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A Note on Lévy Risk Model with Two-Sided Phase-Type Jumps

Tatjana Slijepčević -Manger[a]

[a]*Faculty of Civil Engineering, University of Zagreb, Fra Andrije Kačića-Miošića 26, 10000 Zagreb, Croatia*

Abstract. In this paper we establish some results on a phase-type Lévy risk model with two-sided jumps and a barrier dividend strategy. Following [4] we describe the connection between the ruin problem of the risk model with barrier dividend strategy and the first passage problem of the risk model reflected at its running supremum. Then we give some results for the joint Laplace transform of the upward entrance time and the overshoot for the phase-type jump diffusion reflected at its running supremum. Finally we find some expressions for the Laplace transform of the time of ruin and the expected discounted dividends up to ruin. All our results on the ruin problem are expressed in terms of the solutions to the Cramér-Lundberg equation corresponding to the underlying phase-type jump diffusion.

1. Introduction

De Finetti first considered the problem of finding the optimal dividend-payment strategy in [7]. Since then it has been studied extensively. De Finetti found that, if we want to maximize the expected discounted dividends, we must use a barrier strategy. The optimal dividend problem for general spectrally negative Lévy processes has been studied by Loeffen and Renaud in [9]. They proved that the barrier strategy for spectrally one-sided models is optimal if the tail of the Lévy measure is log-convex.

Recently, many researchers are attracted by risk models with two-sided jumps. In this kind of models, the downward jumps can be interpreted as random losses (from claim or investment indemnity) of an insurance company, while the upward jumps are interpreted as random returns (obtained by investing the initial asset and the insurance premium) of the company. In [6] the threshold dividend strategy for a Lévy process with two sided jumps is studied. The paper by Bo, Song, Tang, Wang and Yang (see [4]) aims at studying risk models with two-sided jumps and a constant barrier dividend strategy. It is worthwhile to note that the techniques used in [6] for studying the threshold dividend strategy are not usable for the barrier dividend strategy because when the surplus process can jump upward, the threshold dividend strategy generates a continuous dividend process, while the barrier dividend strategy creates a discontinuous one.

The purpose of this paper is to present some results on a phase-type Lévy risk model with two-sided jumps and a barrier dividend strategy. Following [4] we relate the ruin problem of the risk model with barrier dividend strategy to the first passage problem of the risk model reflected at its running supremum. For a general Lévy risk model, it is shown that the expected discounted dividends can be expressed in terms of the Laplace transform of the upward entrance time of the unconstrained Lévy process and the joint Laplace transform of the upward entrance time and the overshoot of the Lévy process reflected at its

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Email address: tmanger@grad.hr (Tatjana Slijepčević -Manger)

running supremum. Then we consider the first passage time problem for the phase-type jump diffusion reflected at its running supremum. In order to solve this problem we give the joint moment generating function of the crossing time and the overshoot in section 3.2. The main difference between our results and those given in [3] is that we are considering the case of negative argument for the joint moment generating function. This is very important because it enables us to connect the results from [4] and and our results from 3.2 to find some expressions for the Laplace transform of the time of ruin and the expected discounted dividends up to ruin in section 3.3. All our results on the ruin problem are expressed in terms of the solutions to the Cramér-Lundberg equation corresponding to the underlying phase-type jump diffusion.

2. Lévy Risk Model with Two Sided Jumps and Constant Barrier Dividend Strategy

We consider a Lévy process $X = \{X_t, t \geq 0\}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions of right-continuity and completness. Let $\mathbb{P}_x$, $x \in \mathbb{R}$, be the distribution of $X + x$ under $\mathbb{P}$ and $\mathbb{E}_x$ the expectation operator corresponding to $\mathbb{P}_x$. Denote by T_a the entrance time of the Lévy process X into $(a, +\infty)$:

$$T_a = \inf\{t \geq 0 : X_t > a\}$$

where $\inf \emptyset = \infty$. The running supremum of the process X is defined as

$$S_t = \sup_{s\leq t}(X_s \vee 0)$$

and the Lévy process X reflected at its running supremum S as $Y = S - X$. Note that the reflected Lévy process Y is a Markov process. Further, let τ_a be the entrance time of the reflected Lévy process Y into (a, ∞):

$$\tau_a = \inf\{t \geq 0 : Y_t > a\}.$$

Now we consider a risk problem with constant barrier dividend strategy. An insurance company will pay dividends according to a barrier strategy with parameter $b > 0$. Denote by X the risk process of the company before dividends are deduced. The aggregate dividends paid by time t are

$$L_t^b = \sup_{s\leq t}(X_s - b) \vee 0.$$

Let U^b be the risk process regulated by the dividend payment L^b, that is

$$U_t^b = X_t - L_t^b, \ t \geq 0.$$

We can interpret the dividend process $\{L_t^b\}_{t\geq 0}$ as the magnitude of the displacement which is the minimal amount required to keep $\{U_t^b\}_{t\geq 0}$ always less than or equal to b.

