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Continuity Properties of Discontinuous Homomorphisms and Refinements of Group Topologies

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Abstract. We present several conditions on topological groups G and H under which *every* discontinuous homomorphism of G to H preserves accumulation points of open sets in G. It is also proved that every (locally) precompact abelian group admits a strictly finer zero-dimensional (locally) precompact topological group topology of the same weight as the original one.

1. Introduction

The study of *refinements* of (topological group) topologies has a long tradition. In [13] F. Jones mentions that real-valued functions f defined on the real line and satisfying the equation $f(x+y) = f(x) + f(y)$ were considered by Cauchy who noted before 1821 that such a function f is either continuous or totally discontinuous.

In modern terms Cauchy's remark is equivalent to saying that a homomorphism of topological groups is continuous if it is continuous at some point of the domain. The starting point of Jones' article [13] is the simple but very useful observation that considering the graph $Gr(f) = \{(x, f(x)) : x \in \mathbb{R}\}$ of a discontinuous homomorphism $f\colon \mathbb{R} \to \mathbb{R}$ we obtain a strictly finer topological group topology on the real line. According to [13, Theorem 5], the graph $Gr(f)$ can be connected, even if f is discontinuous. It is natural to ask, therefore, whether the subgroup $Gr(f)$ of $\mathbb{R} \times \mathbb{R}$ can be zero-dimensional, for some discontinuous homomorphism $f\colon \mathbb{R} \to \mathbb{R}$. Example 2.10 in Section 2 answers this question in the affirmative and serves as a key to the proof of Proposition 3.2 on refinements of precompact topologies on abelian groups.

The general problem of finding a finer topological group topology on a topological abelian group G preserving some of the properties of G was considered by several authors in [1, 3–7, 11] and [14, 15, 17, 18]. This study has been mainly focused on the preservation of pseudocompactness or connectedness. Our aim is different. We are interested in finding finer *zero-dimensional* topological group topologies on groups. Clearly every topological group admits the discrete topology which apparently satisfies the conditions of our search. The additional condition on a finer topology not mentioned yet is the preservation of the

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weight of the group. Hence an important special case of the problem can be formulated as follows: *Does every second countable topological abelian group admit a finer second countable zero-dimensional topological group topology?* The affirmative answer to this question would imply that every abelian group endowed with the finest ω-narrow topological group topology is zero-dimensional (see [8, Problem 4.4 c)]).

A natural strategy for solving the aforementioned problem is to fix an appropriate countable base $\mathcal{B}$ at the identity of a second countable topological abelian group G and then construct a (discontinuous) homomorphism $f\colon G \to H$ to a second countable zero-dimensional abelian group H such that the open subset $\{(x, f(x)) : x \in U\}$ of $Gr(f)$ is closed in $Gr(f)$, for each $U \in \mathcal{B}$. Once this is done, $Gr(f)$ will be a zero-dimensional subgroup of $G \times H$.

It turns out, however, that this strategy does not work if G has the Baire property. Even worse, if G is a second countable Baire topological group with topology τ and τ' is a second countable topological group topology on G containing τ, then $cl_{\tau'} O = cl_\tau O$. In other words, if an open set $O \subset G$ is not closed in G, it won't be closed in any finer second countable topological group topology on the same group. These conclusions follow from Theorem 2.7 and Corollary 2.8 of the article. Summing up, the search for finer zero-dimensional, second countable topological group topologies on abelian groups should consist in introducing *new* clopen sets, while almost nothing can be done to open sets in the original topology of the group.

In Section 2 of the article we present different conditions on topological groups G and H guaranteeing that *every*, continuous or discontinuous, homomorphism $f\colon G \to H$ preserves accumulation points of open sets in G, i.e. the inclusion $f(\overline{O}) \subset \overline{f(O)}$ holds for every open subset O of G. We show that this happens in the following cases:

- H is precompact (Theorem 2.5);
- G and H are ω-narrow and G has the Baire property (Theorem 2.7).

We also give several examples of when a homomorphism $f\colon G \to H$ does not preserve accumulation points of open sets.

In Section 3 we study the problem of finding finer zero-dimensional topological group topologies on topological groups formulated above. We prove in Proposition 3.2 that if a topological abelian group G is precompact, then there exists a strictly finer precompact, zero-dimensional topological group topology on G of the same weight as the original one. For locally precompact groups, the conclusion is similar: Every locally precompact abelian group admits a finer locally precompact, zero-dimensional topological group topology of the same weight (see Theorem 3.4).

