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Convergence Theorems for Strict Pseudo-Contractions in CAT(0) Metric Spaces

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Abstract. In this paper, we introduce the notion of strict pseudo-contractive mappings in the framework of CAT(0) metric spaces. Some properties of such mappings including demiclosed principle are investigated. Also, strong convergence and Δ-convergence of the well-known Mann iterative algorithm is established for strict pseudo-contractive mappings.

1. Introduction and Preliminaries

A metric space (X, d) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. For other equivalent definitions and basic properties, we refer the reader to standard texts such as [1, 3, 4, 11]. Complete CAT(0) spaces are often called Hadamard spaces. Let $x, y \in X$ and $\lambda \in [0, 1]$. We write $\lambda x \oplus (1 - \lambda)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = (1 - \lambda)d(x, y) \quad \text{and} \quad d(z, y) = \lambda d(x, y). \tag{1}$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is, $[x, y] = \{\lambda x \oplus (1 - \lambda)y : \lambda \in [0, 1]\}$. A subset C of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

Berg and Nikolaev in [2] have introduced the concept of *quasilinearization*. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is the map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad (a, b, c, d \in X). \tag{2}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a, b)d(c, d) \tag{3}$$

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for all $a, b, c, d \in X$. It known [2, Corollary 3] that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

The concept of Δ-convergence introduced by Lim [12] in 1976 was shown by Kirk and Panyanak [10] in CAT(0) spaces to be very similar to the weak convergence in Hilbert space setting. Next, we give the concept of Δ-convergence and collect some basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [7] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.
A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Uniqueness of asymptotic center implies that CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y). \tag{4}$$

We need following lemmas in the sequel.

Lemma 1.1. [10] *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 1.2. [6] *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 1.3. [9, Theorem 2.6] *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in X$.*

Lemma 1.4. [8, Lemma 2.5] *A geodesic space X is a CAT(0) space if and only if the following inequality*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x, z) + (1-\lambda)d^2(y, z) - \lambda(1-\lambda)d^2(x, y) \tag{5}$$

is satisfied for all $x, y, z \in X$ and $\lambda \in [0, 1]$.

Lemma 1.5. [3, Proposition 2.2] *Let X be a CAT(0) space. Then*

$$d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \leq \lambda d(p, r) + (1-\lambda)d(q, s) \tag{6}$$

for all $p, q, r, s \in X$ and $\lambda \in [0, 1]$.

2. Strict Pseudo-Contractions

In this section, we present an appropriate definition of strict pseudo-contractions in CAT(0) metric spaces and obtain demiclosed principle for such mappings.

Definition 2.1. *Let C be a nonempty subset of a CAT(0) space X. A mapping $T : C \to X$ is called* strict pseudo-contraction *if there exists a constant $0 \leq k < 1$ such that*

$$d^2(Tx, Ty) \leq d^2(x, y) + 4\kappa d^2\left(\frac{1}{2}x \oplus \frac{1}{2}Ty, \frac{1}{2}Tx \oplus \frac{1}{2}y\right) \tag{7}$$

for all $x, y \in C$. If (7) holds, we also say that T is a κ-strict pseudo-contraction.

The definition of pseudo-contraction finds its origin in Hilbert spaces. Note that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. A point $x \in C$ is called fixed point of T if $Tx = x$. We shall denote by $F(T)$ the set of fixed points of T. If T is κ-strict pseudo-contraction and $p \in F(T)$, then it follows from (6) that

$$d^2(Tx, p) \leq d^2(x, p) + \kappa d^2(x, Tx) \tag{8}$$

for all $x \in C$.

Proposition 2.2. *Let C be a nonempty subset of a CAT(0) space X and $T : C \to X$ be a mapping. If T is a κ-strict pseudo-contraction, then T satisfies the Lipschitz condition*

$$d(Tx, Ty) \leq \frac{1+\kappa}{1-\kappa} d(x, y). \tag{9}$$

Proof. Using Cauchy-Schwarz inequality and (5) we have

$$\begin{aligned}
d^2(Tx, Ty) &\leq d^2(x, y) + 4\kappa d^2\left(\frac{1}{2}x \oplus \frac{1}{2}Ty, \frac{1}{2}Tx \oplus \frac{1}{2}y\right) \\
&\leq d^2(x, y) + \kappa\Big(d^2(x, y) + d^2(Tx, Ty) \\
&\quad + d^2(x, Tx) + d^2(y, Ty) - d^2(x, Ty) - d^2(y, Tx)\Big) \\
&= d^2(x, y) + \kappa\left(d^2(x, y) + d^2(Tx, Ty)\right) + 2\kappa\langle \overrightarrow{yx}, \overrightarrow{TxTy}\rangle \\
&\leq d^2(x, y) + \kappa\left(d^2(x, y) + d^2(Tx, Ty)\right) + 2\kappa d(x, y) d(Tx, Ty).
\end{aligned} \tag{10}$$

