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Lipschitz Conditions and the Distance Ratio Metric

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Abstract. We give a study of the Lipschitz continuity of Möbius transformations of a punctured ball onto another punctured ball in $\mathbb{R}^n$ with respect to the distance ratio metric. Some subtle methods are developed, helping to determine the best possible j-Lip constant in this case.

1. Introduction and Main Result

During the past thirty years the theory of quasiconformal maps has been studied in various contexts such as in Euclidean, Banach, or even metric spaces. It has turned out that while some classical tools based on conformal invariants, real analysis and measure theory are no longer useful beyond the Euclidean context, the notion of a metric space and related notions still provide a useful conceptual framework. This has led to the study of the geometry defined by various metrics and to the key role of metrics in recent theory of quasiconformality; see e.g. [4, 7–9, 11, 12, 14]. One of these metrics is the distance ratio metric.

Distance ratio metric. For a proper subdomain G of $\mathbb{R}^n$ and for $x, y \in G$ the *distance ratio metric* j_G is defined by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d_G(x), d_G(y)\}}\right), \tag{1.1}$$

where $d_G(x)$ denotes the Euclidean distance from x to the boundary ∂G of the domain G. If $G_1 \subset G$ is a proper subdomain then for $x, y \in G_1$ clearly

$$j_G(x,y) \le j_{G_1}(x,y)\,. \tag{1.2}$$

Moreover, the numerical value of the metric is highly sensitive to boundary variation, the left and right sides of (1.2) are not comparable even if $G_1 = G \setminus \{p\}, p \in G$, since then

$$j_{G\setminus\{p\}}(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G), d(x,p), d(y,p)\}}\right).$$

¿From the above formula it is easy to see that j-metric depends on the boundary of the domain highly. The aim of this article is to discuss the case more thoroughly and to solve a hypothesis from [13] concerning this matter.

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The distance ratio metric was initially introduced by F.W. Gehring and B.P. Palka [6] and in the above simplified form by M. Vuorinen [15]. This metric is frequently used in the study of hyperbolic type metrics and geometric theory of functions. It is a basic fact that the above j-metric is closely related to the hyperbolic metric both for the unit ball $\mathbb{B}^n$ and for the Poincaré half-space $\mathbb{H}^n$ [16].

Quasi-invariance of j_G. Given domains $G, G' \subset \mathbb{R}^n$ and an open continuous mapping $f : G \to G'$ with $fG \subset G'$ we consider the following condition: there exists a constant $C \geq 1$ such that for all $x, y \in G$ we have

$$j_{G'}(f(x), f(y)) \leq C j_G(x, y)\,, \tag{1.3}$$

or, equivalently, that the mapping

$$f : (G, j_G) \to (G', j_{G'})$$

between metric spaces is Lipschitz continuous with the Lipschitz constant C.

Determination of the best possible Lipschitz constants in this sense, which is the aim of this article, is characterized by a series of very subtle inequalities of various kinds and thus intriguing to study.

Möbius transformation. The characterization of Möbius transformation on the unit ball is given by the following assertion.

Lemma 1.1. [3, Theorem 3.5.1, p.40]. *Let f be a Möbius transformation and $f(\mathbb{B}^n) = \mathbb{B}^n$. Then*

$$f(x) = (\sigma x)A,$$

where σ is an inversion in some sphere orthogonal to S^{n-1} and A is an orthogonal matrix.

In the sequel we shall denote $a^* = \frac{a}{|a|^2}$ for $a \in \mathbb{R}^n \setminus \{0\}$, and $0^* = \infty$, $\infty^* = 0$. A sphere centered at a point $a \in \mathbb{R}^n$ and with the radius $r > 0$ is denoted by $S^{n-1}(a, r)$. For a given point $a \in \mathbb{B}^n \setminus \{0\}$, let

$$\sigma_a(x) = a^* + s^2(x - a^*)^*, \ \ s^2 = |a|^{-2} - 1 \tag{1.4}$$

be an inversion in the sphere $S^{n-1}(a^*, s)$ orthogonal to S^{n-1}. Then $\sigma_a(a) = 0, \sigma_a(0) = a$, $\sigma_a(a^*) = \infty$ and

