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Additive Property of Pseudo Drazin Inverse of Elements in Banach Algebras

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Abstract. This article concerns the pseudo Drazin inverse of the sums (resp. differences) and the products of elements in a Banach algebra $\mathscr{A}$. Some equivalent conditions for the existence of the pseudo Drazin inverse of $a+b$ (resp. $a-b$) are characterized. Moreover, the representations for the pseudo Drazin inverse are given. Some related known results are generalized.

1. Introduction

Throughout this paper, $\mathscr{A}$ is a complex Banach algebra with unity 1. The symbols $J(\mathscr{A})$, $\mathscr{A}^{\#}$, $\mathscr{A}^{\mathrm{nil}}$ denote the Jacobson radical, the sets of all group invertible, nilpotent elements of $\mathscr{A}$, respectively. An element $a \in \mathscr{A}$ is said to have a *Drazin inverse* [10] if there exists $b \in \mathscr{A}$ satisfying

$$ab = ba,\ bab = b,\ a - a^2 b \in \mathscr{A}^{\mathrm{nil}}.$$

The element b above is unique and is denoted by a^D, and the nilpotency index of $a - a^2b$ is called the Drazin index of a, denoted by $\mathrm{ind}(a)$. If $\mathrm{ind}(a) = 1$, then b is the group inverse of a and is denoted by $a^{\#}$. In 2012, Wang and Chen [16] introduced the notion of *pseudo Drazin inverse* (abbr. *p-Drazin inverse*) in associative rings and Banach algebras. An element $a \in \mathscr{A}$ is called *p-Drazin invertible* if there exists $b \in \mathscr{A}$ such that

$$ab = ba,\ bab = b,\ a^k - a^{k+1}b \in J(\mathscr{A}) \text{ for some integer } k \geqslant 1.$$

Any element $b \in \mathscr{A}$ satisfying the conditions above is called a p-Drazin inverse of a, denoted by $a^{\ddagger}$. The set of all p-Drazin invertible elements of $\mathscr{A}$ is denoted by $\mathscr{A}^{pD}$. By $a^{\Pi} = 1 - aa^{\ddagger}$ we mean the strongly spectral idempotent of a.

Representations for the Drazin inverse of the sums and the products of two elements in certain algebras have attracted wide interest. In general, it is a challenging task to characterize the Drazin inverse of $a+b$ or ab without additional hypothesis. Given a and b in a ring R with Drazin inverses a^D and b^D, respectively.

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If $ab = ba = 0$ then it follows that $a + b$ is Drazin invertible with $(a + b)^D = a^D + b^D$ (see [10, Corollary 1]). Wei and Deng [17] presented the expressions on the Drazin inverse of two commutative square complex matrices. Later, Zhuang, Chen et al. [18] extended the results in [17] to the ring case and proved that $a + b$ is Drazin invertible if and only if $1 + a^D b$ is Drazin invertible. Further, Deng [7] explored the Drazin inverse in the ring $\mathcal{B}(X)$ of bounded operators of a Banach space X. Under the condition that $PQ = \lambda QP$, they gave explicit representations of the Drazin inverses $(P - Q)^D$ (resp. $(P + Q)^D$) and $(PQ)^D$ in terms of P, P^D, Q and Q^D. More results on (generalized) Drazin inverse can be found in [1-6, 8, 9, 11-15].

This article is motivated by Deng [7], Wei, Deng [17] and Zhuang et al. [18]. We investigate the representations for p-Drazin inverse of the sums (resp. differences) and the products of two elements in a Banach algebra. Moreover, some equivalent conditions for existence of p-Drazin inverse of the sums and the differences of two elements in a Banach algebra are given.

2. p-Drazin Inverse Under the Condition $ab = ba$

In this section, we give some elementary properties on p-Drazin inverse.

Lemma 2.1. ([16, Proposition 5.2]) *Let $a, b \in \mathscr{A}$. If $ab = ba$ and $a^\ddagger$, $b^\ddagger$ exist, then $(ab)^\ddagger = a^\ddagger b^\ddagger = b^\ddagger a^\ddagger$. Moreover, $ab^\ddagger = b^\ddagger a$.*

In particular, if $a \in \mathscr{A}^{pD}$, then $(a^2)^\ddagger = (a^\ddagger)^2$ by Lemma 2.1.

Lemma 2.2. *Let $a, b \in \mathscr{A}$. The following statements hold:*
(1) *If $a \in J(\mathscr{A})$, then ab, $ba \in J(\mathscr{A})$.*
(2) *If $a, b \in J(\mathscr{A})$, then $(a + b)^k \in J(\mathscr{A})$ for integer $k \geqslant 1$.*

In [10], some properties of Drazin inverse were presented. One may suspect that if the similar properties can be inherited to p-Drazin inverse in a Banach algebra. The following result illustrates the possibility.

