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The Existence of Siegel Disks for the Cremona Map

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Abstract. This paper is concerned with the existence of Siegel disks of the Cremona map $F_\alpha(x, y) = (x\cos\alpha - (y - x^2)\sin\alpha,\ x\sin\alpha + (y - x^2)\cos\alpha)$ with the parameter $\alpha \in [0, 2\pi)$. This problem is reduced to the existence of local invertible analytic solutions to a functional equation with small divisors $\lambda^n + \lambda^{-n} - \lambda - \lambda^{-1}$. The main aim of this paper is to investigate whether this equation with $|\lambda| = 1$ has such a solution under the Brjuno condition.

1. Introduction

Polynomial maps are of interest from a mathematical and physical perspective. Much work has been done on *Cremona maps*, which are polynomial maps with constant Jacobians [10]. If the Jacobian of a map is equal to one, this map is area-preserving. Invariant manifolds of area-preserving maps play an important role in the theory of dynamical systems. One can reduce a dynamical system to a lower dimensional case by restricting the system to the invariant manifold.

The polynomial P_θ is said to be linearizable near the fixed point 0 if there exists a holomorphic change of coordinates Φ in a neighborhood of 0, called a linearizing map, which conjugates P_θ to the rigid rotation $R_\theta : z \mapsto e^{2\pi i\theta}z$. In this case, the maximal linearization domain $\triangle$ around 0 is an open simply-connected set called the Siegel disk of P_θ. Thus $P_\theta : \triangle \to \triangle$ acts as an irrational rotation by the angle θ. In 1990, Marmi [13] proved the existence of an invariant circle in both the modulated singular map and the semistandard map under the Brjuno condition, moreover, for the semistandard map, the Brjuno condition also necessary. And he also estimated the Siegel radius for the complex analytic polynomial maps $f(z) = e^{2\pi i\omega}z + (1/n)z^n$. However his discussion depends on f being entire and the existence of critical points on its boundary. In recent decades, many efforts were made with invariant curves or circles of analytic maps (cf. [2, 6, 7, 9, 14, 16]).).

In this paper, we investigate the existence of Siegel disks of the following Cremona map with one parameter $\alpha \in [0, 2\pi)$

$$F_\alpha(x, y) = \left(x\cos\alpha - (y - x^2)\sin\alpha,\ x\sin\alpha + (y - x^2)\cos\alpha\right), \tag{1}$$

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which is an area-preserving reversible map (cf. [8, 12, 18] and references therein). Let $z = x + \mathrm{i}y$. The planar map F_α can be written as a complex function

$$F_\alpha(z) = \lambda z - \frac{\mathrm{i}\lambda}{4}(z+\bar{z})^2, \quad \lambda = e^{\mathrm{i}\alpha}.$$

Siegel disks correspond to points where the dynamics of $F_\alpha : \mathbb{C} \to \mathbb{C}$ is analytically conjugated to an irrational rotation of the complex disk. For the Cremona map F_α, the existence of Siegel disks is reduced to the existence of local invertible analytic solutions of the functional equation

$$\Phi(\lambda z) - (\lambda + \lambda^{-1})\Phi(z) + \Phi(\lambda^{-1}z) = \frac{\mathrm{i}}{2}(\lambda^{-1} - \lambda)(\Phi(z))^2, \tag{2}$$

When λ is on the unit circle but not a root of unit, the main difficulty to the above problem is that one has to deal with the problem of estimating terms of the form $\lambda^n + \lambda^{-n} - \lambda - \lambda^{-1}$ (the so-called "small divisors problem").

The main aim of this paper is to investigate whether Eq.(2) with $|\lambda| = 1$ has a non-trivial analytic solution under the Brjuno condition.

