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Coupled Fixed Point Theorems for Contractive Mappings Involving New Function Classes and Applications

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Abstract. The aim of this paper is to introduce the notions of α-$(\mathcal{F}, H)$-(φ, ψ) and μ-$(\mathcal{F}, H)$-(φ, ψ) contractions and present several coupled fixed point theorems for this type of contractions in the set of ordered metric spaces. Several examples are offered to illustrate the validity of the obtained results. As an application, the existence of a solution of Fredholm nonlinear integral equations are also investigated.

1. Introduction and Preliminaries

The existence of fixed points in ordered metric spaces was first investigated by Ran and Reurings [13], and then by Nieto and Rodríguez-López [10]. Recently, Samet et al. [22] presented the notions of α-ψ-contractive and α-admissible mappings and proved some fixed point results for such mappings. Their results extend and improve many fixed point results in ordered metric spaces. After that, several authors considered the generalizations of this new approach (see [5, 7, 8, 20]).

One of the interesting and crucial concepts, a coupled fixed point theorem, was introduced and studied by Opoitsev [11, 12] and then by Guo and Lakhsmikantham [4]. Bhaskar and Lakhsmikantham [3] were the first to study coupled fixed points in connection to contractive type conditions. They applied their results to prove the existence and uniqueness of solutions for a periodic boundary value problem. Since then, coupled fixed point theory have been a subject of interest by many authors regarding the application potential of it, for example see [1, 6, 9, 14–19, 21] and references therein.

In the present paper, we introduce the notions of α-$(\mathcal{F}, H)$-(φ, ψ) and μ-$(\mathcal{F}, H)$-(φ, ψ) contractions and prove several coupled fixed point theorems for this type of contractions in ordered metric spaces. We offer several examples to illustrate the validity of the obtained results. As an application, we also give an existence of a solution of a Fredholm nonlinear integral equation.

We begin with giving some notation and preliminaries that we shall need to state our results.

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In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ will denote the set of all real numbers, the set of all non negative real numbers and the set of all natural numbers, respectively. Also, the triple $(X, \preceq, d)$ denotes an ordered metric space where $\preceq$ is a partial order on the set X and d is a metric on X.

The partial order $\preceq$ on X can be induced on X^2 as follows:

$$(x, y), (u, v) \in X^2, \quad (x, y) \preceq (u, v) \Leftrightarrow x \preceq u \text{ and } y \succeq v.$$

Definition 1.1 ([3]). *Let $(X, \preceq)$ be an ordered set and $S : X^2 \to X$. We say that S has the mixed monotone property in X if, for any $x, y \in X$,*

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow S(x_1, y) \preceq S(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow S(x, y_1) \succeq S(x, y_2).$$

Definition 1.2 ([3]). *An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $S : X^2 \to X$ if*

$$x = S(x, y) \quad \text{and} \quad y = S(y, x).$$

Definition 1.3. *Let $(X, \preceq, d)$ be an ordered metric space. We say that $(X, \preceq, d)$ is regular,*

$$\begin{array}{l} \text{if } (x_n) \text{ is a nondecreasing sequence with } x_n \to x \text{ then } x_n \preceq x \text{ for all } n, \\ \text{if } (y_n) \text{ is a nonincreasing sequence with } y_n \to y \text{ then } y_n \succeq y \text{ for all } n. \end{array} \tag{1}$$

Definition 1.4 ([2]). *We say that the function $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a subclass of type I if $x \geq 1$ implies that $H(1, y) \leq H(x, y)$, for all $y \in \mathbb{R}^+$.*

Example 1.5 ([2]). *Define $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $H(x, y) = (y + l)^x$, $l > 1$;
(b) $H(x, y) = x^n y$, $n \in \mathbb{N}$;
(c) $H(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^{n} x^i \right) y$, $n \in \mathbb{N}$;

for all $x, y \in \mathbb{R}^+$. Then H is a subclass of type I.

Definition 1.6 ([2]). *Let $H, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, H)$ is an upper class of type I, if H is a function of subclass of type I and:*

(i) *If $0 \leq s \leq 1$ implies that $\mathcal{F}(s, t) \leq \mathcal{F}(1, t)$,*
(ii) *If $H(1, y) \leq \mathcal{F}(1, t)$ implies that $y \leq t$ for all $t, y \in \mathbb{R}^+$.*

Example 1.7 ([2]). *Define $H, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $H(x, y) = (y + l)^x$, $l > 1$ *and* $\mathcal{F}(s, t) = st + l$;
(b) $H(x, y) = x^m y$, $m \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = st$;
(c) $H(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^{n} x^i \right) y$, $n \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = st$;

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, H)$ is an upper class of type I.

2. Main Results

Let us give useful definitions and lemmas which will be effectively used in the proof of the our main results.

Definition 2.1 ([22]). *Let $S : X^2 \to X$ and $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ be given mappings. We say that S is α-admissible if*

$$\alpha((x,y),(u,v)) \geq 1 \text{ implies } \alpha((S(x,y),S(y,x)),(S(u,v),S(v,u))) \geq 1,$$

for all $(x,y),(u,v) \in X^2$.

For simplify, we will use $\alpha_{xyuv} = \alpha((x,y),(u,v))$.
Let Φ denote all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy

(i)$_\varphi$ φ is continuous and non-decreasing,
(ii)$_\varphi$ $\varphi(t) = 0$ iff $t = 0$,
(iii)$_\varphi$ $\varphi(t+s) \leq \varphi(t) + \varphi(s), \forall t,s \in \mathbb{R}^+$

and following the direction in [23], we denote by Ψ the family of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy

(iv)$_\psi$ ψ is a continuous function with the condition $\varphi(t) > \psi(t)$ for all $t > 0$.