Denote by $\tilde{\tau}_b$ the ruin time of the insurance company with surplus process U^b

$$\tilde{\tau}_b = \inf\{t \geq 0 : U_t^b < 0\}.$$

We give the following results concerning the Laplace transform of the ruin time, the expected discounted dividends and the deficit at ruin which correspond to results from Proposition 2.1 in [4]:

Proposition 1. *Suppose $b > 0$ and $x \in [0, b]$. Then we have the following:*

(a) The Laplace transform of the ruin time is given by

$$\mathbb{E}_x[e^{-r\tilde{\tau}_b}] = \mathbb{E}_{x-b}[e^{-r\tau_b}].$$

(b) The total expected discounted dividends before ruin satisfies

$$\mathbb{E}_x\Big[\int_0^{\tilde{\tau}_b} e^{-rt} dL_t^b\Big] = \mathbb{E}_{x-b}\Big[\int_0^{\tau_b} e^{-rt} dS_t\Big].$$

(c) The deficit at ruin satisfies, for each $y \geq 0$

$$\mathbb{E}_x\Big[e^{-r\tilde{\tau}_b}\mathbb{1}_{\{-U^b_{\tilde{\tau}_b}>y\}}\Big] = \mathbb{E}_{x-b}\Big[e^{-r\tau_b}\mathbb{1}_{\{Y_{\tau_b}-b>y\}}\Big].$$

Proof. Using the spatial homogeneity of the surplus process X, it is not hard to find that $\{U_b, L_b, \tilde{\tau}_b; U_0 = x\}$ has the same distribution as $\{b - Y, S, \tau_b; Y_0 = b - x\}$. Obviously, $Y_0 = b - x$ if $X_0 = x - b$ and the conclusion now easily follows. □

First recall that "0 is regular for $(0, \infty)$ for the Lévy process X" means that a Lévy process started at the origin gets to $(0, \infty)$ at arbitrarily small times (see [1]). Now we give a useful expression for the total expected discounted dividends which corresponds to Theorem 2.1 from [4]:

Theorem 1. *Suppose 0 is regular for $(0, \infty)$ for the Lévy process X and let $b > 0$. The total expected discounted dividends are given by*

$$\mathbb{E}_x\Big[\int_0^{\tilde{\tau}_b} e^{-rt} dL_t^b\Big] = h(x - b) - \mathbb{E}_{x-b}\Big[e^{-r\tau_b} h(-Y_{\tau_b})\Big],$$

for $x \in [0, b]$. Here

$$h(-x) = \mathbb{E}[e^{-rT_x}(X_{T_x} - x)] + \mathbb{E}[e^{-rT_x}]\int_0^\infty \mathbb{E}[e^{-rT_z}]dz$$

for $x \geq 0$.

For the reader's convenience we give the proof following [4].

Proof. From Proposition 1(b) (Proposition 2.1(II) from [4]), we have

$$\begin{aligned}
\mathbb{E}_x\Big[\int_0^{\tilde{\tau}_b} e^{-rt} \mathrm{d}L_t^b\Big] &= \mathbb{E}_{x-b}\Big[\int_0^{\tau_b} e^{-rt}\mathrm{d}S_t\Big] \\
&= \mathbb{E}_{x-b}\Big[\int_0^{\infty} e^{-rt}\mathrm{d}S_t\Big] - \mathbb{E}_{x-b}\Big[e^{-r\tau_b}\mathbb{E}_{-Y_{\tau_b}}\Big(\int_0^{\infty} e^{-rt}\mathrm{d}S_t\Big)\Big] \\
&= h(x - b) - \mathbb{E}_{x-b}\Big[e^{-r\tau_b} h(-Y_{\tau_b})\Big],
\end{aligned}$$