2. Preliminaries

In order to look for a Siegel disk of F_α, it suffices for F_α to be conjugate to the irrational rotation

$$R_\alpha : z \mapsto \lambda z, \quad \lambda = e^{\mathrm{i}\alpha}.$$

This means that there exists a local invertible function $H : \mathbb{C} \to \mathbb{C}$, which is written as $H(z) = \Phi(z) + \mathrm{i}\Psi(z)$, analytic in the disk $D_{R(\alpha)}$ such that

$$F_\alpha \circ H = H \circ R_\alpha.$$

Together with (1), we have

$$\begin{aligned} \Phi(z)\cos\alpha - (\Psi(z) - (\Phi(z))^2)\sin\alpha &= \Phi(\lambda z), \\ \Phi(z)\sin\alpha + (\Psi(z) - (\Phi(z))^2)\cos\alpha &= \Psi(\lambda z). \end{aligned}$$

It follows that $\Psi(\lambda z) = (\Phi(z) - \Phi(\lambda z)\cos\alpha)\csc\alpha$. Thus $\Phi(z)$ must satisfy the functional equation

$$\Phi(\lambda z) - 2\Phi(z)\cos\alpha + \Phi(\lambda^{-1}z) = (\Phi(z))^2 \sin\alpha,$$

which is just Eq.(2). Substituting this expansion $\Phi(z) = \Sigma_{n=1}^{+\infty}\phi_n z^n$ into Eq.(2), we get

$$\sum_{n=1}^{+\infty} D_n\phi_n z^n = \sum_{n=2}^{+\infty}\sum_{j=1}^{n-1} \phi_j\phi_{n-j} z^n \sin\alpha, \tag{3}$$

where $D_n = \lambda^n + \lambda^{-n} - \lambda - \lambda^{-1}$. By comparing coefficients of (3), one easily finds that

$$D_1\phi_1 = 0, \;\; D_n\phi_n = \sum_{j=1}^{n-1} \phi_j\phi_{n-j}\sin\alpha, \;\; n \geq 2.$$

In order to obtain a non-trivial analytic solution, put $\phi_1 \neq 0$.

One can easily obtian the following.

Lemma 2.1. *Eq.*(2) *with* $\Phi(0) = 0$ *and* $\Phi'(0) \neq 0$ *has a formal solution of the form*

$$\Phi(z) = \sum_{n=1}^{+\infty} \phi_n z^n,$$

where ϕ_1 *is any nonzero constant, and*

$$\phi_n = \frac{\lambda - \lambda^{-1}}{2\mathrm{i}} \cdot \frac{1}{\lambda^n + \lambda^{-n} - \lambda - \lambda^{-1}} \sum_{j=1}^{n-1} \phi_j \phi_{n-j}, \ n \geq 2.$$

Here we first introduce the Gauss's continued fraction expansion of an irrational number and define the Brjuno condition. Let $[a_0, a_1, ...]$ be the continued fraction expansion of $\omega \in \mathbb{R}\backslash\mathbb{Q}$, recursively determined by $a_j = \lfloor 1/\omega_j \rfloor$ and $\omega_j = 1/\omega_{j-1} - \lfloor 1/\omega_{j-1} \rfloor$ for all $j \geq 1$, where $\lfloor\ \rfloor$ denotes the integer part, $\omega_0 = \omega - \lfloor \omega \rfloor$ and $a_0 = \lfloor \omega \rfloor$. The partial fractions $p_k/q_k = [a_0, ..., a_k]$ are given by

$$p_k = a_k q_{k-1} + p_{k-2}, \ q_k = a_k q_{k-1} + q_{k-2}$$

for all $k \geq 0$, with initial data $q_{-2} = p_{-1} = 1, q_{-1} = p_{-2} = 0$. As in [5, 11], the sequence $(p_k/q_k)_{k\geq 1}$ has the following important properties:

$$(2q_{k+1})^{-1} < (q_k + q_{k+1})^{-1} < |q_k\omega - p_k| < q_{k+1}^{-1}, \quad \forall k \geq 1 \tag{4}$$

$$q_k \geq \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^{k-1}, \quad k \geq 1 \tag{5}$$

$$\sum_{k\geq 0} \frac{1}{q_k} \leq \frac{\sqrt{5}+5}{2}. \tag{6}$$

As in [20], an irrational number $\omega \in \mathbb{R}\backslash\mathbb{Q}$ is said to satisfy the *Brjuno condition* (or ω is a Brjuno Number) if the sequence of denominators $(p_k/q_k)_{k\geq 0}$ of the convergents of ω satisfies

$$B(\omega) := \sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} < +\infty. \tag{7}$$