All topological groups considered in this article are assumed to be Hausdorff.

2. Accumulation Points of Open Sets

We start with a useful concept which is used in almost all proofs in this section.

Definition 2.1. Let X and Y be topological spaces. A (not necessarily continuous) mapping $f\colon X \to Y$ is called *nearly open* if for every open set $U \subset X$, there exists an open set $V \subset Y$ such that $f(U) \subset V \subset \overline{f(U)}$.

Let us note that if $f\colon X \to Y$ is an open mapping and D is a dense subspace of X, then the restriction of f to D is nearly open. It is worth noting that nearly open mappings were called *d-open* in [16, Definition 5].

Proposition 2.2. ([2, Lemma 10.1.19]) *A continuous mapping $\varphi\colon X \to Y$ is nearly open if and only if the equality $\overline{\varphi^{-1}(V)} = \varphi^{-1}(\overline{V})$ holds for every open set $V \subset Y$.*

Lemma 2.3. *Let $f\colon G \to H$ be a (possibly discontinuous) homomorphism of topological groups. Let also $\Gamma = \{(x, f(x) : x \in G\} \subset G \times H$ be the graph of f and $\pi\colon G \times H \to G$ the projection. If the restriction of π to Γ is a nearly open homomorphism, then the inclusion $f(\overline{O}) \subset \overline{f(O)}$ is valid for every open set $O \subset G$.*

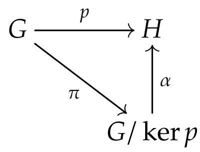
Proof. Denote by φ the restriction of π to Γ. By Proposition 2.2, the equality

$$\overline{\varphi^{-1}(O)} = \varphi^{-1}(\overline{O}) \tag{1}$$

holds for every open set $O \subset G$. Take $x \in \overline{O}$, and let us check that $f(x) \in \overline{f(O)}$. It is clear that $(x, f(x)) \in \varphi^{-1}(\overline{O})$ and by (1), $(x, f(x)) \in \overline{\varphi^{-1}(O)} = \overline{\{(y, f(y)) : y \in O\}}$. Take arbitrary open neighborhoods U and V of x and $f(x)$ in G and H, respectively. Then $(U \times V) \cap \{(y, f(y)) : y \in O\} \neq \emptyset$, whence it follows that there exists $x_0 \in O$ such that $(x_0, f(x_0)) \in U \times V$. In particular $f(x_0) \in V$ and $f(O) \cap V \neq \emptyset$. Therefore, $f(x) \in \overline{f(O)}$. □

Lemma 2.4. *Suppose that G, H are topological groups and $p\colon G \to H$ is an open and continuous epimorphism. If* $\ker p$ *is compact, then p is a perfect mapping.*

Proof. Let $\pi\colon G \to G/\ker\pi$ be the canonical projection. By the first isomorphism theorem [12, 5.27], there exists a topological isomorphism α such that the following diagram commutes.



By [2, Theorem 1.5.6], the homomorphism π is perfect and, by the commutativity of the above diagram, p is perfect too. □

Theorem 2.5. *Let G and H be topological groups and $f\colon G \to H$ an arbitrary (not necessarily continuous) homomorphism. If the group H is precompact, then the inclusion $f(\overline{O}) \subset \overline{f(O)}$ holds for every open set $O \subset G$.*

Proof. Let $\Gamma = \{(x, f(x) : x \in G\}$ be the graph of f. Clearly Γ is a subgroup of $G \times H$. Let also ϱG and ρH be the completions of the groups G and H, respectively. Denote by π the projection of the product $\rho G \times \rho H$ to ρG and let φ be the restriction of π to Γ.

Let $\overline{\Gamma}$ be the closure of Γ in the group $\rho G \times \rho H$. Since the group ρH is compact, Lemma 2.4 implies that π is a perfect homomorphism and in particular π is a closed mapping. It follows that the restriction of π to $\overline{\Gamma}$ is also a closed homomorphism. Since G is dense in ρG and $G \subset \pi(\overline{\Gamma}) \subset \rho G$, we see that $\pi(\overline{\Gamma}) = \rho G$. So $\overline{\varphi} = \pi\restriction_{\overline{\Gamma}}$ is a quotient homomorphism and therefore it is open.

