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Inclusion and Argument Properties for the Srivastava-Khairnar-More Operator

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Abstract. In the present paper, we derive several inclusion relationships and argument properties for certain classes of multivalent analytic functions associated with a family of linear operators involving the Srivastava-Khairnar-More operator. Furthermore, some invariant properties under convolution with convex functions for these classes are investigated. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}.$$

For simplicity, we write $\mathcal{A}_1 = \mathcal{A}$.

Let $f, g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}.$$

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Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \ \ (z \in \mathbb{U}),$$

such that

$$f(z) = g(\omega(z)) \ \ (z \in \mathbb{U}).$$

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see, for details, [3,10]; see also [20]):

$$f(z) \prec g(z) \ \ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad and \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let M be the class of functions $\phi(z)$ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\Re[\phi(z)] > 0$ for $z \in \mathbb{U}$.

By making use of the principle of subordination between analytic functions, Ma and Minda [9] introduced the subclasses $\mathcal{S}_p^*(\varrho;\phi), \mathcal{K}_p(\varrho;\phi), C_p(\varrho,\sigma;\phi,\psi)$ and $QC_p(\varrho,\sigma;\phi,\psi)$ of the class $\mathcal{A}_p$ for $p \in \mathbb{N}$, $0 \le \varrho, \sigma < p$ and $\phi, \psi \in M$, which are defined by

$$\mathcal{S}_p^*(\varrho;\phi) = \left\{ f \in \mathcal{A}_p : \ \frac{1}{p-\varrho}\left(\frac{zf'(z)}{f(z)} - \varrho\right) \prec \phi(z) \ \ in \ \ \mathbb{U} \right\},$$

$$\mathcal{K}_p(\varrho;\phi) = \left\{ f \in \mathcal{A}_p : \ \frac{1}{p-\varrho}\left(1 + \frac{zf''(z)}{f'(z)} - \varrho\right) \prec \phi(z) \ \ in \ \ \mathbb{U} \right\},$$

$$C_p(\varrho,\sigma;\phi,\psi) = \left\{ f \in \mathcal{A}_p : \ \exists\, g \in \mathcal{S}_p^*(\varrho;\phi) \ \ such\ that \ \ \frac{1}{p-\sigma}\left(\frac{zf'(z)}{g(z)} - \sigma\right) \prec \psi(z) \ \ in \ \ \mathbb{U} \right\},$$

and

$$QC_p(\varrho,\sigma;\phi,\psi) = \left\{ f \in \mathcal{A}_p : \ \exists\, g \in \mathcal{K}_p(\varrho;\phi) \ \ such\ that \ \ \frac{1}{p-\sigma}\left(\frac{(zf'(z))'}{g'(z)} - \sigma\right) \prec \psi(z) \ \ in \ \ \mathbb{U} \right\}.$$

We observe that, for special choices for the parameters p, ϱ, σ, and the functions ϕ and ψ involved in these definitions, we can obtain the well-known subclasses of $\mathcal{A}_p$. For example, the classes

$$\mathcal{S}_1^*\left(0; \frac{1+z}{1-z}\right) = \mathcal{S}^*, \ \ \mathcal{K}_1\left(0; \frac{1+z}{1-z}\right) = \mathcal{K},$$

$$C_1\left(0,0; \frac{1+z}{1-z}, \frac{1+z}{1-z}\right) = C, \ \ QC_1\left(0,0; \frac{1+z}{1-z}, \frac{1+z}{1-z}\right) = QC$$

which are starlike, convex, close-to-convex and quasi-convex function in $\mathbb{U}$, respectively.

For parameters

$$a, b \in \mathbb{C} \ \ and \ \ c \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \ (\mathbb{Z}_0^- = \{0, -1, -2, \cdots\}),$$

the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$$ {}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{1.2}$$

where $(\nu)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$(\nu)_k=\frac{\Gamma(\nu+k)}{\Gamma(\nu)}=\begin{cases}1 & (k=0;\nu\in\mathbb{C}\setminus\{0\}),\\ \nu(\nu+1)\cdots(\nu+k-1) & (k\in\mathbb{N};\nu\in\mathbb{C}).\end{cases}$$

The hypergeometric series in (1.2) converges absolutely for all $z\in\mathbb{U}$, so that it represents an analytic function in $\mathbb{U}$. Dziok and Srivastava [4] (see also [5,6]) considered the generalized hypergeometric function ${}_qF_s$ $(q,s\in\mathbb{N}\cup\{0\})$, which is a certain generalization of (1.2).