Theorem 2.3. *Let $a \in \mathscr{A}^{pD}$. Then*
(1) $(a^n)^\ddagger = (a^\ddagger)^n$, $n = 1, 2, \cdots$.
(2) $(a^\ddagger)^\ddagger = a^2 a^\ddagger$.
(3) $((a^\ddagger)^\ddagger)^\ddagger = a^\ddagger$.
(4) $a^\ddagger (a^\ddagger)^\ddagger = aa^\ddagger$.

Proof. (1) It is obvious when $n = 1$.
Assume the result holds for $n - 1$, i.e., $(a^{n-1})^\ddagger = (a^\ddagger)^{n-1}$.
For n, by Lemma 2.1, we have $(a^n)^\ddagger = (aa^{n-1})^\ddagger = a^\ddagger (a^{n-1})^\ddagger = a^\ddagger (a^\ddagger)^{n-1} = (a^\ddagger)^n$.
Hence, a^n is p-Drazin invertible and $(a^n)^\ddagger = (a^\ddagger)^n$.
(2) It is easy to check $a^\ddagger a^2 a^\ddagger = a^2 a^\ddagger a^\ddagger$ and $a^2 a^\ddagger a^\ddagger a^2 a^\ddagger = a^2 a^\ddagger$.
Since $(a^\ddagger)^k - (a^\ddagger)^{k+1} a^2 a^\ddagger = (a^\ddagger)^k - (a^\ddagger)^{k+1} a = (a^\ddagger)^k - (a^\ddagger)^k = 0 \in J(\mathscr{A})$ for some $k \geqslant 1$, it follows that $(a^\ddagger)^\ddagger = a^2 a^\ddagger$.
(3) By (2) and Lemma 2.1.
(4) According to (2). □

Corollary 2.4. *Let $a \in \mathscr{A}^{pD}$. Then $(a^\ddagger)^\ddagger = a$ if and only if $a \in \mathscr{A}^\#$.*

Theorem 2.5. *Let $a, b \in \mathscr{A}^{pD}$ with $ab = ba = 0$. Then $(a + b)^\ddagger = a^\ddagger + b^\ddagger$.*

Proof. Since $ab = ba = 0$, it follows that $ab^\ddagger = ba^\ddagger = 0$ and $a^\ddagger b = b^\ddagger a = 0$.
Thus, we obtain
(i) $(a^\ddagger + b^\ddagger)(a + b) = (a + b)(a^\ddagger + b^\ddagger)$.
(ii) By $a^\ddagger a a^\ddagger = a^\ddagger$, we have

$$\begin{aligned} (a^\ddagger + b^\ddagger)(a + b)(a^\ddagger + b^\ddagger) &= (a^\ddagger a + b^\ddagger b)(a^\ddagger + b^\ddagger) \\ &= a^\ddagger + b^\ddagger. \end{aligned}$$

(iii) According to $a^k - a^{k+1}a^\ddagger \in J(\mathscr{A})$ and $b^k - b^{k+1}b^\ddagger \in J(\mathscr{A})$ for some $k \geqslant 1$, we obtain

$$\begin{array}{rcl}(a+b)^k - (a+b)^{k+1}(a^\ddagger + b^\ddagger) & = & (a^k + b^k) - (a^{k+1} + b^{k+1})(a^\ddagger + b^\ddagger) \\ & = & a^k + b^k - a^{k+1}a^\ddagger - b^{k+1}b^\ddagger \\ & = & a^k - a^{k+1}a^\ddagger + b^k - b^{k+1}b^\ddagger \\ & & \in J(\mathscr{A}).\end{array}$$

Hence, $(a+b)^\ddagger = a^\ddagger + b^\ddagger$. $\square$

Corollary 2.6. *If $a_1, a_2, \cdots, a_n \in \mathscr{A}^{pD}$ such that $a_i a_j = 0$ ($i, j = 1, \cdots, n$; $i \neq j$), then $a_1 + a_2 + \cdots + a_n$ is p-Drazin invertible and $(a_1 + a_2 + \cdots + a_n)^\ddagger = a_1^\ddagger + a_2^\ddagger + \cdots + a_n^\ddagger$.*

Proof. It is true for $n = 2$ by Theorem 2.5.

Assume that the result holds for $n-1$. Then $(a_1 + \cdots + a_{n-1})^\ddagger = a_1^\ddagger + \cdots + a_{n-1}^\ddagger$.