Note that, by $(i)_\varphi$, $(ii)_\varphi$ and $(iv)_\psi$ we have that $\psi(0) = 0$.

Definition 2.2. *Let $(X, \preceq, d)$ be an ordered metric space, $S : X^2 \to X$ and $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ be given mappings. We say that S is an α-$(\mathcal{F}, H)$-(φ, ψ) contractive mapping if*

$$H(\alpha((x,y),(u,v)), \varphi(d(S(x,y), S(u,v)))) \leq \mathcal{F}\left(1, \frac{1}{2}\psi(d(x,u) + d(y,v))\right), \tag{2}$$

for all $(x,y),(u,v) \in X^2$ with $(u,v) \preceq (x,y)$ where $\varphi \in \Phi$, $\psi \in \Psi$ and the pair $(\mathcal{F}, H)$ is upper class of type I.

First of our main results is the following.

Theorem 2.3. *Let $(X, \preceq, d)$ be an ordered complete metric space, $S : X^2 \to X$ be an α-admissible and α-$(\mathcal{F}, H)$-(φ, ψ) contractive mapping with mixed monotone property. Assume that following conditions hold:*

(a) There exist $x_0, y_0 \in X$ such that

$$\begin{aligned} \alpha((S(x_0,y_0), S(y_0,x_0)), (x_0,y_0)) &\geq 1 \text{ and} \\ \alpha((y_0,x_0), S(y_0,x_0), S(x_0,y_0)) &\geq 1, \end{aligned}$$

(b) (i) S is continuous, or
(ii) $(X, \preceq, d)$ is regular and, if (x_n) and (y_n) are sequences in X such that

$$\alpha_{x_{n+1}y_{n+1}x_ny_n} \geq 1 \quad \text{and} \quad \alpha_{y_nx_ny_{n+1}x_{n+1}} \geq 1, \quad \text{for all } n \in \mathbb{N}_0$$

and $x_n \to x$, $y_n \to y$ for all $x, y \in X$, then

$$\alpha_{xyx_ny_n} \geq 1 \quad \text{and} \quad \alpha_{y_nx_nyx} \geq 1.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq S(x_0,y_0)$ and $y_0 \succeq S(y_0,x_0)$, then S has a coupled fixed point.

Proof. Construct the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$x_{n+1} = S(x_n, y_n) \quad \text{and} \quad y_{n+1} = S(y_n, x_n), \quad \text{for all } n \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We shall show that

$$x_n \preceq x_{n+1} \quad \text{and} \quad y_n \succeq y_{n+1}, \quad \text{for all } n \in \mathbb{N}_0. \tag{3}$$

From the hypothesis, since $x_0 \preceq S(x_0, y_0) = x_1$ and $y_0 \succeq S(y_0, x_0) = y_1$, our claim is satisfied for $n = 0$.

Suppose now that (3) holds for some $n \in \mathbb{N}_0$. Then, since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$, and by the mixed monotone property of S, we get

$$x_{n+1} = S(x_n, y_n) \preceq S(x_{n+1}, y_n) \preceq S(x_{n+1}, y_{n+1}) = x_{n+2}$$

and

$$y_{n+1} = S(y_n, x_n) \succeq S(y_{n+1}, x_n) \succeq S(y_{n+1}, x_{n+1}) = y_{n+2}.$$

This proves our claim. On the other hand, by (a), we have

$$\alpha\left((S(x_0, y_0), S(y_0, x_0)), (x_0, y_0)\right) = \alpha_{x_1 y_1 x_0 y_0} \geq 1.$$

Since S is α-admissible

$$\alpha\left(S(x_1, y_1), S(y_1, x_1)), S(x_0, y_0), S(y_0, x_0)\right) = \alpha_{x_2 y_2 x_1 y_1} \geq 1.$$

By induction, we obtain

$$\alpha_{x_{n+1} y_{n+1} x_n y_n} \geq 1, \quad \text{for all } n \in \mathbb{N}_0. \tag{4}$$

Again, since $\alpha\left((y_0, x_0), S(y_0, x_0), S(x_0, y_0)\right) = \alpha_{y_0 x_0 y_1 x_1} \geq 1$, by α-admissibility of S, we deduce

$$\alpha\left((S(y_0, x_0), S(x_0, y_0)), (S(y_1, x_1), S(x_1, y_1))\right) = \alpha_{y_1 x_1 y_2 x_2} \geq 1.$$

Continuing this process, we have

$$\alpha_{y_n x_n y_{n+1} x_{n+1}} \geq 1, \quad \text{for all } n \in \mathbb{N}_0. \tag{5}$$

If for some n, $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then $S(x_n, y_n) = x_n$ and $S(y_n, x_n) = y_n$, that is, S has a coupled fixed point. We now assume that $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \in \mathbb{N}_0$. Since $(x_n, y_n) \preceq (x_{n+1}, y_{n+1})$, by (2) and (4), we deduce

$$\begin{aligned} H(1, \varphi(d(x_{n+2}, x_{n+1}))) &= H(1, \varphi(d(S(x_{n+1}, y_{n+1}), S(x_n, y_n)))) \\ &\leq H\left(\alpha_{x_{n+1} y_{n+1} x_n y_n}, \varphi(d(S(x_{n+1}, y_{n+1}), S(x_n, y_n)))\right) \\ &\leq \mathcal{F}\left(1, \frac{1}{2}\psi(d(x_{n+1}, x_n) + d(y_{n+1}, y_n))\right), \end{aligned}$$

implies that

$$\varphi(d(x_{n+2}, x_{n+1})) \leq \frac{1}{2}\psi(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)). \tag{6}$$

