



Existence and Structure of the Common Fixed Points Based on TVS

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Abstract. In this paper, we investigate the common fixed point property for commutative nonexpansive mappings on τ -compact convex sets in normed and Banach spaces, where τ is a Hausdorff topological vector space topology that is weaker than the norm topology. As a consequence of our main results, we obtain that the set of common fixed points of any commutative family of nonexpansive self-mappings of a nonempty *clm*-compact (resp. weak* compact) convex subset C of $L_1(\mu)$ with a σ -finite μ (resp. the James space J_0) is a nonempty nonexpansive retract of C .

1. Introduction

Let E be a normed space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. E is said to have the fixed point property with respect to τ (τ -fpp) if the following holds: For each nonempty, norm bounded, τ -compact, convex subset C of E , every nonexpansive mapping $T : C \rightarrow C$ (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$) has a fixed point. We say that a nonempty closed and convex subset C of E has the fpp if every nonexpansive mapping $T : C \rightarrow C$ has a fixed point, and also C is said to have the τ -fpp, if each nonempty, norm bounded, τ -compact, convex subset of C has the fpp. It is not clear that a set having the fpp must be bounded. It is known that the norm boundedness assumption for τ -compact, convex sets is redundant (see, for example, [12]). An standard example of such a τ is where τ is the weak topology on E . Another example is where E is a dual Banach space and τ is the weak* topology. Yet another example is when E is $L_1(\mu)$ and τ is the topology *clm* of convergence locally in measure (see, e.g., [18]).

Determining conditions on a Banach space E so that it has the fixed point property has been of considerable interest for many years. Kirk [13] proved that a weakly compact convex subset of a Banach space with weak normal structure has the fpp. It is known that every compact convex subset of a Banach space has normal structure. Kirk's proof of his result also yields that a weak* compact convex subset of a Banach space with weak* normal structure has the fixed point property (see [19]). The condition above that C has normal structure can not be dropped. In fact, Alspach [2] showed that $L_1[0, 1]$ fails the weak fpp.

We say that E has the (common) τ -fpp for commutative semigroups if whenever $\mathcal{S} = \{T_s : s \in S\}$ is a commutative semigroup of nonexpansive self-mappings on a nonempty, τ -compact, convex subset of E , then the common fixed point set of \mathcal{S} , $Fix(\mathcal{S})$, is nonempty. Bruck [4] showed that a Banach space E having the weak fpp has the weak fpp for commutative semigroups. We refer to [23] for a simple proof

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to a more general version of Bruck's result. For a dual Banach space E satisfying the weak* fpp, it is still unknown whether E has the weak* fpp for commutative semigroups. Very recently, Borzdynski and Wisnicki [3] proved that if \mathcal{S} is a commuting family of weak* continuous nonexpansive mappings acting on a weak* compact convex subset C of the dual Banach space E , then the set of common fixed points of \mathcal{S} is a nonempty nonexpansive retract of C . This partially answers a long-standing open problem posed by Lau in [15] (see also [17]). Examples of Banach spaces with the weak* fpp for commutative semigroups include ℓ_1 , trace class operators on a Hilbert space, Hardy space H^1 and the Fourier algebra of a compact group (see [16, 19–22]).

In this paper, by using the retraction tool, we study the common fixed point property for commutative nonexpansive mappings on τ -compact convex sets in E , where τ is a Hausdorff topological vector space topology that is weaker than the norm topology. In Section 2, we shall prove the following: Let E be a Banach space, τ a Hausdorff topological vector space topology on E that is weaker than the norm topology and the norm of E is lsc with respect to τ , and C be a nonempty, τ -compact, separable, convex subset of E which has the τ -fpp. Then any commutative family of nonexpansive self-mappings of C has a common fixed point and the set of common fixed points is a nonexpansive retract of C . In Section 4, we obtain the same result by replacing the separability with the τ -Opial condition. As a consequence, we shall show that the set of common fixed points of any commutative family of nonexpansive self-mappings of a nonempty clm -compact (resp. weak* compact) convex subset C of $L_1(\mu)$ with a σ -finite μ (resp. the James space J_0) is a nonempty nonexpansive retract of C .