As Γ is dense in $\overline{\Gamma}$, we conclude that $\varphi = \overline{\varphi}\restriction_{\Gamma} = \pi\restriction_{\Gamma}$ is a nearly open mapping. Hence Lemma 2.3 implies that $f(\overline{O}) \subset \overline{f(O)}$, for every open set $O \subset G$. □

Let us recall that a topological group G is called *ω-narrow* if G can be covered by countably many translations of an arbitrary neighborhood of the identity in G.

Lemma 2.6. *(See [2, Proposition 4.3.32]) A continuous homomorphism $f\colon G \to H$ of an ω-narrow group G onto a group H with the Baire property is nearly open.*

Theorem 2.7. *Let $f\colon G \to H$ be an arbitrary homomorphism of ω-narrow topological groups. If G has the Baire property, then $f(\overline{O}) \subset \overline{f(O)}$, for every open set $O \subset G$.*

Proof. Again we start as in the proof of Theorem 2.5. Let $\Gamma = \{(x, f(x) : x \in G\}$ be the graph of f. Denote by φ the restriction to Γ of the projection $\pi\colon G \times H \to G$. Clearly Γ is ω-narrow as a subgroup of the ω-narrow group $G \times H$. Since $\varphi(\Gamma) = \pi(\Gamma) = G$, Lemma 2.6 implies that φ is a nearly open mapping. By Lemma 2.3, $f(\overline{O}) \subset \overline{f(O)}$ for every open set $O \subset G$. □

Corollary 2.8. *Let (G, τ) be an ω-narrow group with the Baire property. Suppose that τ' is an ω-narrow topological group topology on G such that $\tau \subset \tau'$. Then the equality $cl_\tau(O) = cl_{\tau'}(O)$ holds for every open set O in (G, τ).*

Proof. Take an open set O in (G, τ). It follows from $\tau \subset \tau'$ that $\mathrm{cl}_\tau(O) \supset \mathrm{cl}_{\tau'}(O)$. Consider the identity mapping $f\colon (G, \tau) \to (G, \tau')$. As (G, τ) is an ω-narrow group with the Baire property and (G, τ') is ω-narrow, Theorem 2.7 implies that

$$\mathrm{cl}_\tau(O) = f\big(\mathrm{cl}_\tau(O)\big) \subset \mathrm{cl}_{\tau'}\big(f(O)\big) = \mathrm{cl}_{\tau'}(O).$$

This completes the proof. □

Problem 2.9. *Let $f\colon G \to H$ be a homomorphism of topological groups, where G is Čech-complete and ω-narrow. What properties of H guarantee that the inclusion $f(\overline{O}) \subset \overline{f(O)}$ holds for every open subset O of G?*

Now we present an example of a homomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that the graph of f is a zero-dimensional subgroup of the plane, as it is mentioned in the introduction.

Example 2.10. There exists a discontinuous homomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $Gr(f)$ is a dense zero-dimensional subgroup of $\mathbb{R} \times \mathbb{R}$. Similarly, there exists a homomorphism $g\colon \mathbb{T} \to \mathbb{T}$ such that $Gr(g)$ is a dense zero-dimensional subgroup of $\mathbb{T} \times \mathbb{T}$.

Indeed, since the subgroup $\mathbb{Q}$ of $\mathbb{R}$ is divisible, there exists a dense subgroup B of $\mathbb{R}$ such that $\mathbb{R} \cong \mathbb{Q} \oplus B$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a homomorphism such that $\ker f = B$ and $f(q) = q$ for each $q \in \mathbb{Q}$. It is easy to see that

$$Gr(f) = \bigcup\{(B + q) \times \{q\} : q \in \mathbb{Q}\}$$

is a dense subgroup of $\mathbb{R}\times\mathbb{R}$. Notice that the subgroup B of $\mathbb{R}$ is a zero-dimensional since $B \cap \mathbb{Q} = \{0\}$. Hence the subgroup $Gr(f)$ of $\mathbb{R} \times \mathbb{R}$ is the union of countably many closed zero-dimensional subsets $(B + q) \times \{q\}$, with $q \in \mathbb{Q}$. Since the plane $\mathbb{R} \times \mathbb{R}$ is a normal space, the countable sum theorem for dimension dim [10, Theorem 7.2.1] implies that the graph of f is also zero-dimensional.