Similarly, since $(y_{n+1}, x_{n+1}) \preceq (y_n, x_n)$, from (2) and (5), we have

$$\begin{aligned} H(1, \varphi(d(y_{n+1}, y_{n+2}))) &= H(1, \varphi(d(S(y_n, x_n), S(y_{n+1}, x_{n+1})))) \\ &\leq H\left(\alpha_{y_n x_n y_{n+1} x_{n+1}}, \varphi(d(S(y_n, x_n), S(y_{n+1}, x_{n+1})))\right) \\ &\leq \mathcal{F}\left(1, \frac{1}{2}\psi(d(y_n, y_{n+1}) + d(x_n, x_{n+1}))\right), \end{aligned}$$

and so,

$$\varphi\left(d\left(y_{n+1}, y_{n+2}\right)\right) \leq \frac{1}{2}\psi\left(d\left(y_n, y_{n+1}\right) + d\left(x_n, x_{n+1}\right)\right). \tag{7}$$

From (6) and (7), we obtain

$$\varphi\left(d\left(x_{n+1}, x_{n+2}\right)\right) + \varphi\left(d\left(y_{n+1,}y_{n+2}\right)\right) \leq \psi(d(x_n, x_{n+1}) + d(y_n, y_{n+1})). \tag{8}$$

By property $(iii)_\varphi$, we get

$$\varphi(d\left(x_{n+1}, x_{n+2}\right) + d\left(y_{n+1,}y_{n+2}\right)) \leq \psi(d(x_n, x_{n+1}) + d(y_n, y_{n+1})). \tag{9}$$

Using the properties of φ and ψ, we deduce

$$d\left(x_{n+1}, x_{n+2}\right) + d\left(y_{n+1,}y_{n+2}\right) \leq d(x_n, x_{n+1}) + d(y_n, y_{n+1}).$$

Set $r_n := d(x_n, x_{n+1}) + d(y_n, y_{n+1})$, then the sequence $\{r_n\}$ is decreasing. Therefore, there is some $r \geq 0$ such that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \left[d(x_n, x_{n+1}) + d(y_n, y_{n+1})\right] = r. \tag{10}$$

Letting $n \to \infty$ in (9), we have that $\varphi(r) \leq \psi(r)$ and so $r = 0$, that is,

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \left[d(x_n, x_{n+1}) + d(y_n, y_{n+1})\right] = 0. \tag{11}$$

We now prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{x_n\}$ or $\{y_n\}$ is not Cauchy sequence. Then, there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{2m_k}\}$, $\{x_{2n_k}\}$ of $\{x_{2n}\}$ and $\{y_{2m_k}\}$, $\{y_{2n_k}\}$ of $\{y_{2n}\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and

$$d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \geq \varepsilon \text{ and } d(x_{n_k-1}, x_{m_k}) + d(y_{n_k-1}, y_{m_k}) < \varepsilon. \tag{12}$$

Using (12) and the triangle inequality, we have

$$\begin{aligned}
\varepsilon &\leq \delta_k := d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \\
&\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) + d(y_{n_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{m_k}) \\
&\leq d(x_{n_k}, x_{n_k-1}) + d(y_{n_k}, y_{n_k-1}) + \varepsilon.
\end{aligned}$$

Taking $k \to \infty$ in the above inequality and using (11), we get

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \left[d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k})\right] = \varepsilon. \tag{13}$$

Again by the triangle inequality, we obtain

$$\begin{aligned}
\delta_k &= d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \\
&\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) \\
&\quad + d(y_{n_k}, y_{n_k+1}) + d(y_{n_k+1}, y_{m_{k+1}}) + d(y_{m_k+1}, y_{m_k}) \\
&= r_{n_k} + r_{m_k} + d(x_{n_k+1}, x_{m_k+1}) + d(y_{n_k+1}, y_{m_k+1}).
\end{aligned}$$

Using the property of φ, we have

$$\begin{aligned}
\varphi(\delta_k) &\leq \varphi\left(r_{n_k} + r_{m_k} + d(x_{n_k+1}, x_{m_k+1}) + d(y_{n_k+1}, y_{m_k+1})\right) \\
&\leq \varphi\left(r_{n_k} + r_{m_k}\right) + \varphi\left(d(x_{n_k+1}, x_{m_k+1})\right) + \varphi\left(d(y_{n_k+1}, y_{m_k+1})\right).
\end{aligned} \tag{14}$$

Since $n_k > m_k$, by (3) we deduce $(x_{m_k}, y_{m_k}) \preceq (x_{n_k}, y_{n_k})$. Then by (2) and (4)

$$\begin{aligned} H(1, \varphi(d(x_{n_k+1}, x_{m_k+1}))) &= H(1, \varphi(d(S(x_{n_k}, y_{n_k}), S(x_{m_k}, y_{m_k})))) \\ &\leq H\left(\alpha_{x_{n_k} y_{n_k} x_{m_k} y_{m_k}}, \varphi(d(S(x_{n_k}, y_{n_k}), S(x_{m_k}, y_{m_k})))\right) \\ &\leq \mathcal{F}\left(1, \frac{1}{2}\psi(d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}))\right) \\ &= \mathcal{F}\left(1, \frac{1}{2}\psi(\delta_k)\right), \end{aligned}$$

which implies

$$\varphi(d(x_{n_k+1}, x_{m_k+1})) \leq \frac{1}{2}\psi(\delta_k). \tag{15}$$

Similarly since $(y_{n_k}, x_{n_k}) \preceq (y_{m_k}, x_{m_k})$ and using (2) and (5)