An irrational number $\omega \in \mathbb{R}\backslash\mathbb{Q}$ is said to satisfy the Diophantine condition (or ω is diophantine) if there exist constants $c > 0$ and $\mu > 2$ such that

$$|\omega - \frac{p}{q}| \geq \frac{c}{q^\mu}$$

for all $p \in \mathbb{Z}, q \in \mathbb{Z}^+$. Another equivalent condition [19] is as follows: there exist constants $c > 0$ and $\beta \geq 0$ such that p_k/q_k the k-th convergent of ω satisfies

$$q_{k+1} \leq c q_k^{1+\beta}, \quad \forall k \geq 0. \tag{8}$$

We call an irrational number $\omega \in \mathbb{R}\backslash\mathbb{Q}$ to satisfy the *strong Brjuno condition* if the sequence of denominators $(p_k/q_k)_{k\geq 0}$ of the convergents of ω satisfies

$$\bar{B}(\omega) := \sum_{k=0}^{+\infty} \frac{\log q_{k+2}}{q_k} < +\infty. \tag{9}$$

This condition is obviously not weaker than the Brjuno condition since $q_{k+1} < q_{k+2}$ for $k \in \mathbb{Z}^+$.

Lemma 2.2. *Let* $\omega \in \mathbb{R}\backslash\mathbb{Q}$ *be diophantine and* p_k/q_k *the k-th convergent of* ω*. Then*

$$\bar{B}(\omega) := \sum_{k=0}^{+\infty} \frac{\log q_{k+2}}{q_k} < +\infty.$$

Proof. By the diophantine condition, there exist constants $c > 0$ and $\mu > 2$ such that

$$|\omega - \frac{p}{q}| \geq \frac{c}{q^{\mu}}$$

for all $p \in \mathbb{Z}, q \in \mathbb{Z}^+$. By (4), $|\omega - \frac{p_k}{q_k}| < \frac{1}{q_k q_{k+1}}$ for all k. Then $q_{k+1} \leq c^{-1} q_k^{\mu-1}$. Thus

$$q_{k+2} \leq c^{-1} q_{k+1}^{\mu-1} \leq c^{-1}(c^{-1} q_k^{\mu-1})^{\mu-1} \leq c^{-\mu} q_k^{(\mu-1)^2}.$$

Therefore

$$\frac{\log q_{k+2}}{q_k} < \frac{\log c^{-\mu}}{q_k} + (\mu-1)^2 \frac{\log q_k}{q_k}.$$

It follows from (5) that $\sum_{k=0}^{+\infty} \frac{\log q_{k+2}}{q_k} < +\infty$. □

Now we construct an irrational number which satisfies the strong Brjuno condition, but is not diophantine. Let the denominators of the convergents of $\omega \in \mathbb{R} \backslash \mathbb{Q}$ satisfy $q_{k+2} = \lfloor e^{\sqrt{q_k}} \rfloor$ for all $k \geq 0$ and then by (5)

$$q_k^{-1} \log q_{k+2} \leq \frac{1}{\sqrt{q_k}} \leq \sqrt{2} \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{k-1}{2}}.$$

Consequently $\sum_{k=0}^{+\infty} \frac{\log q_{k+2}}{q_k} < +\infty$. However this number does not satisfy (8). In fact, assume this number satisfy (8). Then $q_{k+2} \leq c^{2+\beta} q_k^{(1+\beta)^2}$. On the other hand, there exists a fixed small $\varepsilon > 0$ such that $q_{k+2} > e^{q_k^{\varepsilon}}$. For big enough k, we have

$$e^{q_k^{\varepsilon}} > c^{2+\beta} q_k^{(1+\beta)^2}.$$

This is a contradiction.

The next example [19] is given to construct an irrational number which satisfies the Brjuno condition, but not the strong Brjuno condition. Let the denominators of the convergents of $\omega \in \mathbb{R} \backslash \mathbb{Q}$ satisfy $q_{k+1} = \lfloor e^{\sqrt{q_k}} \rfloor$ for all $k \geq 0$ and then by (5)

$$q_k^{-1} \log q_{k+1} \leq \frac{1}{\sqrt{q_k}} \leq \sqrt{2} \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{k-1}{2}}.$$

Consequently $\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} < +\infty$. There exists a fixed small $\varepsilon > 0$ such that $q_{k+1} > e^{q_k^{\varepsilon}}$. For big enough k, we have

$$q_k^{-1} \log q_{k+2} \geq q_k^{-1} \left(e^{q_k^{\varepsilon}} \right)^{\varepsilon} \geq q_k^{-1} (e^{\varepsilon})^{q_k^{\varepsilon}} \geq q_k,$$

Consequently $\sum_{k=0}^{+\infty} \frac{\log q_{k+2}}{q_k} = +\infty$.