$$|\sigma_a(x) - \sigma_a(y)| = \frac{s^2|x-y|}{|x-a^*||y-a^*|}. \tag{1.5}$$

Since j-metric is invariant under orthogonal transformations, by Lemma 1.1, for $x, y, a \in \mathbb{B}^n$, we have

$$j_{\mathbb{B}^n}(f(x), f(y)) = j_{\mathbb{B}^n}(\sigma_a(x), \sigma_a(y)),$$

where $\sigma_a(x)$ is defined as above.

The hyperbolic metric in the unit ball or the half space is Möbius invariant. However, the distance ratio metric j_G is not invariant under Möbius transformations. Therefore, it is natural to ask what is the Lipschitz constant for this metric under conformal mappings or Möbius transformations in higher dimension. F. W. Gehring and B. G. Osgood proved that this metric is not changed by more than a factor 2 under Möbius transformations, see [5, proof of Theorem 4]:

Theorem 1.2. *If D and D' are proper subdomains of $\mathbb{R}^n$ and if f is a Möbius transformation of D onto D', then for all $x, y \in D$*

$$\frac{1}{2} j_D(x, y) \leq j_{D'}(f(x), f(y)) \leq 2 j_D(x, y).$$

On the other hand, the next theorem from [13], conjectured in [10], yields a sharp form of Theorem 1.2 for Möbius automorphisms of the unit ball.

Theorem 1.3. *A Möbius transformation* $f : \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$, $f(0) = a \in \mathbb{B}^n$, *satisfies*

$$j_{\mathbb{B}^n}(f(x), f(y)) \le (1 + |a|) j_{\mathbb{B}^n}(x, y)$$

for all $x, y \in \mathbb{B}^n$. *The constant* $1 + |a|$ *is best possible.*

A similar result for a punctured disk was conjectured in [13]. The next theorem, our main result, settles this conjecture in the affirmative and gives its generalization in higher dimension.

Main Theorem. *Let* $a \in \mathbb{B}^n$ *and* $h : \mathbb{B}^n \to \mathbb{B}^n = h(\mathbb{B}^n)$ *be a Möbius transformation with* $h(0) = a$. *Then* $h(\mathbb{B}^n \setminus \{0\}) = \mathbb{B}^n \setminus \{a\}$ *and for* $x, y \in \mathbb{B}^n \setminus \{0\}$

$$j_{\mathbb{B}^n \setminus \{a\}}(h(x), h(y)) \le C(a) j_{\mathbb{B}^n \setminus \{0\}}(x, y),$$

where the constant $C(a) = 1 + (\log \frac{2+|a|}{2-|a|}) / \log 3$ *is best possible.*

Clearly the constant $C(a) < 1 + |a| < 2$ for all $a \in \mathbb{B}^n$ and hence the constant $C(a)$ is smaller than the constant $1 + |f(0)|$ in Theorem 1.3 and far smaller than the constant 2 in Theorem 1.2.

If $a = 0$ then h is a rotation of the unit ball and hence an Euclidean isometry. Note that $C(0) = 1$, i.e. the result is sharp in this case. For points other than zero, the sharpness is discussed at the end of the proof of the main theorem.

The proof is based on Theorem 2.1 below and on Lemma 2.3, a monotone form of l'Hôpital's rule from [1, Theorem 1.25].

2. Preliminary Results

In view of the definition of the distance ratio metric it is natural to expect that some properties of the logarithm will be needed. In the earlier paper [13], the classical Bernoulli inequality [16, (3.6)] was applied for this purpose. Apparently now some stronger inequalities are needed and we use the following result, which is precise for the case $1 \le C \le 2$ and allows us to get rid of logarithms in further calculations.