For n case, Theorem 2.5 guarantees that

$$\begin{array}{rcl}(a_1 + \cdots + a_{n-1} + a_n)^\ddagger & = & (a_1 + \cdots + a_{n-1})^\ddagger + a_n^\ddagger \\ & = & a_1^\ddagger + \cdots + a_n^\ddagger.\end{array}$$

This completes the proof. $\square$

In [17], Wei and Deng presented the formula for the Drazin inverse of two square matrices that commute with each other. We consider the result in [17] for p-Drazin inverse in a Banach algebra as follows.

Theorem 2.7. *If $a, b \in \mathscr{A}^{pD}$ and $ab = ba$, then $a+b$ is p-Drazin invertible if and only if $1 + a^\ddagger b$ is p-Drazin invertible. In this case, we have*

$$(a+b)^\ddagger = (1 + a^\ddagger b)^\ddagger a^\ddagger + b^\ddagger \sum_{i=0}^{\infty} (-b^\ddagger a a^\Pi)^i a^\Pi,$$

and

$$(1 + a^\ddagger b)^\ddagger = a^\Pi + a^2 a^\ddagger (a+b)^\ddagger.$$

Proof. Suppose that $a + b$ is p-Drazin invertible. We prove that $1 + a^\ddagger b$ is p-Drazin invertible. Write $1 + a^\ddagger b = a_1 + b_1$ with $a_1 = a^\Pi$ and $b_1 = a^\ddagger(a+b)$.

Note that a, b, $a^\ddagger$ and $b^\ddagger$ commute with each other. We obtain $(b_1)^\ddagger = (a^\ddagger(a+b))^\ddagger = a^2 a^\ddagger (a+b)^\ddagger$ by Lemma 2.1 and Theorem 2.3(2).

Since a_1 is idempotent, $(a_1)^\ddagger = a_1 = a^\Pi$. Observing that $a_1 b_1 = b_1 a_1 = 0$, it follows that $(1 + a^\ddagger b)^\ddagger = a^\Pi + a^2 a^\ddagger (a+b)^\ddagger$ by Theorem 2.5.

Conversely, let $\xi = 1 + a^\ddagger b \in \mathscr{A}^{pD}$ and $x = \xi^\ddagger a^\ddagger + b^\ddagger \sum_{i=0}^{\infty} (-b^\ddagger a a^\Pi)^i a^\Pi$.

We prove that x is the p-Drazin inverse of $a + b$ by showing the following conditions hold: (i) $x(a+b) = (a+b)x$, (ii) $x(a+b)x = x$, (iii) $(a+b)^k - (a+b)^{k+1}x \in J(\mathscr{A})$.

(i) By Lemma 2.1, $a + b$ commutes with x.

(ii) Note that $\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i = 1 + \sum_{i=1}^{\infty}(-b^{\ddagger}aa^{\Pi})^i = 1 - b^{\ddagger}aa^{\Pi}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i$. We have

$$\begin{aligned}
x(a+b) &= \left[\xi^{\ddagger}a^{\ddagger} + b^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i a^{\Pi}\right](a+b)\\
&= \xi^{\ddagger}a^{\ddagger}(a+b) + b^{\ddagger}aa^{\Pi}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i + bb^{\ddagger}a^{\Pi}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i\\
&= \xi^{\ddagger}a^{\ddagger}(a+b) + b^{\ddagger}aa^{\Pi}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i + bb^{\ddagger}a^{\Pi}\left[1 - b^{\ddagger}aa^{\Pi}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i\right]\\
&= \xi^{\ddagger}a^{\ddagger}(a+b) + bb^{\ddagger}a^{\Pi}.
\end{aligned}$$

Hence,

$$\begin{aligned}
x(a+b)x &= \left[\xi^{\ddagger}a^{\ddagger}(a+b) + bb^{\ddagger}a^{\Pi}\right]\left[\xi^{\ddagger}a^{\ddagger} + b^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i a^{\Pi}\right]\\
&= (\xi^{\ddagger})^2(a^{\ddagger})^2(a+b) + a^{\Pi}b^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i\\
&= (\xi^{\ddagger})^2a^{\ddagger}\xi + a^{\Pi}b^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i\\
&= \xi^{\ddagger}a^{\ddagger} + b^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}aa^{\Pi})^i a^{\Pi}\\
&= x.
\end{aligned}$$