3. The Siegel Disk

Theorem 3.1. *If $\bar{B}(\alpha/(2\pi)) < +\infty$, then F_α has a Siegel disk in the neighborhood of the origin.*

Proof. In order to prove the convergency of the formal power series $\Phi(z)$ in Lemma 2.1, it suffices to show that

$$\sup_n \frac{1}{|n|} \log |\phi_n| < \infty. \tag{10}$$

Let $\|\cdot\|$ denote the distance from the nearest integer, i.e., $\|y\| := \min_{p \in \mathbb{Z}} |y + p|$. For $n \geq 1$, let

$$\varepsilon_n := \min\{|\lambda^{n+1} - 1|^2, |\lambda^{n-1} - 1|^2\} \leq |(\lambda^{n+1} - 1)(\lambda^{n-1} - 1)| = |D_n|.$$

Put $\omega = \alpha/(2\pi)$, then

$$\varepsilon_n = \min\{4|\sin\pi(n+1)\omega|^2,\ 4|\sin\pi(n-1)\omega|^2\}.$$

Since $2x \le \sin(\pi x) \le 1$ for all $x \in [0, 1/2]$, one immediately has

$$\min\{16\|(n+1)\omega\|^2,\ 16\|(n-1)\omega\|^2\} \le \varepsilon_n \le 4.$$

According to Siegel's ideas (1942) [17], define inductively

$$\sigma_1 = |c_1|,\ \sigma_n = \sum_{l=1}^{n-1} \sigma_l \sigma_{n-l},\ n \ge 2,$$

and

$$\delta_1 = 1,\ \delta_n = \frac{1}{\varepsilon_n} \max_{1\le j\le n-1} \delta_j \delta_{n-j},\ n \ge 2. \tag{11}$$

By induction, we see that

$$\begin{aligned} |\phi_n| &\le \frac{|\sin\alpha|}{|D_n|} \sum_{j=1}^{n-1} |\phi_j||\phi_{n-j}| \le \frac{1}{\varepsilon_n} \sum_{l=1}^{n-1} \sigma_l\sigma_{n-l}\delta_l\delta_{n-l} \le \left(\frac{1}{\varepsilon_n} \max_{1\le l\le n-1} \delta_l\delta_{n-l}\right)\left(\sum_{l=1}^{n-1} \sigma_l\sigma_{n-l}\right) \\ &= \sigma_n\delta_n. \end{aligned}$$

Then

$$\phi_n \le \sigma_n\delta_n,\ \forall n \ge 2. \tag{12}$$

Therefore, to establish (10) it suffices to prove analogous estimates for σ_n and δ_n. To estimate σ_n, let $f(z) = \sum_{n=1}^{+\infty} \sigma_n z^n$, we find that f satisfies the functional equation

$$f(z) = |c_1|z + (f(z))^2.$$

This equation has a unique analytic solution vanishing at zero

$$f(z) = \frac{1 - \sqrt{1 - 4|c_1|z}}{2}, \quad |z| \le \frac{1}{4|c_1|}.$$

By Cauchy's estimate

$$\sigma_n \le (4|c_1|)^n \max_{|z|\le 1/(4|c_1|)} |f(z)| \le (4|c_1|)^n \cdot \frac{1+\sqrt{2}}{2} \le \frac{(4|c_1|)^n(1+\sqrt{2})}{2}. \tag{13}$$

Hence

$$\sup_n \frac{1}{n} \log \sigma_n < \infty.$$

We now consider δ_n. Here we essentially repeat Brjuno's arguments (Brjuno 1971 and 1972 [3, 4]); also refer to Abate 2010[1], Jöschel 1986 [15] for some very readable expositions and for applications of Brjuno's method to the Siegel theorem.