Theorem 2.1. *Let* D *and* D' *be proper subdomains of* $\mathbb{R}^n$. *For an open continuous mapping* $f : D \to D'$ *denote*

$$X = X(z, w) := \frac{|z - w|}{\min\{d_D(z), d_D(w)\}}; \quad Y = Y(z, w) := \frac{|z - w|}{|f(z) - f(w)|} \frac{\min\{d_{D'}(f(z)), d_{D'}(f(w))\}}{\min\{d_D(z), d_D(w)\}}.$$

If there exists a number q, $0 \le q \le 1$, *such that*

$$q \le Y + \frac{Y - 1}{X + 1}, \tag{2.1}$$

then the inequality

$$j_{D'}(f(z), f(w)) \le \frac{2}{1 + q} j_D(z, w),$$

holds for all $z, w \in D$.

Proof. The proof is based on the following assertion.

Lemma 2.2. *For* $a \ge 0, q \in [0, 1]$, *we have*

$$\log\Big(\frac{q + e^a}{1 + qe^a}\Big) \le \frac{1 - q}{1 + q} a \, .$$

Proof. Denote

$$f(a,q) := \log\Big(\frac{q+e^a}{1+qe^a}\Big) - \frac{1-q}{1+q}a\,.$$

By differentiation with respect to the parameter a, we have

$$f_a(a,q) = -\frac{q(1-q)}{1+q}\frac{(e^a-1)^2}{(1+qe^a)(q+e^a)}.$$

Therefore we conclude that

$$f(a,q) \le f(0,q) = 0\,.$$

□

Returning to the proof of Theorem 2.1, observe that

$$X = \frac{|z-w|}{\min\{d_D(z), d_D(w)\}} = \exp(j_D(z,w)) - 1,$$

and

$$Y = \frac{|z-w|}{|f(z)-f(w)|}\frac{\min\{d_{D'}(f(z)), d_{D'}(f(w))\}}{\min\{d_D(z), d_D(w)\}} = \frac{\exp(j_D(z,w))-1}{\exp(j_{D'}(f(z),f(w)))-1}.$$

Hence the condition (2.1) is equivalent to

$$\exp(j_{D'}(f(z),f(w))) \le \exp(j_D(z,w))\Big(\frac{q+e^{j_D(z,w)}}{1+qe^{j_D(z,w)}}\Big).$$

Therefore, by Lemma 2.2, we get

$$j_{D'}(f(z),f(w)) \le j_D(z,w) + \log\Big(\frac{q+e^{j_D(z,w)}}{1+qe^{j_D(z,w)}}\Big)$$

$$\le j_D(z,w) + \frac{1-q}{1+q}j_D(z,w) = \frac{2}{1+q}j_D(z,w)\,.$$

□

In the sequel we shall need the so-called *monotone form of l'Hôpital's rule*.

Lemma 2.3. [1, Theorem 1.25]. *For $-\infty < a < b < \infty$, let f, $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), and let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing(deceasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.3 has found numerous applications recently. See the bibliography of [2] for a long list of applications to inequalities.

Lemma 2.4. *For positive numbers A, B, D and $0 < C < 1, \theta \geq 0$, we have*
1. The inequality

$$1 + \frac{B}{D}\theta\Big(1 + \frac{D}{1+A}\Big)\Big(1 + \frac{B}{1-C}\theta\Big) \leq \Big(1 + \frac{B}{D}\theta\Big)\Big(1 + \frac{B}{1-C}\theta\Big),$$

holds if and only if $B\theta \leq A + C$;
2. The function

$$\frac{\log(1 + \frac{B}{1-C}\theta)}{\log(1 + \frac{B}{D}\theta)}$$

is monotone increasing (decreasing) in θ if $C + D < 1$ $(C + D > 1)$.

Proof. Proof of the first part follows by direct calculation.
For the second part, set

$$f_1(\theta) = \log\Big(1 + \frac{B}{1-C}\theta\Big), f_1(0) = 0; \;\; f_2(\theta) = \log\Big(1 + \frac{B}{D}\theta\Big), f_2(0) = 0.$$

Since

$$\frac{f_1'(\theta)}{f_2'(\theta)} = \frac{D + B\theta}{1 - C + B\theta} = 1 + \frac{C + D - 1}{1 - C + B\theta},$$

the proof follows according to Lemma 2.3. □

Now, we prove our main result.