(iii) We have $(a+b)^k - (a+b)^{k+1}x \in J(\mathscr{A})$. Indeed,

$$\begin{aligned}
a+b-(a+b)^2x &= a+b-(a+b)[\xi^{\ddagger}a^{\ddagger}(a+b) + bb^{\ddagger}a^{\Pi}]\\
&= a+b-\xi^{\ddagger}a^{\ddagger}(a+b)^2 - (a+b)bb^{\ddagger}a^{\Pi}\\
&= a+b-\xi^{\ddagger}(a^{\ddagger}(a+b))^2a - aa^{\Pi}bb^{\ddagger} - b^2b^{\ddagger}(1-aa^{\ddagger})\\
&= a+b-\xi^{\ddagger}(\xi-a^{\Pi})^2a - aa^{\Pi}bb^{\ddagger} - b^2b^{\ddagger} + aa^{\ddagger}b^2b^{\ddagger}\\
&= a+b-\xi^{\ddagger}\xi^2a + \xi^{\ddagger}aa^{\Pi} - aa^{\Pi}bb^{\ddagger} - b^2b^{\ddagger} + aa^{\ddagger}b(1-b^{\Pi})\\
&= b - b^2b^{\ddagger} - aa^{\Pi}bb^{\ddagger} + a + aa^{\ddagger}b - aa^{\ddagger}bb^{\Pi} + \xi^{\ddagger}aa^{\Pi} - \xi^{\ddagger}\xi^2a\\
&= b - b^2b^{\ddagger} - aa^{\Pi}bb^{\ddagger} + a\xi - aa^{\ddagger}bb^{\Pi} + \xi^{\ddagger}aa^{\Pi} - \xi^{\ddagger}\xi^2a\\
&= bb^{\Pi} + (\xi^{\ddagger} - bb^{\ddagger})aa^{\Pi} - aa^{\ddagger}bb^{\Pi} + a\xi\xi^{\Pi}\\
&= a^{\Pi}bb^{\Pi} + (\xi^{\ddagger} - bb^{\ddagger})aa^{\Pi} + a\xi\xi^{\Pi}.
\end{aligned}$$

Since $(aa^{\Pi})^{k_1} \in J(\mathscr{A})$, $(bb^{\Pi})^{k_2} \in J(\mathscr{A})$ and $(\xi\xi^{\Pi})^{k_3} \in J(\mathscr{A})$ for some positive integers k_1, k_2 and k_3, take suitable $k \geqslant k_1 + k_2 + k_3$, it follows that $(a+b)^k - (a+b)^{k+1}x = [a+b-(a+b)^2x]^k \in J(\mathscr{A})$ by Lemma 2.2(2).

The proof is completed. □

3. p-Drazin Inverse Under the Condition $ab = \lambda ba$

In this section, we give some results on p-Drazin inverse under the condition that $ab = \lambda ba$.

Lemma 3.1. *Let $a, b \in \mathscr{A}$ with $ab = \lambda ba$ ($\lambda \neq 0$). Then*

(1) $ab^n = \lambda^n b^n a$, $a^n b = \lambda^n ba^n$.

(2) $(ab)^n = \lambda^{\frac{-n(n-1)}{2}}a^nb^n = \lambda^{\frac{n(n+1)}{2}}b^na^n$.

(3) $(ba)^n = \lambda^{\frac{n(n-1)}{2}}b^na^n = \lambda^{\frac{-n(n+1)}{2}}a^nb^n$.

Proof. By induction, it is easy to obtain the results. $\square$

Let $\mathscr{A}^{\mathrm{qnil}} = \{a \in \mathscr{A} : 1 + ax \in \mathscr{A}^{-1} \ for\ every\ x \in comm(a)\}$. Then we have (see [13, p. 138]) that $a \in \mathscr{A}^{\mathrm{qnil}}$ if and only if $\|a^n\|^{\frac{1}{n}} \to 0$ $(n \to \infty)$ and that $J(\mathscr{A}) \subset \mathscr{A}^{\mathrm{qnil}}$. It is well known that $c^k \in J(\mathscr{A})$ implies that $c \in \mathscr{A}^{\mathrm{qnil}}$ for $k \geqslant 1$. Indeed, for any $x \in \mathrm{comm}(c)$, $1 - (cx)^k = (1 - cx)(1 + cx + \cdots + (cx)^{k-1}) \in \mathscr{A}^{-1}$. Then, $1 - cx \in \mathscr{A}^{-1}$ implies that $c \in \mathscr{A}^{\mathrm{qnil}}$. Hence, we have $a(1 - aa^{\ddagger}) \in \mathscr{A}^{\mathrm{qnil}}$.

Lemma 3.2. *Let $a, b \in \mathscr{A}^{pD}$ with $ab = \lambda ba$ $(\lambda \neq 0)$. Then*
(1) $aa^{\ddagger}b = baa^{\ddagger}$.
(2) $bb^{\ddagger}a = abb^{\ddagger}$.