It follows that

$$(1-\kappa)d^2(Tx, Ty) - 2\kappa d(x, y) d^2(Tx, Ty) - (1+\kappa)d^2(x, y) \leq 0.$$

Solving this quadratic inequality, we obtain the Lipschitz condition (9). □

Theorem 2.3. *Let C be a closed convex subset of a CAT(0) space X and $T : C \to X$ be a κ-strict pseudo-contraction mapping. If $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.*

Proof. From Lipschitz condition (9) it follows that $F(T)$ is closed. We prove convexity. Let $p, q \in F(T)$, $t \in [0, 1]$ and $x = tp \oplus (1-t)q$. By (1), (5) and (8) we have

$$\begin{aligned}
d^2(x, Tx) &\leq td^2(p, Tx) + (1-t)d^2(q, Tx) - t(1-t)d^2(p, q) \\
&\leq t[d^2(x, p) + \kappa d^2(x, Tx)] + (1-t)[d^2(x, q) + \kappa d^2(x, Tx)] - t(1-t)d^2(p, q) \\
&= t[(1-t)^2 d^2(p, q) + \kappa d^2(x, Tx)] + (1-t)[t^2 d^2(p, q) + \kappa d^2(x, Tx)] \\
&\quad - t(1-t)d^2(p, q) \\
&= [t(1-t)^2 + (1-t)t^2 - t(1-t)]d^2(p, q) + \kappa d^2(x, Tx) \\
&= \kappa d^2(x, Tx).
\end{aligned}$$

Since $0 \leq \kappa < 1$, then $d(x, Tx) = 0$. □

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that "$I - T$ is demiclosed at zero" if the conditions, $\{x_n\} \subseteq C$ Δ- converges to x^* and $d(x_n, Tx_n) \to 0$ imply $x^* \in F(T)$.

Theorem 2.4. ***(Demiclosed principle)*** *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T : C \to X$ be a mapping. If T is a κ-strict pseudo-contraction, then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\} \subseteq C$ Δ-converges to x^*. It follows from Lemma 1.2 that $x^* \in C$. For each $x \in X$, set

$$f(x) := \limsup_{n\to\infty} d^2(x_n, x).$$

By definition of quasilinearization we see that

$$d^2(x_n, x^*) + d^2(x, x^*) = 2\langle \overrightarrow{x^* x_n}, \overrightarrow{x^* x}\rangle + d^2(x_n, x).$$

This together with Lemma 1.3 implies that

$$f(x^*) + d^2(x, x^*) \le f(x), \quad \forall x \in X.$$

In particular,

$$f(x^*) + d^2(Tx^*, x^*) \le f(Tx^*). \tag{11}$$

On the other hand, using similar method as in (10), we have

$$\begin{aligned} d^2(Tx_n, Tx^*) &\le d^2(x_n, x^*) + \kappa\left(d^2(x_n, Tx_n) + d^2(x^*, Tx^*)\right) \\ &\quad +2\kappa d(x_n, Tx_n)d(x^*, Tx^*). \end{aligned}$$

It follows from the assumption $d(x_n, Tx_n) \to 0$ that

$$\begin{aligned} f(Tx^*) &= \limsup_{n\to\infty} d^2(x_n, Tx^*) \le \limsup_{n\to\infty} d^2(Tx_n, Tx^*) \\ &\le f(x^*) + \kappa d^2(x^*, Tx^*). \end{aligned}$$

This together with (11) implies that $Tx^* = x^*$. □

3. Mann's Algorithm

We recall that given a self-mapping T of a closed convex subset C of a CAT(0) space X, Mann's algorithm generates a sequence $\{x_n\}$ in C by the recursive formula