$$\begin{aligned} H(1, \varphi(d(y_{m_k+1}, y_{n_k+1}))) &= H(1, \varphi(d(S(y_{m_k}, x_{m_k}), S(y_{n_k}, x_{n_k})))) \\ &\leq H\left(\alpha_{y_{m_k} x_{m_k} y_{n_k}, x_{n_k}}, \varphi(d(S(y_{m_k}, x_{m_k}), S(y_{n_k}, x_{n_k})))\right) \\ &\leq \mathcal{F}\left(1, \frac{1}{2}\psi(d(y_{m_k}, y_{n_k}) + d(x_{m_k}, x_{n_k}))\right) \\ &= \mathcal{F}\left(1, \frac{1}{2}\psi(\delta_k)\right), \end{aligned}$$

and so

$$\varphi(d(y_{m_k+1}, y_{n_k+1})) \leq \frac{1}{2}\psi(\delta_k). \tag{16}$$

Letting (15) and (16) in (14), we obtain

$$\varphi(\delta_k) \leq \varphi(r_{n_k} + r_{m_k}) + \psi(\delta_k).$$

Taking $k \to \infty$ in the above inequality and using (11) and (13), we get

$$\varphi(\varepsilon) \leq \varphi(0) + \psi(\varepsilon) = \psi(\varepsilon).$$

From the properties of φ and ψ, we get $\varepsilon = 0$, which is a contradiction. This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

Suppose now assumption (i) holds. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} S(x_n, y_n) = S\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = S(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} S(y_n, x_n) = S\left(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\right) = S(y, x).$$

Therefore $x = S(x, y)$ and $y = S(y, x)$.

Suppose that the assumption (ii) is satisfied. Since $\{x_n\}$ is a nondecreasing sequence that converges to x, we have that $x_n \preceq x$ for all n. Similarly, $y_n \succeq y$ for all n, that is, $(x_n, y_n) \preceq (x, y)$. Also, by (4) and our assumption, we deduce $\alpha_{xyx_ny_n} \geq 1$ and $\alpha_{y_nx_nyx} \geq 1$.

From (2),

$$\begin{aligned} H(1, \varphi(d(S(x,y), x_{n+1}))) &= H(1, \varphi(d(S(x,y), S(x_n, y_n)))) \\ &\leq H(\alpha_{xyx_ny_n}, \varphi(d(S(x,y), S(x_n, y_n)))) \\ &\leq \mathcal{F}\left(1, \frac{1}{2}\psi(d(x, x_n) + d(y, y_n))\right), \end{aligned}$$

and so,

$$\varphi(d(S(x,y), x_{n+1})) \leq \frac{1}{2}\psi(d(x_n, x) + d(y_n, y)).$$

Letting $n \to \infty$ in the last inequality and using the property of φ, we have

$$\varphi(d(S(x,y), x)) \leq \frac{1}{2}\psi(0) = 0,$$

which implies $\varphi(d(S(x,y), x)) = 0$. Thus $d(S(x,y), x) = 0$ or equivalently, $x = S(x,y)$. Similarly, one can easily show that $y = S(y,x)$. This completes the proof. □

For the uniqueness of the coupled fixed point in Theorem 2.3, we consider the following hypothesis:

(K1) for arbitrary points $(x,y), (u,v) \in X^2$ there exists the pair $(z,t) \in X^2$ which is comparable with both (x,y) and (u,v) such that

$$\alpha_{ztxy} \geq 1, \quad \alpha_{yxtz} \geq 1, \quad \alpha_{ztuv} \geq 1 \quad \text{and} \quad \alpha_{vutz} \geq 1.$$

Theorem 2.4. *Adding condition (K1) to the hypotheses of Theorem 2.3, we obtain uniqueness of the coupled fixed point of S.*

Proof. From Theorem 2.3, the set of coupled fixed points of S is non-empty. Suppose (x,y) and (u,v) are coupled fixed points of S. We shall show that $x = u$ and $y = v$.

By condition (K1), there exists the pair $(z,t) \in X^2$ that is comparable to (x,y) and (u,v), and we also have $\alpha_{ztxy} \geq 1$, $\alpha_{yxtz} \geq 1$, $\alpha_{ztuv} \geq 1$ and $\alpha_{vutz} \geq 1$.

We define sequences $\{z_n\}$ and $\{t_n\}$ as follows

$$z_0 = z, \;\; t_0 = t, \;\; z_{n+1} = S(z_n, t_n) \;\text{ and }\; t_{n+1} = S(t_n, z_n) \;\text{ for all } n.$$

Since (z,t) is comparable with (x,y), we may assume that $(x,y) \preceq (z,t) = (z_0, t_0)$. By using the mathematical induction, it is easy to prove that $(x,y) \preceq (z_n, t_n)$ and $\alpha_{z_nt_nxy} \geq 1$ for all n. Then, by (2), we get

$$\begin{aligned} H(1, \varphi(d(z_{n+1}, x))) &= H(1, \varphi(d(S(z_n, t_n), S(x,y)))) \\ &\leq H(\alpha_{z_nt_nxy}, \varphi(d(S(z_n, t_n), S(x,y)))) \\ &\leq \mathcal{F}(1, \frac{1}{2}\psi(d(z_n, x) + d(t_n, y))), \end{aligned}$$

implies

$$\varphi(d(z_{n+1}, x)) \leq \frac{1}{2}\psi(d(x, z_n) + d(y, t_n)). \tag{17}$$

Similarly, we have that $(t_n, z_n) \preceq (y,x)$ and $\alpha_{yxt_nz_n} \geq 1$. Again, from (2)

$$\begin{aligned} H(1, \varphi(d(y, t_{n+1}))) &= H(1, \varphi(d(S(y,x), S(t_n, z_n)))) \\ &\leq H(\alpha_{yxt_nz_n}, \varphi(d(S(y,x), S(t_n, z_n)))) \\ &\leq \mathcal{F}(1, \frac{1}{2}\psi(d(y, t_n) + d(x, z_n))), \end{aligned}$$

and so

$$\varphi\left(d(y, t_{n+1})\right) \le \frac{1}{2}\psi(d(y, t_n) + d(x, z_n)). \tag{18}$$