Proof. (1) Since $a(1 - aa^{\ddagger}) \in \mathscr{A}^{\mathrm{qnil}}$, it follows that $\|(a(1 - aa^{\ddagger}))^n\|^{\frac{1}{n}} \to 0$ $(n \to \infty)$. Suppose $p = aa^{\ddagger}$. We have

$$\begin{aligned}
\|pb - pbp\|^{\frac{1}{n}} &= \|a^{\ddagger}ab(1 - aa^{\ddagger})\|^{\frac{1}{n}} \\
&= \|(a^{\ddagger})^n a^n b(1 - aa^{\ddagger})\|^{\frac{1}{n}} \\
&= \|(a^{\ddagger})^n \lambda^n b a^n (1 - aa^{\ddagger})\|^{\frac{1}{n}} \\
&= \|\lambda^n (a^{\ddagger})^n b(a(1 - aa^{\ddagger}))^n\|^{\frac{1}{n}} \\
&\leqslant |\lambda| \, \|a^{\ddagger}\| \|b\|^{\frac{1}{n}} \|(a(1 - aa^{\ddagger}))^n\|^{\frac{1}{n}}.
\end{aligned}$$

Hence, $\|pb - pbp\|^{\frac{1}{n}} \to 0$ $(n \to \infty)$, it follows that $pb = pbp$.
Similarly, $bp = pbp$.
Thus, $aa^{\ddagger}b = baa^{\ddagger}$.
(2) The proof is similar to the proof of (1). $\square$

Authors [16] proved that $(ab)^{\ddagger} = b^{\ddagger}a^{\ddagger}$ under the condition $ab = ba$ in a Banach algebra. We can obtain some generalized results under weak commutative condition $ab = \lambda ba$.

Theorem 3.3. *Let $a, b \in \mathscr{A}^{pD}$ with $ab = \lambda ba$ $(\lambda \neq 0)$. Then*
(1) $a^{\ddagger}b = \lambda^{-1}ba^{\ddagger}$.
(2) $ab^{\ddagger} = \lambda^{-1}b^{\ddagger}a$.
(3) $(ab)^{\ddagger} = b^{\ddagger}a^{\ddagger} = \lambda^{-1}a^{\ddagger}b^{\ddagger}$.

Proof. (1) Since $aa^{\ddagger}b = baa^{\ddagger}$, we have

$$\begin{aligned}
a^{\ddagger}b &= a^{\ddagger}aa^{\ddagger}b = a^{\ddagger}baa^{\ddagger} = a^{\ddagger}\lambda^{-1}aba^{\ddagger} \\
&= \lambda^{-1}a^{\ddagger}aba^{\ddagger} = \lambda^{-1}baa^{\ddagger}a^{\ddagger} \\
&= \lambda^{-1}ba^{\ddagger}.
\end{aligned}$$

(2) The proof is similar to (1).
(3) We first prove that $b^{\ddagger}a^{\ddagger} = \lambda^{-1}a^{\ddagger}b^{\ddagger}$. By Lemma 3.2, it follows that

$$\begin{aligned}
b^{\ddagger}a^{\ddagger} &= b^{\ddagger}(aa^{\ddagger})a^{\ddagger} = (aa^{\ddagger})b^{\ddagger}a^{\ddagger} \\
&= a^{\ddagger}(ab^{\ddagger})a^{\ddagger} = a^{\ddagger}(\lambda^{-1}b^{\ddagger}a)a^{\ddagger} \\
&= \lambda^{-1}a^{\ddagger}b^{\ddagger}(aa^{\ddagger}) = \lambda^{-1}a^{\ddagger}(aa^{\ddagger})b^{\ddagger} \\
&= \lambda^{-1}a^{\ddagger}b^{\ddagger}.
\end{aligned}$$

We now prove that $x = b^{\ddagger}a^{\ddagger}$ is the p-Drazin inverse of ab.
(i) By Lemma 3.2, we obtain

$$\begin{aligned}
(ab)x &= abb^{\ddagger}a^{\ddagger} = aa^{\ddagger}b^{\ddagger}b \\
&= b^{\ddagger}aa^{\ddagger}b = b^{\ddagger}a^{\ddagger}ab \\
&= x(ab).
\end{aligned}$$

(ii) $x(ab)x = b^{\ddagger}(a^{\ddagger}a)bb^{\ddagger}a^{\ddagger} = b^{\ddagger}bb^{\ddagger}(a^{\ddagger}a)a^{\ddagger} = b^{\ddagger}a^{\ddagger} = x$.