In (11) the maximum is attained for some decomposition

$$\delta_1 = 1,\ \delta_n = \frac{1}{\varepsilon_n}\delta_{j_n}\delta_{n-j_n},\ \text{where } 1 \le j_n \le n-1. \tag{14}$$

Decomposing δ_{j_n}, δ_{n-j_n} in the same manner, and proceeding like this we will finally obtain some well defined decomposition

$$\delta_n = \prod_{k=1}^{l(n)} \varepsilon_{i_k}^{-1}, \text{ where } \varepsilon_{i_1} = \varepsilon_n,\ 2 \le i_2 \le \cdots \le i_{l(n)} \le n-1. \tag{15}$$

We claim that $l(n) = n-1$. In fact, as $\delta_1 = 1$, $\delta_2 = 1/\varepsilon_2$, one has $l(1) = 0$, $l(2) = 1$. By induction, for $n \ge 2$ from (14) one has $l(n) = 1 + l(j_n) + l(n-j_n) = 1 + j_n - 1 + n - j_n - 1 = n-1$.

It is natural to introduce the function $\Omega_\lambda : \mathbb{N} \setminus \{1\} \to \mathbb{R}^+$ defined as follows:

$$\Omega_\lambda(m) := \min_{1\le j\le m-1} \varepsilon_j.$$

Clearly Ω_λ is non-increasing and, since $\omega \in \mathbb{R}\backslash\mathbb{Q}$, $\lim_{m\to+\infty} \Omega_\lambda(m) = 0$. From (4) one has

$$\Omega(q_k) \ge \min_{1\le j\le q_k-1} \min\{16\|(j+1)\omega\|^2,\ 16\|(j-1)\omega\|^2\} \ge 16|q_k\omega - p_k|^2 \ge \frac{4}{q_{k+1}^2}. \tag{16}$$

Let $N_m(n)$ denote the number of factor $\varepsilon_{i_k}^{-1}$ in the expression (15) satisfying

$$\varepsilon_{i_k} < \frac{1}{4}\Omega_\lambda(m).$$

The next lemma contains the key estimate.

Theorem 3.2. *For $m \ge 2$, we have*

$$\begin{aligned} N_m(n) &= 0 && \text{if } n \le m, \\ N_m(n) &\le \frac{2n}{m} - 1 && \text{if } n > m. \end{aligned}$$

Proof of Lemma 3.2. We argue by induction on n. If $i_k \le n \le m$ we have $\varepsilon_{i_k} \ge \Omega_\lambda(m)$, and hence $N_m(n) = 0$.

Assume now $n > m$. Write δ_n as in (15), we have a few cases to consider.

Case 1: $\varepsilon_n \ge \frac{1}{4}\Omega_\lambda(m)$. Then by (14)

$$N_m(n) = N_m(n_1) + N_m(n_2), \quad n_1 + n_2 = n,$$

and applying the induction hypothesis to each term we get $N_m(n) \le 2n/m - 1$.

Case 2: $\varepsilon_n < \frac{1}{4}\Omega_\lambda(m)$. Then

$$N_m(n) = 1 + N_m(n_1) + N_m(n_2), \quad n_1 + n_2 = n, \quad n_1 \ge n_2,$$

and there are three subcases.

Case 2.1: $n_1 \le m$. Then

$$N_m(n) = 1.$$

Case 2.2: $n_1 \ge n_2 \ge m$. Then by induction

$$N_m(n) = 1 + N_m(n_1) + N_m(n_2) \le 1 + \frac{2n}{m} - 2 = \frac{2n}{m} - 1.$$

Case 2.3: $n_1 \ge m > n_2$. Then

$$N_m(n) = 1 + N_m(n_1),$$

and we have two different subsubcases.

Case 2.3.1: $n_1 \le n - m + 1$. Then by induction

$$N_m(n) \le 1 + 2\frac{n-m+1}{m} - 1 \le \frac{2n}{m} - 1,$$

and we are done in this case too.

Case 2.3.2: $n_1 > n - m + 1$. The crucial remark here is that $\varepsilon_{n_1}^{-1}$ gives no contribute to $N_m(n_1)$. Indeed, assume by contradiction that $\varepsilon_{n_1} < \frac{1}{4}\Omega_\lambda(m)$. Then

$$\begin{aligned}\frac{1}{2}\Omega_\lambda(m) &> \varepsilon_n + \varepsilon_{n_1} \ge \min\{|\lambda^{n-n_1+2} - 1|^2, |\lambda^{n-n_1} - 1|^2, |\lambda^{n-n_1-2} - 1|^2\} \\ &\ge \Omega_\lambda(n - n_1 + 2) \ge \Omega_\lambda(m).\end{aligned}$$

This is a contradiction.