Proof of the Main Theorem. For the proof, without loss of generality we may assume that $h(z) = \sigma_a(z)$ and suppose in the sequel that $|z| \geq |w|$. Let $G = \mathbb{B}^n \setminus \{0\}$ and $G' = \mathbb{B}^n \setminus \{a\}$. Then

$$j_G(z, w) = \log\Big(1 + \frac{|z - w|}{\min\{|z|, |w|, 1 - |z|, 1 - |w|\}}\Big) = \log\Big(1 + \frac{|z - w|}{\min\{|w|, 1 - |z|\}}\Big),$$

and

$$j_{G'}(\sigma_a(z), \sigma_a(w)) = \log\Big(1 + \frac{|\sigma_a(z) - \sigma_a(w)|}{T}\Big),$$

where

$$T = T_a(z, w) := \min\{|\sigma_a(z) - a|, |\sigma_a(w) - a|, 1 - |\sigma_a(z)|, 1 - |\sigma_a(w)|\}.$$

In concert with the definition of the number T, the proof is divided into four cases. We shall consider each case separately applying Bernoulli inequality in the first case, its stronger form from Theorem 2.1 in the second one and a direct approach in the last two cases.

1. $T = |\sigma_a(z) - a|$.

Since

$$|\sigma_a(z) - a| = |\sigma_a(z) - \sigma_a(0)| = \frac{s^2|z|}{|a^*||z - a^*|}$$

and

$$|\sigma_a(z) - \sigma_a(w)| = \frac{s^2|z - w|}{|z - a^*||w - a^*|},$$

we have

$$j_{G'}(\sigma_a(z), \sigma_a(w)) = \log\Big(1 + \frac{|z - w|}{|a||z||w - a^*|}\Big).$$

Suppose firstly that $|w| \le 1-|z|$. Since also $|w| \le 1-|z| \le 1-|w|$, we conclude that $0 \le |w| \le 1/2$. Hence, by the Bernoulli inequality (see e.g. [16, (3.6)]), we get

$$j_{G'}(\sigma_a(z),\sigma_a(w)) \le \log\Big(1+\frac{|z-w|}{|z|(1-|a||w|)}\Big) \le \log\Big(1+\frac{|z-w|}{|w|(1-\frac{|a|}{2})}\Big)$$

$$\le \frac{1}{1-\frac{|a|}{2}}\log\Big(1+\frac{|z-w|}{|w|}\Big) = \frac{1}{1-\frac{|a|}{2}}\, j_G(z,w).$$

Suppose now $1-|z| \le |w|(\le |z|)$. Then $1/2 \le |z| < 1$.

Since in this case $(|z|-\frac{1}{2})(2-|a|(1+|z|)) \ge 0$, we easily obtain that

$$\frac{1}{|z|(1-|a||z|)} \le \frac{1}{(1-\frac{|a|}{2})(1-|z|)}.$$

Hence,

$$j_{G'}(\sigma_a(z),\sigma_a(w)) \le \log\Big(1+\frac{|z-w|}{|z|(1-|a||w|)}\Big) \le \log\Big(1+\frac{|z-w|}{|z|(1-|a||z|)}\Big)$$

$$\le \log\Big(1+\frac{|z-w|}{(1-\frac{|a|}{2})(1-|z|)}\Big) \le \frac{1}{1-\frac{|a|}{2}}\log\Big(1+\frac{|z-w|}{1-|z|}\Big) = \frac{1}{1-\frac{|a|}{2}}\, j_G(z,w).$$

2. $T = |\sigma_a(w) - a|$.