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)Tx_n, \quad n \ge 0, \tag{12}$$

where the initial guess x_0 is arbitrary and $\{\alpha_n\}$ is a real control sequence in the interval $(0, 1)$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space X, $T : C \to C$ be a κ-strict pseudo-contraction for some $0 \le \kappa < 1$ such that the fixed point set $F(T)$ is nonempty. Let $\{x_n\}$ be the sequence generated by Mann's algorithm (12). If $\alpha_n \subset [\alpha, \beta]$ for some $\alpha, \beta \in (\kappa, 1)$ and for all $n \ge 0$, then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Let $p \in F(T)$. It follows from (5) and (8) that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n x_n \oplus (1 - \alpha_n)Tx_n, p) \\ &\le \alpha_n d^2(x_n, p) + (1 - \alpha_n)d^2(Tx_n, p) - \alpha_n(1 - \alpha_n)d^2(x_n, Tx_n) \\ &\le \alpha_n d^2(x_n, p) + (1 - \alpha_n)(d^2(x_n, p) + \kappa d^2(x_n, Tx_n)) - \alpha_n(1 - \alpha_n)d^2(x_n, Tx_n) \\ &= d^2(x_n, p) - (\alpha_n - \kappa)(1 - \alpha_n)d^2(x_n, Tx_n). \end{aligned} \tag{13}$$

Since $\kappa < \alpha_n < 1$ for all $n \ge 0$, we have $d(x_{n+1}, p) \le d(x_n, p)$, that is, the sequence $\{d(x_n, p)\}$ is decreasing and so $\lim_{n\to\infty} d(x_n, p)$ exists. Moreover, utilizing (13) and considering $\kappa < \alpha \le \alpha_n \le \beta < 1$, we have

$$\begin{aligned} (\alpha - \kappa)(1 - \beta)d^2(x_n, Tx_n) &\le (\alpha_n - \kappa)(1 - \alpha_n)d^2(x_n, Tx_n) \\ &\le d^2(x_n, p) - d^2(x_{n+1}, p). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{14}$$

Since $\{x_n\}$ is bounded, it follows from Lemma 1.1 that $\omega_\Delta(x_n) \neq \emptyset$, where

$$\omega_\Delta(x_n) = \{x \in X : x_{n_i} \ \Delta\text{-converges to } x \text{ for some subsequence } \{n_i\} \text{ of } \{n\}\}.$$

Let $p \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which Δ-converges to p. Using (14) and Theorem 2.4 (demiclosedness of $I - T$), we get $p \in F(T)$ and so $\omega_\Delta(x_n) \subset F(T)$. We show that $\omega_\Delta(x_n)$ is singleton. Let $p, q \in \omega_\Delta(x_n)$ and let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ which Δ-converge to p and q, respectively. If $p \neq q$, then from (4) and the fact that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$, we have

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, p) &= \limsup_{i\to\infty} d(x_{n_i}, p) < \limsup_{i\to\infty} d(x_{n_i}, q) \\
&= \lim_{n\to\infty} d(x_n, q) = \limsup_{j\to\infty} d(x_{n_j}, q) \\
&< \limsup_{j\to\infty} d(x_{n_j}, p) = \lim_{n\to\infty} d(x_n, p),
\end{aligned}$$

which is a contradiction. Hence, $p = q$ and the proof is complete. □

Remark 3.2. *Theorem 3.1 generalizes Marino and Xu's result [13, Theorem 3.1] to CAT(0) metric spaces which are more general than Hilbert spaces. Note that our strong assumption on control sequence $\{\alpha_n\}$ is not restrictive. Also, Theorem 3.1 includes Corollary 3.1 of [5], where $x_n \in C$ for $n \geq 2$ and it is not needed projecting x_n on C.*

The following theorem gives a sufficient condition for strong convergence of $\{x_n\}$, which is an extension of Corollary 3.3 of [5].

Theorem 3.3. *With the assumptions of Theorem 3.1, $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T))$ denotes the metric distance from the point x_n to $F(T)$.*

Proof. The necessity is apparent. We show the sufficiency. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As in proof of Theorem 3.1, we have $d(x_{n+1}, p) \leq d(x_n, p)$. Taking infimum over all $p \in F(T)$, we have $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$. Thus $\lim_{n\to\infty} d(x_n, F(T))$ exists and so $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Let $n, m \geq 1$ and $p \in F(T)$ be arbitrary. Then we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p) + d(x_n, p) \leq 2d(x_n, p),$$

which follows that $d(x_{n+m}, x_n) \leq 2d(x_n, F(T))$. Thus $\{x_n\}$ is a Cauchy sequence. Let $x_n \to q \in C$. Therefore,

$$d(q, F(T)) \leq d(q, x_n) + d(x_n, F(T)) \to 0.$$

Since by Theorem 2.3, $F(T)$ is closed, then $q \in F(T)$ and the proof is complete. □

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