \{0\}, \quad \forall y, z \in K, \lambda \in (0,1].$$

This completes the proof. □

As a consequence of Theorem 3.2, we have the following corollary.

Corollary 3.5. *Let $T : K \longrightarrow 2^{L(X,Y)}$ be a C-strong pseudomonotone and generalized hemicontinuous set-valued mapping with nonempty compact values where $L(X,Y)$ is equipped with the topology of bounded convergence. Then problem* (3) *admits a solution.*

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