This case can be treated by means of Theorem 2.1 with the same resulting constant $C_1(a) = \frac{2}{2-|a|}$.
Indeed, in terms of Theorem 2.1, we consider firstly the case $|w| \le 1-|z|$.
We get

$$X = \frac{|z-w|}{|w|} \ge \frac{|z|-|w|}{|w|} = \frac{|z|}{|w|} - 1 = X^*,$$

and

$$Y = \frac{|z-a^*|}{|a^*|} \ge 1-|a||z| = Y^*.$$

Therefore,

$$Y + \frac{Y-1}{X+1} \ge Y^* - \frac{1-Y^*}{1+X^*} = 1-|a|(|w|+|z|) \ge 1-|a| = q\,.$$

In the second case, i.e. when $1-|z| \le |w|$, we want to show that

$$Y + \frac{Y-1}{X+1} \ge 1-|a|$$

which is equivalent to

$$(Y-(1-|a|))(1+X) + Y \ge 1.$$

Since in this case

$$X = \frac{|z-w|}{1-|z|} \ge \frac{|z|-|w|}{1-|z|} := X^*$$

and

$$Y = \frac{|w||z-a^*|}{|a^*|(1-|z|)} \geq (1-|a||z|)\frac{|w|}{1-|z|} = 1-|a||z| + (|w|+|z|-1)\frac{1-|a||z|}{1-|z|} := Y^*,$$

we get

$$(Y-(1-|a|))(1+X)+Y-1 \geq (Y^*-(1-|a|))(1+X^*)+Y^*-1$$
$$= \Big[|a|(1-|z|)+(|w|+|z|-1)\frac{1-|a||z|}{1-|z|}\Big]\frac{1-|w|}{1-|z|} - |a||z| + (|w|+|z|-1)\frac{1-|a||z|}{1-|z|}$$
$$\geq |a|(1-|w|-|z|)+(|w|+|z|-1)\frac{1-|a||z|}{1-|z|} = (|w|+|z|-1)\frac{1-|a|}{1-|z|} \geq 0.$$

Therefore by Theorem 2.1, in both cases we get

$$j_{G'}(\sigma_a(z),\sigma_a(w)) \leq \frac{2}{1+q}j_G(z,w) = \frac{2}{2-|a|}j_G(z,w) = C_1(a)j_G(z,w).$$

3. $T = 1-|\sigma_a(z)|$.

In this case, applying well-known assertions (e.g. [13, Lemma 3.2], [16, Exercise 2.52(2)])

$$|a|^2|z-a^*|^2-|z-a|^2 = (1-|a|^2)(1-|z|^2); \;\; |\sigma_a(z)| \leq \frac{|a|+|z|}{1+|a||z|},$$

and

$$|a||w-a^*| \geq 1-|a||w| (\geq 1-|a||z|),$$

we get

$$j_{G'}(\sigma_a(z),\sigma_a(w)) = \log\Big(1+\frac{|\sigma_a(z)-\sigma_a(w)|}{1-|\sigma_a(z)|}\Big)$$
$$= \log\Big(1+\frac{|a|s^2|z-w|}{|w-a^*|(|a||z-a^*|-|z-a|)}\Big) = \log\Big(1+\frac{|z-w|(|a||z-a^*|+|z-a|)}{|a||w-a^*|(1-|z|^2)}\Big)$$
$$= \log\Big(1+\frac{|z-w|}{1-|z|^2}\left|\frac{z-a^*}{w-a^*}\right|\Big(1+\frac{|z-a|}{|a||z-a^*|}\Big)\Big)$$
$$\leq \log\Big(1+\frac{|z-w|}{1-|z|^2}\Big(1+\frac{|z-w|}{|w-a^*|}\Big)\Big(1+\frac{|a|+|z|)}{1+|a||z|}\Big)\Big)$$
$$\leq \log\Big(1+\frac{|z-w|}{1-|z|}\Big(1+\frac{|a||z-w|}{1-|a||z|}\Big)\Big(1+\frac{|a|(1-|z|)}{1+|a||z|}\Big)\Big).$$

Applying here Lemma 2.4, part 1., with

$$A = |a||z|,\; B = |a|,\; C = |a||w|, D = |a|(1-|z|),\; \theta = |z-w|,$$

we obtain

$$j_{G'}(\sigma_a(z),\sigma_a(w)) \leq \log\Big[\Big(1+\frac{|z-w|}{1-|z|}\Big)\Big(1+\frac{|a||z-w|}{1-|a||w|}\Big)\Big]. \tag{2.2}$$