where, for $x \geq 0$,

$$\begin{aligned}
h(-x) &= \mathbb{E}_{-x}\Big[\int_0^{\infty} e^{-rt}\mathrm{d}S_t\Big] \\
&= \mathbb{E}_{-x}[e^{-rT_0}X_{T_0}] + \mathbb{E}_{-x}[e^{-rT_0}]\mathbb{E}\Big[\int_0^{\infty} e^{-rt}\mathrm{d}S_t\Big] \\
&= \mathbb{E}[e^{-rT_x}(X_{T_x} - x)] + \mathbb{E}[e^{-rT_x}]\int_0^\infty \mathbb{E}[S_t]re^{-rt}\mathrm{d}t
\end{aligned}$$

$$= \mathbb{E}[e^{-rT_x}(X_{T_x} - x)] + \mathbb{E}[e^{-rT_x}] \int_0^\infty re^{-rt}\mathrm{d}t \int_0^\infty \mathbb{P}(S_t \geq z)\mathrm{d}z$$
$$= \mathbb{E}[e^{-rT_x}(X_{T_x} - x)] + \mathbb{E}[e^{-rT_x}] \int_0^\infty re^{-rt}\mathrm{d}t \int_0^\infty \mathbb{P}(T_z \leq t)\mathrm{d}z$$
$$= \mathbb{E}[e^{-rT_x}(X_{T_x} - x)] + \mathbb{E}[e^{-rT_x}] \int_0^\infty \mathbb{E}[e^{-rT_z}]\mathrm{d}z.$$

The second equality holds since $Y_{T_0} = 0$ and the penultimate one follows from the regularity of 0 for $(0, \infty)$. The proof is now complete. □

Theorem 1 (Theorem 2.1 from [4]) shows that if we want to compute the expected discounted dividends when the underlying risk model is a general Lévy process X with two sided jumps, we have to know the Laplace transform of the one-sided first (upward) passage time of X as well as the joint distribution of the (upward) entrance time and the overshoot of the reflected Lévy process $Y = S - X$.

3. Phase-Type Double Jump-Diffusion Model

3.1. *Lévy phase-type model*

Denote by $J = \{J_t\}_{t\geq 0}$ a finite state continuous time Markov process with one state Δ absorbing and the remaining ones $1, \dots, m$ transient. The distribution F on $(0, \infty)$ of the absorption time ζ in $J = \{J_t\}_{t\geq 0}$ is called a *phase-type* distribution. This means that $F(t) = \mathbb{P}(\zeta \leq t)$ where $\zeta = \inf\{s > 0 : J_s = \Delta\}$. Let $\mathbf{T}$ be the restriction of the full intensity matrix to the m transient states and let $\boldsymbol{\alpha} = (\alpha_1 \dots \alpha_m)$ be the initial probability row vector where $\alpha_i = \mathbb{P}(J_0 = i)$. For any $i = 1, \dots, m$ denote by t_i the intensity of a transition $i \to \Delta$ and write $\mathbf{t} = (t_1 \dots t_m)^T$ for the column vector of such intensities. We know that $\mathbf{t} = -\mathbf{T}\mathbf{1}$, where $\mathbf{1}$ stands for a column vector of ones. The cumulative phase-type distribution F is given by

$$F(x) = 1 - \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{1}, \tag{1}$$

the density is $f(x) = \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{t}$ and the Laplace transform is given by

$$\hat{F}[s] = \int_0^\infty e^{-sx} F(\mathrm{d}x) = \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}.$$

The Laplace transform $\hat{F}[s]$ can be extended to the complex plane except at a finite number of poles (the eigenvalues of $\mathbf{T}$). If there exists no number $l < m$, l-vector $\mathbf{v}$ and $l \times l$-matrix $\mathbf{G}$ such that $F(x) = 1 - \mathbf{v}e^{\mathbf{G}x}\mathbf{1}$ a representation of form (1) to the distribution function F is called *minimal*.

Phase-type distributions include and generalize exponential distributions. They also form a dense class in the set of all distributions on $(0, \infty)$ and have many applications in applied probability, see for example [2] for surveys. The applicability of the class comes from the fact that the overshoot distribution is again phase-type with the same m and $\mathbf{T}$ but α_i replaced by $\mathbb{P}(J_x = i | \zeta > x)$. This is similar to the memoryless property of the exponential distribution ($m = 1$) and explains the existence of many formulas which generalize the scalar exponential case.

Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process defined on probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions. We consider X of the form:

$$X_t = \mu t + \sigma W_t + \sum_{k=1}^{N^+(t)} U_k^+ - \sum_{l=1}^{N^-(t)} U_l^-, \tag{2}$$

where W is standard Brownian motion, $N^\pm$ are Poisson processes with rates of arrival $\lambda^\pm$ and $U^\pm$ are i.i.d. random variables with respective jump size distributions $F^\pm$ of phase-type with parameters $m^\pm$, $\mathbf{T}^\pm$, $\boldsymbol{\alpha}^\pm$.

All processes are assumed to be independent. For any $s \in i\mathbb{R}$, the Lévy exponent G of X, defined by $G(s) = \log \mathbb{E}[e^{sX_1}]$, is

$$G(s) = s\mu + s^2\frac{\sigma^2}{2} + \lambda^+(\hat{F}^+[-s] - 1) + \lambda^-(\hat{F}^-[s] - 1), \tag{3}$$

where $\hat{F}^{\pm}[s] = \boldsymbol{\alpha}^{\pm}(s\mathbf{I} - \mathbf{T}^{\pm})^{-1}\mathbf{t}^{\pm}$. As above, $G(s)$ can be extended to the complex plane except a finite number of poles (the eigenvalues of $\mathbf{T}^{\pm}$); this extension will also be denoted by G. Any Lévy process may be approximated arbitrarily closely by processes of form (2) (see [3, Proposition 1]). Many of the computations involving Lévy processes are based on finding the roots of the "Cramér-Lundberg equation"

$$G(s) = r, \tag{4}$$

for some riskless discount rate r.

3.2. First passage time

We consider the process $Y = S - X$ defined as the process (2) reflected at its running supremum S. To solve the first passage time problem for Y we have to compute the joint moment generating function

$$v_b(y) = v_b(y, r, k) = \mathbb{E}_y[e^{-r\tau + k(Y_\tau - b)}] \tag{5}$$

of the crossing time

$$\tau = \tau_b = \inf\{t > 0 : Y_t > b\}$$

and the overshoot $Y_\tau - b$, where $r, b \geq 0$ and the number k is such that $v_b(y)$ is finite. Under the measure $\mathbb{P}_y$ the process Y starts in $Y_0 = y \vee 0 - y$.

At the crossing time τ_b the component $\mu t + \sigma W_t$ must either take the process Y to the barrier b, or we must have a downward jump of X. Denote by M^- the set of all phases of the underlaying phase process $J = \{J_t\}_{t \geq 0}$ for the jump, during which upcrossing may occur (calling the non-jumping time phase 0) and by m^- the number of elements of M^-. We write M^+ for the set of phases during which upcrossing can not occur and m^+ for the number of elements of M^+. Let H_i denote the event that the upcrossing of b occurs in phase $i \in M^-$ or the event of crossing the supremum in phase $i \in M^+$. Denote by

$$\pi_i = \mathbb{E}_y[e^{-r\tau}\mathbb{1}_{H_i}]$$

the (discounted) probability of upcrossing in phase $i \in M^-$. We write $\boldsymbol{\pi} = (\pi_i, i \in M^-)$ and let $\hat{\mathbf{f}}^-[k]$ be the vector of analytic continuations of Laplace transforms at k of the overshoot $Y_{\tau_b} - b$ depending on the initial starting state. For the running supremum $S_t = \sup_{s \leq t}(X_s \vee 0)$ of X, let $\Delta S_t = S_t - S_{t-}$ be the jump of S at time t and S_t^c the continuous part of S. Introduce the dummy-variables $\delta_0 = \mathbb{E}_y[\int_0^{\tau_b} e^{-rs}\mathrm{d}S_s^c]$ and

$$\delta_i = \mathbb{E}_y\Big[\sum_{0 < s \leq \tau_b} e^{-rs}\mathbb{1}_{\{\Delta S_s > 0, H_i\}}\Big], \quad i = 1, \ldots, m^+.$$

Denote by $\boldsymbol{\delta}$ the row vector $\boldsymbol{\delta} = (\delta_i, i \in M^+)$ and write $(\mathbf{g}[\rho]_i, i \in M^+)$ where $g[\rho]_0 = \rho$ and $g[\rho]_i = \rho\mathbf{1}_i(-\rho\mathbf{I} - \mathbf{T}^+)^{-1}\mathbf{1}$. Here $\mathbf{1}_i$ denote a row vector of zeros with a 1 on the ith position. Let p denote the number of roots of the Cramér-Lundberg equation $G(\rho) = r$. The next result gives an expression for the moment-generating function $v_b(y)$ in terms of the roots of the Cramér-Lundberg equation.