(iii) We first present an useful equality, i.e.,

$$\begin{aligned} b^{k+1}b^{\ddagger}a^{k+1}a^{\ddagger} &= b^kbb^{\ddagger}a^kaa^{\ddagger} = b^ka^kaa^{\ddagger}bb^{\ddagger} \\ &= b^ka^{k+1}a^{\ddagger}bb^{\ddagger} = b^ka^{k+1}bb^{\ddagger}a^{\ddagger} \\ &= b^k\lambda^{k+1}ba^{k+1}b^{\ddagger}a^{\ddagger} \\ &= \lambda^{k+1}b^{k+1}a^{k+1}b^{\ddagger}a^{\ddagger}. \end{aligned}$$

Hence, we have

$$\begin{aligned} (ab)^k - (ab)^{k+1}b^{\ddagger}a^{\ddagger} &= \lambda^{\frac{k(k+1)}{2}}b^ka^k - \lambda^{\frac{(k+1)(k+2)}{2}}b^{k+1}a^{k+1}b^{\ddagger}a^{\ddagger} \\ &= \lambda^{\frac{k(k+1)}{2}}(b^ka^k - \lambda^{k+1}b^{k+1}a^{k+1}b^{\ddagger}a^{\ddagger}) \\ &= \lambda^{\frac{k(k+1)}{2}}(b^ka^k - b^{k+1}b^{\ddagger}a^{k+1}a^{\ddagger}) \\ &= \lambda^{\frac{k(k+1)}{2}}[-(b^k - b^{k+1}b^{\ddagger})(a^k - a^{k+1}a^{\ddagger}) + (b^k - b^{k+1}b^{\ddagger})a^k \\ &\quad + b^k(a^k - a^{k+1}a^{\ddagger})] \\ &\in J(\mathscr{A}) \end{aligned}$$

for some $k \geqslant 1$.

Therefore, $(ab)^{\ddagger} = b^{\ddagger}a^{\ddagger} = \lambda^{-1}a^{\ddagger}b^{\ddagger}$. □

Corollary 3.4. *Let $a, b \in \mathscr{A}^{pD}$ with $ab = \lambda ba$ ($\lambda \neq 0$). Then*

(1) $(a^{\ddagger}b)^n = \lambda^{\frac{n(n-1)}{2}}(a^{\ddagger})^nb^n = \lambda^{\frac{-n(n+1)}{2}}b^n(a^{\ddagger})^n$.

(2) $(ab^{\ddagger})^n = \lambda^{\frac{n(n-1)}{2}}a^n(b^{\ddagger})^n = \lambda^{\frac{-n(n+1)}{2}}(b^{\ddagger})^na^n$.

Proof. By induction and Theorem 3.3. □

Theorem 3.5. *Let $a, b \in \mathscr{A}^{pD}$ with $ab = \lambda ba$ ($\lambda \neq 0$). Then $a-b$ is p-Drazin invertible if and only if $w = aa^{\ddagger}(a-b)bb^{\ddagger}$ is p-Drazin invertible. In this case, we have*

$$(a-b)^{\ddagger} = w^{\ddagger} + a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^ib^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^ib^{\ddagger},$$

and $w^{\ddagger} = aa^{\ddagger}(a-b)^{\ddagger}bb^{\ddagger}$.

Proof. Assume that $a-b$ is p-Drazin invertible. Since $aa^{\ddagger}$ is idempotent, $aa^{\ddagger} \in \mathscr{A}^{pD}$. By Lemma 3.2, we have $aa^{\ddagger}(a-b) = (a-b)aa^{\ddagger}$. Hence, $aa^{\ddagger}(a-b) \in \mathscr{A}^{pD}$ according to Lemma 2.1.

Again, Lemma 3.2 guarantees that $aa^{\ddagger}(a-b)bb^{\ddagger} = bb^{\ddagger}aa^{\ddagger}(a-b)$. Thus, $aa^{\ddagger}(a-b)bb^{\ddagger} \in \mathscr{A}^{pD}$ and $w^{\ddagger} = aa^{\ddagger}(a-b)^{\ddagger}bb^{\ddagger}$ according to Lemma 2.1.

Conversely, let

$$x = w^{\ddagger} + a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^ib^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^ib^{\ddagger}.$$

We prove that x is the p-Drazin inverse of $a-b$, i.e., the following conditions hold: (i) $x(a-b) = (a-b)x$, (ii) $x(a-b)x = x$, (iii) $(a-b)^k - (a-b)^{k+1}x \in J(\mathscr{A})$.