Suppose that $1-|z|\le|w| \quad (\le |z|)$. By Lemma 2.4, part 2., with

$$B=|a|,\ C=|a||z|,\ D=|a|(1-|z|),\ \theta=|z-w|,$$

we get

$$J(z,w;a):=\frac{j_{G'}(\sigma_a(z),\sigma_a(w))}{j_G(z,w)}\le 1+\frac{\log(1+\frac{|a||z-w|}{1-|a||z|})}{\log(1+\frac{|z-w|}{1-|z|})}$$

$$\le 1+\frac{\log(1+\frac{2|a||z|}{1-|a||z|})}{\log(1+\frac{2|z|}{1-|z|})},$$

because in this case we have $C+D=|a|<1$ and $|z-w|\le 2|z|$.

Since the last function is monotonically decreasing in $|z|$ and $|z|\ge 1/2$, we obtain

$$J(z,w;a)\le 1+\frac{\log(\frac{1+\frac{1}{2}|a|}{1-\frac{1}{2}|a|})}{\log(\frac{1+\frac{1}{2}}{1-\frac{1}{2}})}=1+(\log\frac{2+|a|}{2-|a|})/\log 3:=C_2(a).$$

Let us now consider $|w|\le 1-|z|(\le 1-|w|)$. The estimate (2.2) and Lemma 2.4, part 2., with

$$B=|a|,\ C=D=|a||w|,\ \theta=|z-w|,$$

yield

$$J(z,w;a)\le\frac{\log\Big[\Big(1+\frac{|z-w|}{1-|z|}\Big)\Big(1+\frac{|a||z-w|}{1-|a||w|}\Big)\Big]}{\log\Big(1+\frac{|z-w|}{|w|}\Big)}\le\frac{\log\Big[\Big(1+\frac{|z-w|}{|w|}\Big)\Big(1+\frac{|a||z-w|}{1-|a||w|}\Big)\Big]}{\log\Big(1+\frac{|z-w|}{|w|}\Big)}$$

$$=1+\frac{\log\Big(1+\frac{|a||z-w|}{1-|a||w|}\Big)}{\log\Big(1+\frac{|z-w|}{|w|}\Big)}\le 1+\frac{\log\Big(1+\frac{|a|}{1-|a||w|}\Big)}{\log\Big(1+\frac{1}{|w|}\Big)},$$

since $C+D=2|a||w|\le|a|<1$ and $0\le|z-w|\le|z|+|w|\le 1$.

Denote the last function as $g(|w|)$ and let $|w|=r,\ 0<r\le 1/2$. Since

$$g'(r)=\frac{|a|^2}{(1-r|a|)(1+(1-r)|a|)\log(1+1/r)}+\frac{\log\Big(1+\frac{|a|}{1-|a|r}\Big)}{r(1+r)\log^2(1+1/r)}>0,$$

it follows that $g(r)$ is a monotonically increasing function and we finally obtain

$$J(z,w;a)\le 1+\frac{\log\Big(1+\frac{|a|}{1-|a|/2}\Big)}{\log(1+2)}=C_2(a).$$

4. $T=1-|\sigma_a(w)|$.

This case can be treated analogously with the previous one. Now,

$$j_{G'}(\sigma_a(z),\sigma_a(w))=\log\Big(1+\frac{|a|s^2|z-w|}{|z-a^*|(|a||w-a^*|-|w-a|)}\Big)=\log\Big(1+\frac{|z-w|(|a||w-a^*|+|w-a|)}{|a||z-a^*|(1-|w|^2)}\Big)$$

$$\leq \log\Big(1 + \frac{|z-w|}{1-|w|}\Big(1 + \frac{|a||z-w|}{1-|a||z|}\Big)\Big(1 + \frac{|a|(1-|w|)}{1+|a||w|}\Big)\Big).$$

Applying Lemma 2.4, part 1., with

$$A = |a||w|,\ B = |a|,\ C = |a||z|, D = |a|(1-|w|),\ \theta = |z-w|,$$

we obtain

$$j_{G'}(\sigma_a(z), \sigma_a(w)) \leq \log\Big[\Big(1 + \frac{|z-w|}{1-|w|}\Big)\Big(1 + \frac{|a||z-w|}{1-|a||z|}\Big)\Big]. \tag{2.3}$$