2. The τ -fpp for Commutative Mappings on Separable Subsets

Recall some general concepts and definitions. Let E be a normed space and C be a nonempty subset of E . A mapping T on C is said to be a retraction if $T^2 = T$. A subset F of C is called a nonexpansive retract of C if either $F = \emptyset$ or there exists a retraction of C onto F which is a nonexpansive mapping. Nonexpansive retract plays an important role in the study of the structure of fixed point sets of nonexpansive mappings. We refer the reader to [4–7] for more information concerning nonexpansive retracts.

Let E be a normed space, and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. The purpose of this section is to study the τ -fpp (which implies the weak* fpp) for commutative semigroups of nonexpansive mappings. In fact, we give some partial answers to the following question:

If a dual Banach space E has the τ -fpp, does E have the τ -fpp for commuting semigroups?

The following theorem is essential to get the main results.

Theorem 2.1. *Let E be a normed space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose that the norm of E is lsc with respect to τ . Let C be a nonempty, τ -compact, convex subset of E with the τ -fpp. If $\{T_1, \dots, T_n\}$ is a commutative family of nonexpansive mappings on C , then $\bigcap_{i=1}^n \text{Fix}(T_i)$ is a nonempty nonexpansive retract of C .*

Proof. First, we prove that for each nonexpansive mapping $T : C \rightarrow C$, $\text{Fix}(T)$ is a nonempty nonexpansive retract of C . To this purpose, consider C^C with the product topology induced by the topology τ on C . Then by Tychonoff's theorem C^C is compact. Now, consider a nonexpansive mapping $T : C \rightarrow C$ and define

$$\mathfrak{R} := \{S \in C^C : S \text{ is nonexpansive, } \text{Fix}(T) \subset \text{Fix}(S)\}.$$

We show that \mathfrak{R} is closed in C^C . Suppose that $\{U_\lambda : \lambda \in \Lambda\}$ is a net in \mathfrak{R} which converges to U in C^C . Then for $z \in \text{Fix}(T)$, $U_\lambda(z) = z$ so $U(z) = \tau - \lim_\lambda U_\lambda(z) = z$. By the lower semi-continuity of the norm with respect to τ , for any $x, y \in C$, $\|Ux - Uy\| \leq \liminf_\lambda \|U_\lambda x - U_\lambda(y)\| \leq \|x - y\|$. So we have shown that $U \in \mathfrak{R}$, hence that \mathfrak{R} is closed in C^C . Since C^C is compact, therefore \mathfrak{R} is compact (the topology on \mathfrak{R} is that of τ -pointwise convergence). Define a preorder \leq in \mathfrak{R} by $S \leq U$ if $\|Sx - Sy\| \leq \|Ux - Uy\|$ for all $x, y \in C$ and using the Bruck's method [7] we obtain a minimal element $R \in \mathfrak{R}$. Indeed, by considering Zorn's lemma it suffices to show every linearly ordered subset of \mathfrak{R} has a lower bound in \mathfrak{R} . If $\{U_\lambda\}$ is a linearly ordered subset of \mathfrak{R} by \leq , the family of sets $\{S \in \mathfrak{R} : S \leq U_\lambda\}$ is linearly ordered by inclusion. The proof that \mathfrak{R} is

closed in C^C can be repeated to show that these sets are closed in \mathfrak{K} , and hence compact. So there exists $U \in \bigcap_{\lambda} \{S \in \mathfrak{K} : S \leq U_{\lambda}\}$ with $U \leq U_{\lambda}$ for each λ . Now, we have shown the existence of a minimal element $P \in \mathfrak{K}$ in the following sense:

$$\begin{aligned} \text{if } S \in \mathfrak{K} \text{ and } \|S(x) - S(y)\| &\leq \|P(x) - P(y)\|, \forall x, y \in C, \\ \text{then } \|S(x) - S(y)\| &= \|P(x) - P(y)\|. \quad (*) \end{aligned}$$