(i) By Lemma 3.2, we have

$$\begin{aligned} (a-b)w &= (a-b)aa^{\ddagger}(a-b)bb^{\ddagger} \\ &= aa^{\ddagger}(a-b)(a-b)bb^{\ddagger} \\ &= aa^{\ddagger}(a-b)bb^{\ddagger}(a-b) \\ &= w(a-b). \end{aligned}$$

Hence, $(a-b)w^{\ddagger} = w^{\ddagger}(a-b)$. Moreover,

$$\begin{aligned}
(a-b)a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} &= (aa^{\ddagger}-ba^{\ddagger})\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} = aa^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - ba^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi}\\
&= aa^{\ddagger}(1+ba^{\ddagger}+(ba^{\ddagger})^2+\cdots)b^{\Pi} - ba^{\ddagger}(1+ba^{\ddagger}+(ba^{\ddagger})^2+\cdots)b^{\Pi}\\
&= (aa^{\ddagger}+ba^{\ddagger}+(ba^{\ddagger})^2+\cdots)b^{\Pi} - (ba^{\ddagger}+(ba^{\ddagger})^2+\cdots)b^{\Pi}\\
&= aa^{\ddagger}b^{\Pi}.
\end{aligned}$$

Similarly, $(a-b)\left(a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}\right) = -bb^{\ddagger}a^{\Pi}$.

We have

$$\begin{aligned}
(a-b)x &= (a-b)(w^{\ddagger}+a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger})\\
&= (a-b)w^{\ddagger} + aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi}.
\end{aligned}$$

Note that Lemma 3.2. We get

$$\begin{aligned}
&a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi}(a-b) = a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i(a-b)b^{\Pi} = a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i ab^{\Pi} - a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i bb^{\Pi}\\
=\;& a^{\ddagger}(1+ba^{\ddagger}+(ba^{\ddagger})^2+\cdots)ab^{\Pi} - a^{\ddagger}(1+ba^{\ddagger}+(ba^{\ddagger})^2+\cdots)bb^{\Pi}\\
=\;& a^{\ddagger}(ab^{\Pi}+ba^{\ddagger}ab^{\Pi}+(ba^{\ddagger})^2ab^{\Pi}+\cdots) - a^{\ddagger}(bb^{\Pi}+ba^{\ddagger}bb^{\Pi}+(ba^{\ddagger})^2bb^{\Pi}+\cdots)\\
=\;& a^{\ddagger}(ab^{\Pi}+a^{\ddagger}abb^{\Pi}+aa^{\ddagger}ba^{\ddagger}bb^{\Pi}+aa^{\ddagger}(ba^{\ddagger})^2bb^{\Pi}+\cdots)\\
& -a^{\ddagger}(bb^{\Pi}+ba^{\ddagger}bb^{\Pi}+(ba^{\ddagger})^2bb^{\Pi}+\cdots)\\
=\;& a^{\ddagger}(ab^{\Pi}+bb^{\Pi}+ba^{\ddagger}bb^{\Pi}+(ba^{\ddagger})^2bb^{\Pi}+\cdots)\\
& -a^{\ddagger}(bb^{\Pi}+ba^{\ddagger}bb^{\Pi}+(ba^{\ddagger})^2bb^{\Pi}+\cdots)\\
=\;& aa^{\ddagger}b^{\Pi}.
\end{aligned}$$

Similarly, $a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}(a-b) = -bb^{\ddagger}a^{\Pi}$.

Therefore, we have

$$\begin{aligned}
x(a-b) &= \left(w^{\ddagger}+a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}\right)(a-b)\\
&= (a-b)w^{\ddagger} + aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi}\\
&= (a-b)x.
\end{aligned}$$

(ii) $aa^{\ddagger}$, $bb^{\ddagger}$ indeed commute with any elements of $\mathscr{A}$ from Lemma 2.1. Since $aa^{\ddagger}$ and $bb^{\ddagger}$ are idempotents, $w^{\ddagger} = aa^{\ddagger}w^{\ddagger}bb^{\ddagger}$ and

$$\begin{aligned}
w^{\ddagger}(a-b)w^{\ddagger} &= aa^{\ddagger}w^{\ddagger}bb^{\ddagger}(a-b)aa^{\ddagger}w^{\ddagger}bb^{\ddagger}\\
&= (w^{\ddagger})^2w\\
&= w^{\ddagger}.
\end{aligned}$$

Hence, we get $w^{\ddagger}a^{\Pi} = a^{\Pi}w^{\ddagger} = 0$ and $w^{\ddagger}b^{\Pi} = b^{\Pi}w^{\ddagger} = 0$. Also, $w^{\ddagger}(aa^{\ddagger}b^{\Pi}+bb^{\ddagger}a^{\Pi}) = w^{\ddagger}b^{\Pi}aa^{\ddagger} + w^{\ddagger}a^{\Pi}bb^{\ddagger} = 0$.