Suppose that $1-|z| \leq |w| (\leq |z|)$. We get

$$J(z,w;a) := \frac{j_{G'}(\sigma_a(z), \sigma_a(w))}{j_G(z,w)} \leq \frac{\log\Big[\Big(1 + \frac{|z-w|}{1-|z|}\Big)\Big(1 + \frac{|a||z-w|}{1-|a||z|}\Big)\Big]}{\log\Big(1 + \frac{|z-w|}{1-|z|}\Big)}$$

$$= 1 + \frac{\log(1 + \frac{|a||z-w|}{1-|a||z|})}{\log\Big(1 + \frac{|z-w|}{1-|z|}\Big)}$$

and this inequality is already considered above.

In the case $|w| \leq 1-|z| \leq 1-|w|$, we have

$$J(z,w;a) \leq \frac{\log\Big[\Big(1 + \frac{|z-w|}{1-|w|}\Big)\Big(1 + \frac{|a||z-w|}{1-|a||z|}\Big)\Big]}{\log\Big(1 + \frac{|z-w|}{|w|}\Big)} \leq \frac{\log\Big[\Big(1 + \frac{|z-w|}{|w|}\Big)\Big(1 + \frac{|a||z-w|}{1-|a|(1-|w|)}\Big)\Big]}{\log\Big(1 + \frac{|z-w|}{|w|}\Big)}$$

$$= 1 + \frac{\log\Big(1 + \frac{|a||z-w|}{1-|a|(1-|w|)}\Big)}{\log\Big(1 + \frac{|z-w|}{|w|}\Big)} \leq 1 + \frac{\log\Big(1 + \frac{|a|}{1-|a|(1-|w|)}\Big)}{\log\Big(1 + \frac{1}{|w|}\Big)},$$

where the last inequality follows from Lemma 2.4, part 2., with

$$B = |a|,\ C = |a|(1-|w|),\ D = |a||w|,\ \theta = |z-w|,$$

since $C + D = |a| < 1$ and $|z-w| \leq |z| + |w| \leq 1$.

Denote now $|w| = r$ and let $k(r) = k_1(r)/k_2(r)$ with

$$k_1(r) = \log\Big(1 + \frac{|a|}{1-|a|(1-r)}\Big);\ k_2(r) = \log\Big(1 + \frac{1}{r}\Big).$$

We shall show now that the function $k(r)$ is monotonically increasing on the positive part of real axis. Indeed, since $k_1(\infty) = k_2(\infty) = 0$ and

$$k_1'(r)/k_2'(r) = \frac{|a|^2 r(1+r)}{(1+|a|r)(1-|a|+|a|r)} = \frac{|a|(1+r)}{1+|a|r}\frac{|a|r}{1-|a|+|a|r}$$

$$= \Big(1 - \frac{1-|a|}{1+|a|r}\Big)\Big(1 - \frac{1-|a|}{1-|a|+|a|r}\Big),$$

with both functions in parenthesis evidently increasing on $\mathbb{R}^+$, the conclusion follows from Lemma 2.3.

Since in this case $0 < r \leq 1/2$, we also obtain that

$$J(z,w;a) \le 1 + \frac{\log\Big(1+\frac{|a|}{1-|a|/2}\Big)}{\log\Big(1+2\Big)} = C_2(a).$$

To verify that the constant $C_2(a)$ is sharp, we firstly calculate $T_a(\frac{a}{2|a|}, \frac{-a}{2|a|}) = \frac{1-|a|}{2+|a|}$. Then, it follows that

$$J\left(\frac{a}{2|a|}, \frac{-a}{2|a|}; a\right) = C_2(a).$$

Because $C_2(a) = 1 + (\log \frac{2+|a|}{2-|a|})/\log 3 > \frac{1}{1-\frac{|a|}{2}} = C_1(a)$, we conclude that the best possible upper bound C is $C = C_2(a)$. □

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