We shall prove that $P(x) \in \text{Fix}(T)$ for all $x \in C$. For a given $x \in C$, consider the set $K = \{S(P(x)) : S \in \mathfrak{K}\}$. Then K is a nonempty τ -compact convex subset of C , because \mathfrak{K} is convex and compact. On the other hand, $TS \in \mathfrak{K}$, $\forall S \in \mathfrak{K}$. Therefore, we have $T(K) \subseteq K$, and then, by the τ -fpp, there exists $h \in \mathfrak{K}$ with $h(P(x)) \in \text{Fix}(T)$. Let $y = h(P(x))$. Then $P(y) = h(y) = y$, and by using the minimality of P , we have $\|P(x) - y\| = \|P(x) - P(y)\| = \|h(P(x)) - h(P(y))\| = \|h(P(x)) - y\| = 0$. So $P(x) = y \in \text{Fix}(T)$. Since this is so for each $x \in C$ and P belongs to \mathfrak{K} , it follows that $P^2 = P$. So, we have shown that $\text{Fix}(T)$ is a nonexpansive retract of C . Now, let $\{T_1, \dots, T_n\}$ be a commuting family of nonexpansive mappings on C . We prove that $\bigcap_{i=1}^n \text{Fix}(T_i)$ is a nonempty nonexpansive retract of C . The proof is by induction n . If $n = 1$, then $\text{Fix}(T_1)$ is a nonempty nonexpansive retract of C by the above discussion. Now suppose that $\bigcap_{j=1}^n \text{Fix}(T_j)$ is a nonempty nonexpansive retract of C and $R : C \rightarrow \bigcap_{j=1}^n \text{Fix}(T_j)$ a nonexpansive retraction. Then, it is easy to check that $\text{Fix}(T_{n+1}R) = \bigcap_{j=1}^{n+1} \text{Fix}(T_j)$ (see, e.g., [4] for the details). Another application of the first part of the proof implies that $\text{Fix}(T_{n+1}R) = \bigcap_{j=1}^{n+1} \text{Fix}(T_j)$ is a nonempty nonexpansive retract of C , which completes the induction. \square

We will also need the following lemma, due to Bruck [4], as an intermediary step.

Lemma 2.2. *If C is a bounded closed convex subset of a Banach space E and $\{F_n\}$ is a descending sequence of nonempty nonexpansive retracts of C , then $\bigcap_{n=1}^{\infty} F_n$ is the fixed point set of some nonexpansive $r : C \rightarrow C$.*

Theorem 2.3. *Let E be a Banach space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose the norm of E is lsc with respect to τ , and C be a nonempty, τ -compact, separable, convex subset of E which has the τ -fpp. Then any commutative family of nonexpansive self-mappings of C has a common fixed point and the set of common fixed points is a nonexpansive retract of C .*

Proof. Let $\mathcal{S} = \{T_i\}_{i \in I}$ be a commutative family of nonexpansive mappings on C , and let \mathcal{F} be the family of the finite intersections of fixed point sets of mappings in the commutative family \mathcal{S} . We have shown, in Theorem 2.1, that \mathcal{F} is a family of nonempty nonexpansive retracts of C , and \mathcal{F} is obviously directed by \supset . Now, since C is separable, there is a countable subfamily \mathcal{F}' of \mathcal{F} such that

$$\text{Fix}(\mathcal{S}) = \bigcap \{F : F \in \mathcal{F}\} = \bigcap \{F : F \in \mathcal{F}'\}.$$

Using the fact that \mathcal{F} is directed by \supset we can therefore find a descending sequence $\{F_n\}$ in \mathcal{F} with $\text{Fix}(\mathcal{S}) = \bigcap_n F_n$. But, by Lemma 2.2, $\text{Fix}(\mathcal{S}) = \bigcap_n F_n = \text{Fix}(r)$, for some nonexpansive $r : C \rightarrow C$. Since C has the τ -fpp, Theorem 2.1 implies $\text{Fix}(\mathcal{S})$ is a nonempty nonexpansive retract of C . This completes the proof. \square