Therefore case 1 applies to δ_{n_1} and we have

$$N_m(n_1) = 1 + N_m(k_{l_1}) + N_m(n_1 - k_{l_1}).$$

We can repeat the argument for this decomposition, and we finish unless we run into case 2.3.2 again. However, this loop cannot happen more than $m+1$ times, and we eventually have to land into a different case. The proof is completed. □

We can now complete the proof of the theorem. Define the sequences of index set

$$\begin{aligned} I(0) &:= \{k = 1, ..., l(n) \,\text{in}\, (15) \,|\, \frac{1}{4}\Omega_\lambda(q_1) \le \varepsilon_{i_k} \le 4\}, \\ I(v) &:= \{k = 1, ..., l(n) \,\text{in}\, (15) \,|\, \frac{1}{4}\Omega_\lambda(q_{v+1}) \le \varepsilon_{i_k} < \frac{1}{4}\Omega_\lambda(q_v)\}, \end{aligned}$$

where the sequence $(q_v)_{v=1}^{+\infty}$ is the sequence of the denominators of the partial fractions of ω. By $l(n) = n - 1$, we see that $\text{card}\, I(0) \le n - 1$, and if $v \ge 1$ $\text{card}\, I(v) \le \frac{2n}{q_v} - 1$. Thus by (15), (16), (5) and (6)

$$\begin{aligned}\frac{1}{n}\log\delta_n &= \sum_{k=1}^{l(n)} \frac{1}{n}\log\varepsilon_{i_k}^{-1} \le \sum_{v=0}^{+\infty} \frac{1}{n}\text{card}I(v)\log\frac{4}{\Omega_\lambda(q_{v+1})} \le \sum_{v=0}^{+\infty} \frac{1}{n}\cdot\frac{2n}{q_v}\log q_{v+2}^2 \\ &= 4\sum_{v=0}^{+\infty}\frac{\log q_{v+2}}{q_v}.\end{aligned}$$

Together with (12) and (13), we have

$$\begin{aligned}\sup_{n\ge1}\frac{1}{n}\log|\phi_n| &\le \sup_{n\ge1}\frac{1}{n}(\log\sigma_n + \log\delta_n) \le \log|4c_1| + \log\frac{1+\sqrt{2}}{2} + 4\sum_{v=0}^{+\infty}\frac{\log q_{v+2}}{q_v} \\ &< +\infty,\end{aligned}$$

which implies that power series $\Phi(z) = \Sigma_{n=1}^{+\infty}\phi_n z^n$ converges in a neighborhood of the origin. The theorem is proved. □

4. Numerical Evidences

In this section, some numerical experiments will be presented under different parameter values.

Figs.1-3 present some orbits of the Cremona map for three values of $\alpha/(2\pi)$ near rational number 1/3, and show that a family of invariant curves at a neighborhood of the origin exist when $\alpha = 2\pi/3 - 1/1000$, disappear when $\alpha = 2\pi/3$, and appear again when $\alpha = 2\pi/3 + 1/1000$.

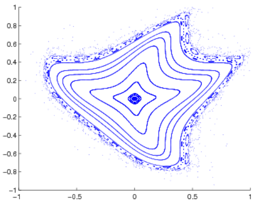

Figure 1: $\alpha = \frac{\pi}{2} - \frac{1}{1000}$

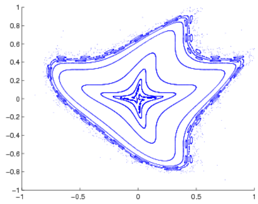

Figure 2: $\alpha = \frac{\pi}{2}$

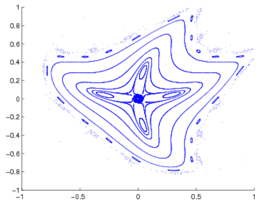

Figure 3: $\alpha = \frac{\pi}{2} + \frac{1}{1000}$

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