Our construction of a homomorphism $g\colon \mathbb{T} \to \mathbb{T}$ is very close to that of f. Let us identify the circle group $\mathbb{T}$ with the quotient group $\mathbb{R}/\mathbb{Z}$ under the canonical projection $x \mapsto x + \mathbb{Z}$, where $x \in \mathbb{R}$. Since the subgroup $\mathbb{Q}/\mathbb{Z}$ of $\mathbb{T}$ is divisible, there exists a dense subgroup C of $\mathbb{T}$ such that $\mathbb{T} \cong \mathbb{Q}/\mathbb{Z} \oplus C$, algebraically. Let $g\colon \mathbb{T} \to \mathbb{T}$ be a homomorphism such that $\ker f = C$ and $g(z) = z$ for each $z \in \mathbb{Q}/\mathbb{Z}$. A simple verification similar to the one given above in the case of f shows that $Gr(g)$ is a dense zero-dimensional subgroup of $\mathbb{T} \times \mathbb{T}$.

In the following example we construct two topologies τ, τ' on the real line $\mathbb{R}$ such that the identity mapping $f\colon (\mathbb{R}, \tau) \to (\mathbb{R}, \tau')$ does not preserve accumulation points of open sets in the domain of f.

Example 2.11. The exist two Hausdorff topological group topologies τ and τ' on the additive group $\mathbb{R}$ of reals such that the identity mapping $f\colon (\mathbb{R}, \tau) \to (\mathbb{R}, \tau')$ does not preserve accumulation points of open sets in $(\mathbb{R}, \tau)$.

Let τ^* be the usual interval topology on $\mathbb{R}$ and $\mathbb{Q}$ the group of rational numbers endowed with the topology σ inherited from $(\mathbb{R}, \tau^*)$. Denote by τ the finest topological group topology on $\mathbb{R}$ that induces the topology σ on $\mathbb{Q}$. A local base at zero of $(\mathbb{R}, \tau)$ is formed by the sets $(-1/n, 1/n) \cap \mathbb{Q}$, where $n \in \mathbb{N}^+$. In particular, the subgroup $\mathbb{Q}$ of $\mathbb{R}$ is open in $(\mathbb{R}, \tau)$.

Since the group $\mathbb{Q}$ is divisible it splits in $\mathbb{R}$. In other words, there exists a subgroup B of $\mathbb{R}$ such that the mapping $m\colon \mathbb{Q} \times B \to \mathbb{R}$ defined by $m(q, b) = q + b$ for all $q \in \mathbb{Q}$ and $b \in B$ is an isomorphism. Denote by τ' the finest topological group topology on $\mathbb{R}$ which induces the usual interval topology on B, i.e. $\tau'{\restriction}_B = \tau^*{\restriction}_B$. Then B is open in $(\mathbb{R}, \tau')$.

We consider the identity mapping $f\colon (\mathbb{R}, \tau) \to (\mathbb{R}, \tau')$. Take an open set $O = (a, b) \cap \mathbb{Q}$ in $(\mathbb{R}, \tau)$, where $0 < a < b$ and $a \in \mathbb{Q}$. It is clear that $a \in \mathrm{cl}_\tau(O)$. Choose a positive integer n such that $0 \notin a + (-1/n, 1/n)$. The neighborhood $U = a + ((-1/n, 1/n) \cap B)$ of the point a in $(\mathbb{R}, \tau')$ satisfies the equality

$$U \cap f(O) = U \cap \big((a, b) \cap \mathbb{Q}\big) = \Big(a + \big((-1/n, 1/n) \cap B\big)\Big) \cap \big((a, b) \cap \mathbb{Q}\big) = \emptyset.$$

Hence $a \notin \mathrm{cl}_{\tau'}(f(O))$ and $f(\mathrm{cl}_\tau(O)) \not\subset \mathrm{cl}_{\tau'}(f(O))$.

Remark 2.12. Example 2.11 is a (more complicated) form of a much simpler fact. Let $\mathbb{Q}$ be the group of rationals with the usual interval topology and $\mathbb{Q}_d$ the same group with the discrete topology. Then evidently the identity isomorphism of $G = \mathbb{Q} \times \mathbb{Q}_d$ onto $H = \mathbb{Q}_d \times \mathbb{Q}$ is not nearly open. Clearly the groups G and H are non-discrete and countable (hence ω-narrow.) Notice that the group $(\mathbb{R}, \tau)$ in Example 2.11 is not ω-narrow.

In Example 2.13 below we present countable topological abelian groups G and H such that no non-trivial homomorphism $f: G \to H$ preserves accumulation points of open sets in G.