An standard example of such a pair (E, τ) is where E is a dual Banach space and τ is the weak* topology. Thus, Theorem 2.3 yields the following result:

Theorem 2.4. *Let E be a dual Banach space and suppose C is a nonempty, weak* compact, separable, convex subset of E which has the weak*-fpp. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C .*

Example 2.5. *The James space J_0 , ℓ_1 and L_1 over a separable measure space are separable [1]. Moreover, it is well known that in ℓ_1 , and in the James space J_0 , a nonexpansive self-mapping of a weak* compact convex subset has a fixed point [11]. Thus, by Theorem 2.4, we deduce that for any nonempty, weak* compact, convex subset C of ℓ_1 , or the James space J_0 , the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C .*

For commutative sequences of nonexpansive self-mappings it is possible to say even more (cf. [24]).

Theorem 2.6. *Let E be a Banach space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose the norm of E is lsc with respect to τ , and C be a nonempty, τ -compact, convex subset of E which has the τ -fpp. Then the set of common fixed points of any commutative sequence of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C .*

Proof. Let $\{T_n\}$ be a commutative sequence of nonexpansive mappings on C . Then, by Theorem 2.1, for each natural number n , $F_n = \bigcap_{j=1}^n \text{Fix}(T_j)$ is a nonempty nonexpansive retract of C . Thus, applying Lemma 2.2, we deduce that $\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$ is the fixed point set of some nonexpansive mapping $r : C \rightarrow C$. Therefore, by Theorem 2.1, $\text{Fix}(r) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$ is a nonempty nonexpansive retract of C . \square

3. Fixed Point Property with Respect to clm -Topology in $L_1(\mu)$

In this section, we will use the topology of convergence in measure which we now recall for the convenience of the reader. Let (Ω, Σ, μ) be a positive σ -finite measure space and $L_0(\mu)$ be the set of all scalar-valued Σ -measurable functions on Ω . The topological vector space topology clm , of convergence locally in measure on $L_0(\mu)$, is generated by the following translation-invariant metric: Let $(A_n)_{n=1}^{\infty}$ be a $\widetilde{\Sigma}$ -partition of Ω , where $\widetilde{\Sigma} := \{A \in \Sigma : \mu(A) \in (0, \infty)\}$. Define d_0 by

$$d_0(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(A_n)} \int_{A_n} \frac{|f - g|}{1 + |f - g|} d\mu, \text{ for all } f, g \in L_0(\mu).$$

If $\mu(\Omega) < \infty$, then the simpler metric

$$d_0(f, g) = \int_{\Omega} \frac{|f - g|}{1 + |f - g|} d\mu, \text{ for all } f, g \in L_0(\mu),$$

generates the clm topology. In this case, we simply refer to clm as the topology of convergence in measure, denoted by cm . $L_0(\mu)$ is complete with respect to the above metric. For sequences, the clm -topology reduces, in a sense, to almost everywhere convergence. Indeed, any sequence in $L_0(\mu)$ that converges almost everywhere to $f \in L_0(\mu)$ must converge to f locally in measure. On the other hand, every clm -convergent sequence of scalar-valued measurable functions has a subsequence that converges almost everywhere to the same limit function. Note that when we discuss $L_1(\mu)$, clm or cm will denote the topologies introduced above, restricted to $L_1(\mu)$. Further, the $L_1(\mu)$ -norm is clm -lower semicontinuous. This follows from Fatou's lemma and the fact that clm is a metric topology. Thus, an example of the pair (E, τ) in Section 2 is where E is $L_1(\mu)$ and τ is the topology clm of convergence locally in measure.

We remark that one can show that in every $L_1(\mu)$, μ σ -finite, clm -compact sets must be norm separable. Besides, Lami Dozo and Turpin [14] showed that $L_1(\mu)$ has the fpp with respect to the topology clm of convergence locally in measure. Combining the above discussion with Theorem 2.3 yields the following result:

Theorem 3.1. *Let C be a nonempty, clm -compact, convex subset of $L_1(\mu)$, where μ is σ -finite. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C .*

In the particular case that μ is finite, Theorem 3.1 is a consequence of a result in [22] (see also [21]).