Proposition 2. *The joint moment generating function $v_b(y)$ defined in (5) for the process (2) is given by*

$$v_b(y) = \boldsymbol{\pi}\hat{\mathbf{f}}^-[-k]$$

where $y < b$ *and* $\boldsymbol{\pi} = (\pi_i, i \in M^-)$, $\boldsymbol{\delta} = (\delta_i, i \in M^+)$ *solve the system*

$$e^{-\rho_i b}\boldsymbol{\pi}\hat{\mathbf{f}}^-[\rho_i] - \boldsymbol{\delta}\mathbf{g}[\rho_i] = e^{-\rho_i(y\vee 0-y)}, \quad i = 1,\dots,p. \tag{6}$$

If the vectors $\mathbf{k}^i := (e^{-\rho_i b}\hat{\mathbf{f}}^-[\rho_i]', \mathbf{g}[\rho_i]')$, $i = 1,\dots,p$ *are linearly independent and all roots* ρ_i *of* $G(\rho) = r$ *are distinct, then* $(\boldsymbol{\pi}, \boldsymbol{\delta})$ *uniquely solve system (6).*

Proof. We split the probability space in $H_0,\dots,H_{m^-}$ and use the fact that, conditionally on the phase in which the upcrossing occurs, the time of overshoot τ_b and the overshoot $Y_{\tau_b} - b$ are independent to get the decomposition

$$\begin{aligned} v_b(y) &= \mathbb{E}_y[e^{-r\tau_b}e^{k(Y_{\tau_b}-b)}] \\ &= \mathbb{E}_y[e^{-r\tau_b}\mathbf{1}_{H_0}] + \sum_{i=1}^{m^-}\mathbb{E}_y[e^{-r\tau_b}\mathbf{1}_{H_i}]\mathbb{E}_i[e^{k(Y_{\tau_b}-b)}] \\ &= \boldsymbol{\pi}\hat{\mathbf{f}}^-[-k], \end{aligned}$$

where $\mathbb{E}_i$ stands for the expectation under $\mathbb{P}$ conditioned on H_i.

We use the optional stopping approach to the reflected process Y and the martingale introduced in [8] to compute the vector $\boldsymbol{\pi}$. Note that $\Delta S_t = S_t - S_{t-}$ and S^c have finite number of jumps in each finite time interval and finite expected variation respectively. Applying [8] we find that for $\gamma \in i\mathbb{R}$ and $r > 0$

$$N_t = (G(-\gamma) - r)\int_0^t (-rs + \gamma Y_s)\mathrm{d}s + e^{\gamma Y_0} - e^{-rt+\gamma Y_t} + \gamma\int_0^t e^{-rs}\mathrm{d}S^c_s + \sum_{0<s\le t} e^{-rs}[1 - e^{-\gamma\Delta S_s}]$$

is a zero mean martingale. Here we used that if ΔS_s or $\mathrm{d}S_s$ is positive then $Y_s = 0$. It is easy to check that $|N_{\tau_b\wedge t}|$ can be dominated by an integrable function. We conclude that $\mathbb{E}_y[N_{\tau_b}] = 0$ by applying Doob's optional stopping theorem (see [1]) with the stopping time $\tau_b \wedge t$. Expanding the equality $\mathbb{E}_y[N_{\tau_b}] = 0$ for $y < b$ leads to

$$0 = (G(-\gamma) - r)\mathbb{E}_y\Big[\int_0^{\tau_b} e^{-rs+\gamma Y_s}\mathrm{d}s\Big] + e^{\gamma(y\vee 0-y)} - e^{\gamma b}\boldsymbol{\pi}\hat{\mathbf{f}}^-[-\gamma] + \gamma\delta_0 + \sum_{j=1}^{m^+}\delta_j(1 - \hat{\mathbf{f}}^+[\gamma]_j). \tag{7}$$

Identity (7) can be extended to hold for γ in the complex plane except finitely many poles (the eigenvalues of $-\mathbf{T}^-, \mathbf{T}^+$) by analytic continuation. Letting $\gamma = -\rho_i$, where ρ_i is a root of $G(\rho) = r$, we find system (6). Since for minimal representations of $F^\pm$ the number of unknowns is equal to the number of equations, the last assertion follows. □

3.3. *The ruin time and the expected discounted dividends*

Now we can present an expression for the Laplace transform of the ruin time.