Example 2.13. Let $H_1 = \left\{a/2^k : a \in \mathbb{Z},\ k \in \mathbb{N}\right\}$ and $H_2 = \left\{a/5^k : a \in \mathbb{Z},\ k \in \mathbb{N}\right\}$ and consider the topological group refinements τ_i of the usual topology on $\mathbb{Q}$ obtained by declaring each of the groups H_i open in $(\mathbb{Q}, \tau_i)$, where $i = 1, 2$. Then for every non-trivial homomorphism $h: (\mathbb{Q}, \tau_1) \to (\mathbb{Q}, \tau_2)$, there exists an open set $U \subset (\mathbb{Q}, \tau_1)$ for which $h(\mathrm{cl}_{\tau_1}(U)) \not\subset \mathrm{cl}_{\tau_2}(h(U))$.

Indeed, since each homomorphism $h: \mathbb{Q} \to \mathbb{Q}$ has the form $h(x) = qx$ for some fixed rational number q, it suffices to verify our claim only for the identity homomorphism. In this case observe that while $1 \in cl_{\tau_1}((1, 2) \cap H_1)$, the neighborhood $(0.5, 1.5) \cap H_2$ of 1 in $(\mathbb{Q}, \tau_2)$ separates 1 from $(1, 2) \cap H_1$ in $(\mathbb{Q}, \tau_2)$. Indeed, we have that

$$\big((1,2) \cap H_1\big) \cap \Big(\big(0.5, 1.5\big) \cap H_2\Big) = \big(1, 1.5\big) \cap (H_1 \cap H_2) = \big(1, 1.5\big) \cap \mathbb{Z} = \emptyset.$$

Since the index of H_i in $\mathbb{Q}$ is countable, $(\mathbb{Q}, \tau_i)$ is a second countable topological group which is of course of the first category in itself.

3. Refining Group Topologies

The next result is immediate from Example 2.10. In the case of the circle group $\mathbb{T}$, it sheds some light on [9, Theorem 4.8] which states that every discrete abelian group endowed with the *Bohr topology* is zero-dimensional (see also [2, Theorem 9.9.31]).

Lemma 3.1. *The additive group of reals $\mathbb{R}$ admits a finer second countable, zero-dimensional, locally precompact topological group topology, while the circle group $\mathbb{T}$ admits a finer precompact, second countable, zero-dimensional topological group topology.*

In the next two results we refine (locally) precompact topological group topologies.

Proposition 3.2. *Every precompact abelian group G admits a finer precompact zero-dimensional topological group topology of the same weight.*

Proof. The completion of G, say, ρG is a compact topological abelian group. Hence ρG is topologically isomorphic to a subgroup of $\mathbb{T}^\kappa$, where κ is the weight of ρG (which coincides with the weight of G). By Lemma 3.1, $\mathbb{T}$ admits a finer precompact, second countable, zero-dimensional topological group topology which is denoted by τ. Let $L = (\mathbb{T}, \tau)$. The topology of G inherited from L^κ is as required since the latter group is precompact, zero-dimensional and has weight κ. □

We do not know whether the above result extends to non-abelian groups:

Problem 3.3. *Does every non-abelian compact group admit a finer precompact zero-dimensional topological group topology?*

Theorem 3.4. *Let G be a locally precompact abelian group. Then there exists a finer locally precompact, zero-dimensional group topology on G having the same weight as the original topology.*

Proof. Since G is locally precompact it can be considered as a dense subgroup of a locally compact abelian group G^*. It is clear that the groups G and G^* have the same weight, say, κ. By [12, Theorem 24.30], the group G^* is topologically isomorphic to $\mathbb{R}^n \times H$, where n is a non-negative integer and H contains a compact open subgroup K. Notice that the index of K in H is not greater than κ.

Since $w(K) \leq w(H) \leq \kappa$, it follows from Proposition 3.2 that the compact group K admits a finer precompact, zero-dimensional topological group topology σ of weight $\leq \kappa$. Denote by $\mathcal{T}$ the coarsest topological group topology on H which contains the original topology of H and such that (K, σ) is an open subgroup of $(H, \mathcal{T})$. Then the topology $\mathcal{T}$ is zero-dimensional and locally precompact. Since the index of K in H is at most κ, the weight of $(H, \mathcal{T})$ does not exceed κ. Now we take the second countable, zero-dimensional, locally precompact topological group topology τ' on $\mathbb{R}$ provided by Lemma 3.1. Consider the topology λ on G which the group G inherits from $(\mathbb{R}, \tau')^n \times (H, \mathcal{T})$. It is clear the group (G, λ) has the required properties. □

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