4. The τ -fpp for Commutative Mappings Under the τ -Opial Condition

In this section, we study the common fpp for commutative semigroups by considering conditions under which the fixed point sets of nonexpansive mappings are τ -closed. For our purposes it will be convenient to prove the following result:

Theorem 4.1. *Let E be a normed space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose that the norm of E is lsc with respect to τ . Let C be a nonempty, τ -compact, convex subset of E with the τ -fpp. Then any commutative family of nonexpansive self-mappings of C such that their fixed point sets are τ -closed has a common fixed point and the set of common fixed points is a nonexpansive retract of C .*

Proof. Let $\mathcal{S} = \{T_i\}_{i \in I}$ be a commutative family of nonexpansive mappings on C , and let \mathcal{F} be the family of the finite intersections of fixed point sets of mappings in the commutative family \mathcal{S} . Since, by assumption, the fixed point sets are τ -closed, Theorem 2.1 implies that \mathcal{F} is a family of nonempty τ -compact subsets of C that is directed by \supset . Hence,

$$\text{Fix}(\mathcal{S}) = \bigcap \{F : F \in \mathcal{F}\} \neq \emptyset.$$

Now, defining

$$\mathfrak{K} := \{T \in C^C : T \text{ is nonexpansive, } \text{Fix}(\mathcal{S}) \subset \text{Fix}(T)\},$$

and using an argument quite similar to the one used in the proof of Theorem 2.1, it is easy to show that \mathfrak{K} contains a nonexpansive retraction P from C onto $\text{Fix}(\mathcal{S})$. \square

Let E be a normed space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. We say that a nonempty set $C \subset E$ satisfies the τ -Opial condition if whenever the bounded sequence $\{x_n\}$ of elements of C converges to $x \in C$, with respect to τ , we have

$$\liminf_n \|x_n - x\| < \liminf_n \|x_n - y\|,$$

for $y \in C \setminus \{x\}$. We say that C satisfies the τ -Opial condition for nets, if whenever the bounded net $\{x_\alpha\}_{\alpha \in I}$ of elements of C converges to $x \in C$, with respect to τ , we have

$$\liminf_{\alpha \in I} \|x_\alpha - x\| < \liminf_{\alpha \in I} \|x_\alpha - y\|,$$

for $y \in C \setminus \{x\}$.

Spaces with the weak Opial condition have weak normal structure [10] and hence the weak fpp. Similarly dual spaces with the weak* Opial condition can contain no nontrivial separable weak* sequentially compact convex diametral sets; in particular, separable duals with the weak* Opial condition have weak* normal structure [18], and hence the weak* fpp. Using the following argument, it is possible to say more:

Suppose T is a nonexpansive self mapping of a nonempty bounded closed convex subset C of a Banach space E . It is well known that T admits an approximate fixed point sequence; that is, a sequence (x_n) in C with $\|Tx_n - x_n\| \rightarrow 0$. Now suppose C is, in addition, τ -compact, where τ is a Hausdorff topological vector space topology on E that is weaker than the norm topology. If the norm of E satisfies τ -Opial condition for nets (for sequences), then the fixed point set of T is nonempty and τ -compact (provided C is metrizable with respect to τ). In fact, if (x_i) is net (sequence) in C such that converges to x , with respect to τ , and $\|Tx_i - x_i\| \rightarrow 0$, then

$$\limsup_i \|Tx - x_i\| = \liminf_i \|Tx - Tx_i\| \leq \liminf_i \|x - x_i\|,$$

contradicting the τ -Opial condition unless $Tx = x$. Moreover, if $\{z_i\}$ is a net (sequence) in $\text{Fix}(T)$ converging in τ to some $z \in C$, then

$$\liminf_i \|z_i - Tz\| = \liminf_i \|Tz_i - Tz\| \leq \liminf_i \|z_i - z\|,$$

contradicting the τ -Opial condition for nets (for sequences) unless $Tz = z$. This implies that if either the norm of E satisfies τ -Opial condition for nets, or the norm of E satisfies τ -Opial condition for sequences and C is metrizable with respect to τ , then $\text{Fix}(T)$ is nonempty and τ -closed. Thus, Banach spaces with the τ -Opial condition for nets (for sequences) have the τ -fpp (provided τ is metrizable), and in this case the fixed point sets of nonexpansive mappings are τ -compact.