From (17), (18) and the property of φ, we get

$$\begin{aligned} \varphi\left(d(x, z_{n+1}) + d(y, t_{n+1})\right) &\le \varphi\left(d(x, z_{n+1})\right) + \varphi\left(d(y, t_{n+1})\right) \\ &\le \psi(d(x, z_n) + d(y, t_n)). \end{aligned} \tag{19}$$

This gives us that $\{d(x, z_n) + d(y, t_n)\}$ is a nonnegative decreasing sequence, and consequently, there exists $\gamma \ge 0$ such that

$$\lim_{n\to\infty} [d(x, z_n) + d(y, t_n)] = \gamma. \tag{20}$$

Suppose that $\gamma > 0$. Letting $n \to \infty$ in (19), we obtain $\varphi(\gamma) \le \psi(\gamma)$ which implies that $\gamma = 0$. It follows

$$\lim_{n\to\infty} d(x, z_n) = \lim_{n\to\infty} d(y, t_n) = 0.$$

Similarly, one can show

$$\lim_{n\to\infty} d(u, z_n) = \lim_{n\to\infty} d(v, t_n) = 0.$$

From the triangle inequality, we have

$$\begin{aligned} d(x, u) &\le d(x, z_n) + d(z_n, u), \\ d(y, v) &\le d(y, t_n) + d(t_n, v). \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$d(x, u) = d(y, v) = 0$$

and so, $x = u$ and $y = v$. □

We now give an example in order to demonstrate the validity of offered results.

Example 2.5. *Let $X = \mathbb{R}$ with the usual metric and partial order be defined by $x \preceq y \Leftrightarrow x \le y$. Then $(X, \preceq, d)$ is regular and complete ordered metric space. Let $S : X^2 \to X$ be given by*

$$S(x, y) = \begin{cases} \frac{x-y}{8}, & \text{if } x \ge y \\ 0, & \text{otherwise.} \end{cases}$$

Clearly S is mixed monotone. Define $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ by

$$\alpha\left((x, y), (u, v)\right) = \begin{cases} 2, & \text{if } x \ge u,\ y \le v \\ \frac{1}{8}, & \text{otherwise.} \end{cases}$$

Also, we define the mappings $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t$, $\psi(t) = \frac{t}{2}$ and H: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}$: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

$$H(s, t) = \begin{cases} st, & s \ge 1 \\ s, & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{F}(s, t) = st.$$

Then, the pair $(\mathcal{F}, H)$ is an upper class of type I. We now prove that S is an α-$(\mathcal{F}, H)$-(φ, ψ) contraction. If $\alpha((x,y),(u,v)) \leq 1$, the result is clear. Supposse $\alpha((x,y),(u,v)) \geq 1$. Then

$$\begin{aligned} H(\alpha((x,y),(u,v)), \varphi(d(S(x,y), S(u,v)))) &= \alpha((x,y),(u,v))\varphi(d(S(x,y), S(u,v))) \\ &= 2\left|\frac{x-y}{8} - \frac{u-v}{8}\right| \\ &\leq \frac{|x-u| + |y-v|}{4} \\ &= \mathcal{F}\left(1, \frac{1}{2}\psi(d(x,u) + d(y,v))\right), \end{aligned}$$

for all $(u,v) \leq (x,y)$. It can be easily proved that all other conditions of Theorems 2.3 and 2.4 is satisfied. Therefore, S has a unique coupled fixed point which is $(0,0)$.

Remark 2.6. *Assume that all hypotheses of Theorems 2.3 and 2.4 are satisfied. If we replace the inequality (2) with following contractions, then $S : X^2 \to X$ has separately a unique coupled fixed point for each contraction.*

(i) There exist $l > 1$, $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $(x,y),(u,v) \in X^2$ with $(u,v) \leq (x,y)$

$$(\varphi(d(S(x,y), S(u,v)) + l)^{\alpha((x,y),(u,v))} \leq \frac{1}{2}\psi(d(x,u) + d(y,v)) + l.$$

(ii) There exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $(x,y),(u,v) \in X^2$ with $(u,v) \leq (x,y)$

$$\alpha((x,y),(u,v))\varphi(d(S(x,y), S(u,v)) \leq \frac{1}{2}\psi(d(x,u) + d(y,v)).$$

(iii) There exist $\psi \in \Psi$ such that for all $(x,y),(u,v) \in X^2$ with $(u,v) \leq (x,y)$

$$\alpha((x,y),(u,v))d(S(x,y), S(u,v) \leq \frac{1}{2}\psi(d(x,u) + d(y,v)).$$

(iv) There exist $k \in [0,1)$ such that for all $(x,y),(u,v) \in X^2$ with $(u,v) \leq (x,y)$

$$d(S(x,y), S(u,v) \leq \frac{k}{2}(d(x,u) + d(y,v)).$$

Note that all inequalities in Remark 2.6 are obtained from the inequality (2).

Definition 2.7. *Let $S : X^2 \to X$ and $\mu : X^2 \times X^2 \to \mathbb{R}^+$ be given mappings. We say that S is μ-subadmissible if*

$$\mu((x,y),(u,v)) \leq 1 \text{ implies } \mu((S(x,y), S(y,x)),(S(u,v), S(v,u))) \leq 1,$$

for all $(x,y),(u,v) \in X^2$.

For simplify, we will use $\mu_{xyuv} = \mu((x,y),(u,v))$.