According to Lemma 3.2 and $(a-b)w^{\ddagger} = w^{\ddagger}(a-b)$, we have

$$\begin{aligned}
&\left[a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}\right](a-b)w^{\ddagger} = \left[a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}\right]w^{\ddagger}(a-b)\\
= \;& 0
\end{aligned}$$

and

$$\begin{aligned}
&[a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}](aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi}) = a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi}aa^{\ddagger}b^{\Pi}\\
&+a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi}bb^{\ddagger}a^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}aa^{\ddagger}b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}bb^{\ddagger}a^{\Pi}\\
= \;& a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&x(a-b)x = [w^{\ddagger} + a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}][(a-b)w^{\ddagger} + aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi}]\\
= \;& w^{\ddagger}(a-b)w^{\ddagger} + w^{\ddagger}(aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi}) + [a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}](a-b)w^{\ddagger}\\
&+[a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}](aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi})\\
= \;& w^{\ddagger} + [a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}](aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi})\\
= \;& w^{\ddagger} + a^{\ddagger}\sum_{i=0}^{\infty}(ba^{\ddagger})^i b^{\Pi} - a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger}a)^i b^{\ddagger}\\
= \;& x.
\end{aligned}$$

(iii) Note that $w^{\ddagger} = aa^{\ddagger}w^{\ddagger}bb^{\ddagger}$. Then $w^{\ddagger}(a-b)^2 = w^{\ddagger}aa^{\ddagger}(a-b)^2bb^{\ddagger} = w^{\ddagger}w^2$ and $(a-b)(aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi}) = aa^{\ddagger}(a-b)b^{\Pi} + a^{\Pi}(a-b)bb^{\ddagger}$.

Writing $w^{\Pi} = 1 - ww^{\ddagger}$, we obtain

$$\begin{aligned}
(a-b)^2x &= (a-b)(w^{\ddagger}(a-b) + aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi})\\
&= w^{\ddagger}(a-b)^2 + (a-b)(aa^{\ddagger}b^{\Pi} + bb^{\ddagger}a^{\Pi})\\
&= w^{\ddagger}w^2 + aa^{\ddagger}(a-b)b^{\Pi} + a^{\Pi}(a-b)bb^{\ddagger}\\
&= w - ww^{\Pi} + aa^{\ddagger}(a-b)b^{\Pi} + a^{\Pi}(a-b)bb^{\ddagger}\\
&= aa^{\ddagger}(a-b)bb^{\ddagger} + aa^{\ddagger}(a-b)b^{\Pi} + a^{\Pi}(a-b)bb^{\ddagger} - ww^{\Pi}\\
&= aa^{\ddagger}(a-b) + a^{\Pi}(a-b)bb^{\ddagger} - ww^{\Pi}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(a-b) - (a-b)^2x &= (a-b) - (aa^{\ddagger}(a-b) + a^{\Pi}(a-b)bb^{\ddagger} - ww^{\Pi})\\
&= (a-b) - aa^{\ddagger}(a-b) - a^{\Pi}(a-b)bb^{\ddagger} + ww^{\Pi}\\
&= a^{\Pi}(a-b) - a^{\Pi}(a-b)bb^{\ddagger} + ww^{\Pi}\\
&= a^{\Pi}(a-b)b^{\Pi} + ww^{\Pi}\\
&= a^{\Pi}ab^{\Pi} - a^{\Pi}bb^{\Pi} + ww^{\Pi}.
\end{aligned}$$

There exist some integers k_1, k_2 and k_3 such that $(aa^\Pi)^{k_1} \in J(\mathscr{A})$, $(bb^\Pi)^{k_2} \in J(\mathscr{A})$ and $(ww^\Pi)^{k_3} \in J(\mathscr{A})$. It follows from Lemma 2.2 that $(a^\Pi ab^\Pi - a^\Pi bb^\Pi)^{k_4} \in J(\mathscr{A})$ for integer $k_4 \geqslant k_1 + k_2$. Finally, as ww^Π commutes with $a^\Pi ab^\Pi - a^\Pi bb^\Pi$, we have $(a^\Pi ab^\Pi - a^\Pi bb^\Pi + ww^\Pi)^k \in J(\mathscr{A})$ for integer $k \geqslant k_1 + k_2 + k_3$. Hence, $(a-b)^k - (a-b)^{k+1}x \in J(\mathscr{A})$ for some integer k.

Therefore, $a - b \in \mathscr{A}^{pD}$ and

$$(a-b)^\ddagger = w^\ddagger + a^\ddagger \sum_{i=0}^{\infty} (ba^\ddagger)^i b^\Pi - a^\Pi \sum_{i=0}^{\infty} (b^\ddagger a)^i b^\ddagger.$$

This completes the proof. □

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