Combining the above facts with Theorem 4.1, we obtain the following results:

Theorem 4.2. *Let E be a Banach space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose that the norm of E is lsc with respect to τ and satisfies τ -Opial condition. Let C be a nonempty, τ -compact, convex subset of E and let C be metrizable with respect to τ . Then any commuting family of nonexpansive self-mappings of C has a common fixed point and the set of common fixed points is a nonexpansive retract of C .*

Let (Ω, Σ, μ) be a finite measure space. It follows from Proposition 5.2 in [21] (see also Lemma 2.6 in [22]) that $L_1(\mu)$ satisfies cm -Opial condition. Further, cm -topology is metrizable and the norm of $L_1(\mu)$ is lsc with respect to the topology of convergence in measure. Thus, $L_1(\mu)$ is an example that satisfies the assumptions of Theorem 4.2.

Theorem 4.3. *Let E be a Banach space and τ be a Hausdorff topological vector space topology on E that is weaker than the norm topology. Suppose that the norm of E is lsc with respect to τ and satisfies τ -Opial condition for nets. Let C be a nonempty, τ -compact, convex subset of E . Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C .*

Corollary 4.4. *Let C be a nonempty, weak* compact and convex subset of a dual Banach space E . Suppose that C satisfies weak*-Opial condition for nets. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C .*

Let E be a Banach space and let Γ be a nonempty subspace of its dual E^* . If

$$\sup\{x^*(x) : x^* \in \Gamma, \|x^*\| = 1\} = \|x\|,$$

for each $x \in E$, then we say that Γ is a norming set for E . It is obvious that a norming set generates a Hausdorff linear topology $\sigma(E, \Gamma)$ which is weaker than the weak topology $\sigma(E, E^*)$. It is worth noting here that $n(E) \subseteq E^{**}$ is a norming set for E^* , where n is a natural embedding of E into E^{**} , and hence, for $\Gamma = n(E)$, $\sigma(E, \Gamma)$ is the weak* topology on E^* . Throughout, Γ denotes a norming set for E . It is easy to observe that the norm of E is lower semicontinuous with respect to the $\sigma(E, \Gamma)$ -topology [8]. It is shown in [8] that if E is a Banach space, Γ is a norming set for E and C is a nonempty, bounded and Γ -sequentially compact subset of E , then in C the Γ -Opial condition for nets is equivalent to the Γ -Opial condition. Thus, we obtain the following:

Corollary 4.5. *Let E be a Banach space, Γ be a norming set for E , C be a nonempty, Γ -compact and Γ -sequentially compact convex subset of E . Suppose that C satisfies Γ -Opial condition. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C .*

Corollary 4.6. *Let E be a dual Banach space with a separable predual space, C be a nonempty, weak* compact and convex subset of E . Suppose that C satisfies weak*-Opial condition. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of C is a nonempty nonexpansive retract of C .*

Finally, it is worth mentioning that some of the well-known classical dual Banach spaces satisfy weak*-Opial condition.

Example 4.7. *The following dual Banach spaces satisfy the weak*-Opial condition for nets:*

- (i) ℓ_1 ;
- (ii) the James space J_0 ;
- (iii) $B(G)$, the Fourier-Stieltjes algebra of a compact group G ;

see [1, 9, 16, 19], for details. Hence, by Corollary 4.4 (or 4.6), the common fixed point set of any commutative family of nonexpansive self-mappings of a nonempty weak* compact convex subset C in the above spaces is a nonempty nonexpansive retract of C .

Problem: Can Theorems 4.2 and 4.3 be extended to left reversible or amenable semigroups?

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