Definition 2.8. *Let $(X, \leq, d)$ be an ordered metric space, $S : X^2 \to X$ and $\mu : X^2 \times X^2 \to \mathbb{R}^+$ be given mappings. We say that S is a μ-$(\mathcal{F}, H)$-(φ, ψ) contractive mapping if*

$$H(1, \varphi(d(S(x,y), S(u,v)))) \leq \mathcal{F}\left(\mu((x,y),(u,v)), \frac{1}{2}\psi(d(x,u) + d(y,v))\right), \tag{21}$$

for all $(x,y),(u,v) \in X^2$ with $(u,v) \leq (x,y)$ where $\varphi \in \Phi$, $\psi \in \Psi$ and the pair $(\mathcal{F}, H)$ is upper class of type I.

Another of our main results is the following.

Theorem 2.9. *Let $(X, \preceq, d)$ be an ordered complete metric space, $S : X^2 \to X$ be a μ-subadmissible and μ-$(\mathcal{F}, H)$-(φ, ψ) contractive mapping with mixed monotone property. Assume that following conditions hold:*

(*a*) *There exist $x_0, y_0 \in X$ such that*

$$\begin{aligned} \mu\left((S(x_0, y_0), S(y_0, x_0)), (x_0, y_0)\right) &\leq 1 \text{ and} \\ \mu\left((y_0, x_0), S(y_0, x_0), S(x_0, y_0)\right) &\leq 1, \end{aligned}$$

(*b*) (*i*) *S is continuous, or*
(*ii*) *$(X, \preceq, d)$ is regular and, if (x_n) and (y_n) are sequences in X such that*

$$\mu_{x_{n+1}y_{n+1}x_ny_n} \leq 1 \quad \text{and} \quad \mu_{y_nx_ny_{n+1}x_{n+1}} \leq 1, \quad \text{for all } n \in \mathbb{N}_0$$

and $x_n \to x$, $y_n \to y$ for all $x, y \in X$, then

$$\mu_{xyx_ny_n} \leq 1 \quad \text{and} \quad \mu_{y_nx_nyx} \leq 1.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq S(x_0, y_0)$ and $y_0 \succeq S(y_0, x_0)$, then S has a coupled fixed point.

Proof. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X as in Theorem 2.3. Following the similar procedures in Theorem 2.3, we can obtain

$$\mu_{x_{n+1}y_{n+1}x_ny_n} \leq 1 \quad \text{and} \quad \mu_{y_nx_ny_{n+1}x_{n+1}} \leq 1, \quad \text{for all } n \in \mathbb{N}_0.$$

The rest of proof can be completed by following the analogous processes in Theorem 2.3. □

Let us give following condition for the uniqueness of the coupled fixed point in Theorem 2.9:

(K2) for arbitrary points $(x, y), (u, v) \in X^2$ there exists the pair $(z, t) \in X^2$ which is comparable with both (x, y) and (u, v) such that

$$\mu_{ztxy} \leq 1, \quad \mu_{yxtz} \leq 1, \quad \mu_{ztuv} \leq 1 \quad \text{and} \quad \mu_{vutz} \leq 1.$$

Theorem 2.10. *Adding condition (K2) to the hypotheses of Theorem 2.9, we obtain uniqueness of the coupled fixed point of S.*

Proof. Similar to the proof of Theorem 2.4. □

Example 2.11. *In the setting of Example 2.5, replace the mappings α, H and $\mathcal{F}$ by the following, besides retaining the rest:*

$$\mu\left((x, y), (u, v)\right) = \begin{cases} \frac{1}{2}, & \text{if } x \geq u,\ y \leq v \\ 8, & \text{otherwise} \end{cases}$$

and

$$H(s, t) = st \quad \text{and} \quad \mathcal{F}(s, t) \begin{cases} st, & s \leq 1 \\ s, & \text{otherwise.} \end{cases}$$

We now prove that S is a μ-$(\mathcal{F}, H)$-(φ, ψ) contraction. If $\mu((x, y), (u, v)) \geq 1$, the result is clear. Suppose $\mu((x, y), (u, v)) \leq 1$. Then

$$\begin{aligned} H\left(1, \varphi\left(d\left(S(x, y), S(u, v)\right)\right)\right) &= \varphi\left(d\left(S(x, y), S(u, v)\right)\right) \\ &= \left|\frac{x-y}{8} - \frac{u-v}{8}\right| \\ &\leq \frac{|x-u| + |y-v|}{8} \\ &= \mathcal{F}\left(\mu\left((x, y), (u, v)\right), \frac{1}{2}\psi\left(d(x, u) + d(y, v)\right)\right), \end{aligned}$$

for all $(u, v) \preceq (x, y)$. It can be easily proved that all other conditions of Theorems 2.9 and 2.10 is satisfied. Therefore, S has a unique coupled fixed point which is $(0, 0)$.

Remark 2.12. *Assume that all hypotheses of Theorems 2.9 and 2.10 are satisfied. If we replace the inequality (21) with following contractions, then $S : X^2 \to X$ has separately a unique coupled fixed point for each contraction.*

(i) There exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $(x, y), (u, v) \in X^2$ with $(u, v) \preceq (x, y)$

$$\varphi\left(d(S(x,y), S(u,v)\right) \leq \frac{1}{2}\mu((x,y),(u,v))\psi\left(d(x,u) + d(y,v)\right).$$

(ii) There exist $\psi \in \Psi$ such that for all $(x, y), (u, v) \in X^2$ with $(u, v) \preceq (x, y)$

$$d(S(x,y), S(u,v) \leq \frac{1}{2}\mu((x,y),(u,v))\psi\left(d(x,u) + d(y,v)\right).$$

(iii) There exist $k \in [0, 1)$ such that for all $(x, y), (u, v) \in X^2$ with $(u, v) \preceq (x, y)$

$$d(S(x,y), S(u,v) \leq \frac{k}{2}\left(d(x,u) + d(y,v)\right).$$

3. Application to Integral Equations

In this section, we investigate the existence of solution to a Fredholm nonlinear integral equation, as an application to the coupled fixed point theorem proved in previous section.