Theorem 2. *The Laplace transform of the ruin time for the process (2) can be expressed as*

$$\mathbb{E}_x[e^{-r\tau_b}] = \boldsymbol{\pi}\mathbf{1}$$

for $x \in [0, b]$, *where* $\boldsymbol{\pi}$ *solves the system (6) for* $y = x - b$ *and* $\mathbf{1}$ *denotes a column vector of ones.*

Proof. From Proposition 1(a) (Proposition 2.1(I) in [4]) we have

$$\mathbb{E}_x[e^{-r\tilde{\tau}_b}] = \mathbb{E}_{x-b}[e^{-r\tau_b}].$$

Decomposing the probability space in $H_0, \dots, H_{m^-}$ and using the fact that the time of overshoot τ_b and the phase in which the upcrossing occurs are independent, yields the decomposition

$$\begin{aligned}\mathbb{E}_{x-b}[e^{-r\tau_b}] &= \mathbb{E}_{x-b}[e^{-r\tau_b}\mathbf{1}_{H_0}] + \sum_{i=1}^{m^-} \mathbb{E}_{x-b}[e^{-r\tau_b}\mathbf{1}_{H_i}]\mathbb{E}_i[\mathbf{1}_\Omega] \\ &= \boldsymbol{\pi}\mathbf{1},\end{aligned}$$

where $\mathbb{E}_i$ denotes the expectation under $\mathbb{P}$ conditioned on H_i and the theorem is proved. □

We next present an expression for the total expected discounted dividends.

Theorem 3. *Let $b > 0$. For $x \in [0, b]$ the total expected discounted dividends for the process (2) can be expressed as*

$$\mathbb{E}_x\Big[\int_0^{\tilde{\tau}_b} e^{-rt} dL_t^b\Big] = h(x-b) - \boldsymbol{\pi}\boldsymbol{\mathcal{G}},$$

where

$$h(-x) = \mathbb{E}[e^{-rT_x}(X_{T_x} - x)] + \mathbb{E}[e^{-rT_x}] \int_0^\infty \mathbb{E}[e^{-rT_z}]dz,$$

$\boldsymbol{\pi}$ solves the system (6) for $y = x - b$ and $\boldsymbol{\mathcal{G}} = (\mathbb{E}_i[h(-Y_{\tau_b})], i \in M^-)$.

Proof. From Theorem 1 (Theorem 2.1 in [4]) we have

$$\mathbb{E}_x\Big[\int_0^{\tilde{\tau}_b} e^{-rt} \mathrm{d}L_t^b\Big] = h(x-b) - \mathbb{E}_{x-b}\Big[e^{-r\tau_b} h(-Y_{\tau_b})\Big],$$

where, for $x \geq 0$,

$$h(-x) = \mathbb{E}[e^{-rT_x}(X_{T_x} - x)] + \mathbb{E}[e^{-rT_x}] \int_0^\infty \mathbb{E}[e^{-rT_z}]\mathrm{d}z.$$

Now, as before, we split the probability space in $H_0, \dots, H_{m^-}$ and use the fact that, conditionally on the phase in which the upcrossing occurs, the time of overshoot τ_b and the overshoot $Y_{\tau_b} - b$ are independent, to find the decomposition

$$\begin{aligned}\mathbb{E}_{x-b}\Big[e^{-r\tau_b} h(-Y_{\tau_b})\Big] &= \mathbb{E}_{x-b}[e^{-r\tau_b}\mathbf{1}_{H_0}] + \sum_{i=1}^{m^-} \mathbb{E}_{x-b}[e^{-r\tau_b}\mathbf{1}_{H_i}]\mathbb{E}_i[h(-Y_{\tau_b})] \\ &= \boldsymbol{\pi}\boldsymbol{\mathcal{G}},\end{aligned}$$

where $\mathbb{E}_i$ stands for the expectation under $\mathbb{P}$ conditioned on H_i and the theorem is proved. □

Note that if we knew the distribution of the reflected process Y for the Lévy phase-type process we could calculate the Laplace transform of the ruin time using Theorem 2 and the total expected discounted dividends using Theorem 3. This could be the topic of some further research.

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