Consider the following integral equation:

$$x(r) = \int_a^b (H_1(r,s) + H_2(r,s))(f(s,x(s)) + g(s,x(s)))ds + k(r), \tag{22}$$

for all $r \in I = [a, b]$.

Define now the mapping $S : X^2 \to X$ by

$$\begin{aligned} S(x,y)(r) &= \int_a^b H_1(r,s)\left(f\left(s,x(s)\right) + g\left(s,y(s)\right)\right)ds \\ &\quad + \int_a^b H_2(r,s)(f\left(s,y(s)\right) + g\left(s,x(s)\right))ds + k(r), \quad \forall r \in I. \end{aligned}$$

Assume that the following conditions are satisfied:

(A) $H_1(r, s) \geq 0$ and $H_2(r, s) \leq 0$ for all $r, s \in I$;

(B) there exist $\xi : X^2 \times X^2 \to \mathbb{R}$ where $X := C(I, \mathbb{R})$ such that if $\xi((x, y), (u, v)) \geq 0$ (shortly, $\xi_{xyuv} \geq 0$) for all $(x, y), (u, v) \in X^2$ with $(u, v) \preceq (x, y)$, then for every $\lambda, \mu > 0$,

$$0 \leq f(r,x) - f(r,u) \leq \lambda \ln\left(|x-u| + 1\right)$$

and

$$-\mu \ln\left(\left|y - v\right| + 1\right) \leq g(r,y) - g(r,v) \leq 0;$$

(C) if, for all $(x, y), (u, v) \in X^2$

$$\xi\left((x,y),(u,v)\right) \geq 0 \text{ implies } \xi\left((S(x,y), S(y,x)), (S(u,v), S(v,u))\right) \geq 0;$$

(D) there exist $x_0, y_0 \in X$ such that

$$\begin{aligned} \xi\left((S(x_0,y_0), S(y_0,x_0)), (x_0,y_0)\right) &\geq 0 \text{ and} \\ \xi\left((y_0,x_0), S(y_0,x_0), S(x_0,y_0)\right) &\geq 0; \end{aligned}$$

(E) if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\xi_{x_{n+1}y_{n+1}x_ny_n} \geq 0 \quad \text{and} \quad \xi_{y_nx_ny_{n+1}x_{n+1}} \geq 0, \quad \text{for all } n \in \mathbb{N}_0$$

and $x_n \to x$, $y_n \to y$ for all $x, y \in X$, then

$$\xi_{xyx_ny_n} \geq 0 \quad \text{and} \quad \xi_{y_nx_nyx} \geq 0;$$

(F) $4 \cdot \max(\lambda, \mu)\|H_1 - H_2\|_\infty \leq 1$, where

$$\|H_1 - H_2\|_\infty = sup\{(H_1(r,s) - H_2(r,s)) : r, s \in I\}.$$

Definition 3.1 ([9]). *A pair $(p, q) \in X^2$ is called a coupled lower and upper solution of Eq. (22) if, for all $r \in I$, $p(r) \leq q(r)$ and*

$$\begin{aligned} p(r) \quad \leq \quad & \int_a^b H_1(r,s)(f(s,p(s)) + g(s,q(s)))ds \\ & + \int_a^b H_2(r,s)(f(s,q(s)) + g(s,p(s)))ds + k(r) \end{aligned}$$

and

$$\begin{aligned} q(r) \quad \geq \quad & \int_a^b H_1(r,s)(f(s,q(s)) + g(s,p(s)))ds \\ & + \int_a^b H_2(r,s)(f(s,p(s)) + g(s,q(s)))ds + k(r). \end{aligned}$$

Theorem 3.2. *Consider the integral equation (22) with*

$$H_1, H_2 \in C(I \times I, \mathbb{R}) \ \text{ and } \ f, g \in C(I \times \mathbb{R}, \mathbb{R}), \ \text{ and } \ k \in C(I, \mathbb{R}).$$

With the conditions (A)-(F), if Eq. (22) has a coupled lower and upper solution then it has a solution in X.

Proof. The function $d : X^2 \to \mathbb{R}^+$ defined by

$$d(x, y) = \sup_{r \in I} |x(r) - y(r)|, \quad \forall x, y \in X.$$

Then it is obvious that $(X, \leq, d)$ is a complete ordered metric space if, for $x, y \in X$

$$x \leq y \Leftrightarrow x(r) \leq y(r), \quad \forall r \in I.$$

Suppose that $\{c_n\}$ is a monotone nondecreasing in X that converges to $c \in X$ and $\{d_n\}$ is a monotone nonincreasing sequence in X that converges to $d \in X$, then $c_n \leq c$ and $d_n \geq d$ for all n and hence $(X, \leq, d)$ is regular. Also, X^2 is a partially ordered set if we define the following order relation in X^2 with

$$(x, y) \leq (u, v) \Leftrightarrow x(r) \leq u(r) \quad \text{and} \quad y(r) \geq v(r), \quad \forall r \in I.$$

It is not difficult to prove, like in [9], that S has the mixed monotone property.

Let $(x, y), (u, v) \in X^2$ with $(u, v) \leq (x, y)$ and $\xi_{xyuv} \geq 0$. Since S has the mixed monotone property, we have $S(u, v) \leq S(x, y)$ and

$$\begin{aligned} & d(S(x,y), S(u,v)) \\ & \quad = \sup_{r \in I} |S(x,y)(r) - S(u,v)(r)| \end{aligned}$$

$$= \sup_{r\in I} (S(x,y)(r) - S(u,v)(r))$$
$$= \sup_{r\in I}\Big(\int_a^b H_1(r,s)[(f(s,x(s)) - f(s,u(s))) - (g(s,v(s)) - g(s,y(s)))]ds$$
$$- \int_a^b H_2(r,s)[(f(s,v(s)) - f(s,y(s))) - (g(s,x(s)) - g(s,u(s)))]ds\Big)$$
$$\leq \sup_{r\in I}\Big(\int_a^b H_1(r,s)[\lambda \ln(|x(s)-u(s)|+1)) + \mu \ln\left(\left|y(s)-v(s)\right|+1\right))]ds$$
$$+ \int_a^b (-H_2(r,s))\,[\lambda \ln\left(\left|v(s)-y(s)\right|+1\right)) + \mu \ln(|x(s)-u(s)|+1))]ds\Big)$$
$$\leq \max(\lambda,\mu)\sup_{r\in I}\Big(\int_a^b (H_1(r,s) - H_2(r,s)) \ln(|x(s)-u(s)|+1)\,ds$$
$$+ \int_a^b (H_1(r,s) - H_2(r,s)) \ln\left(\left|y(s)-v(s)\right|+1\right) ds\Big). \tag{23}$$

Defining

$$(I) = \int_a^b (H_1(r,s) - H_2(r,s)) \ln(|x(s)-u(s)|+1)\,ds,$$

and

$$(II) = \int_a^b (H_1(r,s) - H_2(r,s)) \ln\left(\left|y(s)-v(s)\right|+1\right) ds,$$

and using the Cauchy–Schwartz inequality in (I) we obtain

$$\begin{aligned}(I) &\leq \left(\int_a^b (H_1(r,s)-H_2(r,s))^2 ds\right)^{\frac{1}{2}} \cdot \left(\int_a^b (\ln(|x(s)-u(s)|+1))^2 ds\right)^{\frac{1}{2}} \\ &\leq \|H_1 - H_2\|_\infty \cdot (\ln(|x-u|+1)) = \|H_1 - H_2\|_\infty \cdot (\ln(d(x,u)+1)).\end{aligned} \tag{24}$$

Similarly, we can obtain the following estimate for (II) :

$$(II) \leq \|H_1 - H_2\|_\infty \cdot (\ln(d(y,v)+1)). \tag{25}$$

By (23)-(25) and assumption (G), we get

$$\begin{aligned} d(S(x,y),S(u,v)) &\leq \max(\lambda,\mu)\,\|H_1 - H_2\|_\infty\,[\ln(d(x,u)+1) + \ln(d(y,v)+1)] \\ &\leq \max(\lambda,\mu)\,\|H_1 - H_2\|_\infty\,[\ln(d(x,u)+d(y,v)+1) \\ &\quad + \ln(d(y,v)+d(x,u)+1)] \\ &= 2\max(\lambda,\mu)\,\|H_1 - H_2\|_\infty\,[\ln(d(x,u)+d(y,v)+1)] \\ &\leq \frac{1}{2}\ln(d(x,u)+d(y,v)+1). \end{aligned} \tag{26}$$

Now, define $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ by

$$\alpha((x,y),(u,v)) = \begin{cases} 1 & \text{if } \xi((x,y),(u,v)) \geq 0 \text{ where } x,y,u,v \in X \\ 0 & \text{otherwise.} \end{cases}$$

Also, define $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t$ and $\psi(t) = \ln(1+t)$. Therefore, using (26), we obtain

$$\alpha((x,y),(u,v))\,\varphi(d(S(x,y),S(u,v))) \leq \frac{1}{2}\psi(d(x,u)+d(y,v)).$$

It easily shows that all the hypotheses of Remark 2.6 (ii) are satisfied. Therefore S has a coupled fixed point, that is, integral equation (22) has a solution. □

We now consider following assumptions:

(B') there exist $\zeta : X^2 \times X^2 \to \mathbb{R}$ such that if $\zeta((x,y),(u,v)) \leq 0$ (shortly, $\zeta_{xyuv} \leq 0$) for all $(x,y),(u,v) \in X^2$ with $(u,v) \preceq (x,y)$, then for every $\lambda, \mu > 0$,

$$0 \leq f(r,x) - f(r,u) \leq \lambda \ln(|x-u|+1)$$

and

$$-\mu \ln\left(\left|y-v\right|+1\right) \leq g(r,y) - g(r,v) \leq 0;$$

(C') if, for all $(x,y),(u,v) \in X^2$

$$\zeta((x,y),(u,v)) \leq 0 \text{ implies } \zeta((S(x,y),S(y,x)),(S(u,v),S(v,u))) \leq 0;$$

(D') there exist $x_0, y_0 \in X$ such that

$$\begin{aligned} \zeta((S(x_0,y_0),S(y_0,x_0)),(x_0,y_0)) &\leq 0 \text{ and} \\ \zeta((y_0,x_0),S(y_0,x_0),S(x_0,y_0)) &\leq 0; \end{aligned}$$

(E') if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\zeta_{x_{n+1}y_{n+1}x_ny_n} \leq 0 \quad \text{and} \quad \zeta_{y_nx_ny_{n+1}x_{n+1}} \leq 0, \quad \text{for all } n \in \mathbb{N}_0$$

and $x_n \to x$, $y_n \to y$ for all $x, y \in X$, then

$$\zeta_{xyx_ny_n} \leq 0 \quad \text{and} \quad \zeta_{y_nx_nyx} \leq 0.$$

Theorem 3.3. *With the conditions (A), (B')-(E'), and (F), if Eq. (22) has a coupled lower and upper solution then, it has a solution in X.*

Proof. Following the same lines in Theorem 3.2 and using Remark 2.12 (i), we obtain the desired result. □

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