We now introduce a function $f^{\delta}_{\mu,p}(a,b,c)(z)$ defined by

$$f^{\delta}_{\mu,p}(a,b,c)(z)=(1-\mu+\delta)z^p\cdot{}_2F_1(a,b;c;z)+(\mu-\delta)z[z^p\cdot{}_2F_1(a,b;c;z)]'+\mu\delta z^2[z^p\cdot{}_2F_1(a,b;c;z)]''$$

$$(z\in\mathbb{U};\ \mu,\delta\geq 0). \tag{1.3}$$

We note that, for $p=1$ and $\delta=0$, we have $f^{0}_{\mu,1}(a,b,c)(z)=f_{\mu}(a,b,c)(z)$, which was studied by Skukla and Skukla [16], and for $\mu=\delta=0$ and $b=1$, we obtain

$$f^{0}_{0,p}(a,1,c)(z)=\phi_p(a,c)(z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}z^{k+p},$$

which was introduced by Saitoh [15].

Next, we introduce the following family of linear operators $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a,b,c):\ \mathcal{A}_p\to\mathcal{A}_p$, defined by

$$\mathcal{I}^{\lambda,\delta}_{\mu,p}(a,b,c)f(z)=f^{\lambda,\delta}_{\mu,p}(a,b,c)(z)*f(z)\ \ (\lambda>-p;\ \mu,\delta\geq 0;\ z\in\mathbb{U}), \tag{1.4}$$

where $f^{\lambda,\delta}_{\mu,p}(a,b,c)(z)$ is the function defined in terms of the Hadamard product (or convolution) as follows:

$$f^{\delta}_{\mu,p}(a,b,c)(z)*f^{\lambda,\delta}_{\mu,p}(a,b,c)(z)=\frac{z^p}{(1-z)^{\lambda+p}}\ \ (\lambda>-p;\ \mu,\delta\geq 0), \tag{1.5}$$

where $f^{\delta}_{\mu,p}(a,b,c)(z)$ is given by (1.3).

We also note that the operator $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a,b,c)$ generalizes several previously studied familiar operators, and we will show some of the interesting particular cases as follows.

(i) $\mathcal{I}^{\lambda,0}_{\mu,1}(a,b,c)=\mathcal{I}^{\lambda}_{\mu}(a,b,c)$, where $\mathcal{I}^{\lambda}_{\mu}(a,b,c)$ is the Srivastava-Khairnar-More operator [19];

(ii) $\mathcal{I}^{\lambda,0}_{0,1}(a,b,c)=\mathcal{I}_{\lambda}(a,b,c)$, where the operator $\mathcal{I}_{\lambda}(a,b,c)$ was introduced by Noor [12];

(iii) $\mathcal{I}^{\lambda,0}_{0,p}(a,1,c)=\mathcal{I}^{\lambda}_{p}(a,c)$, where $\mathcal{I}^{\lambda}_{p}(a,c)$ is the Cho-Kwon-Srivastava operator [1];

(iv) $\mathcal{I}^{n,0}_{0,1}(a,n+1,a)=\mathcal{I}_n$, where $\mathcal{I}_n$ is the Noor integral operator [11].

Since

$$\frac{z^p}{(1-z)^{\lambda+p}}=\sum_{k=0}^{\infty}\frac{(\lambda+p)_k}{k!}z^{k+p}\ \ \ (\lambda>-p;\ z\in\mathbb{U}), \tag{1.6}$$

by using (1.2), (1.3) and (1.6) in (1.5), we get

$$\left(\sum_{k=0}^{\infty} \frac{[1+(k+p-1)(\mu\delta(k+p)+\mu-\delta)](a)_k(b)_k}{(c)_k} \frac{z^{k+p}}{k!}\right) * f_{\mu,p}^{\lambda,\delta}(a,b,c)(z) = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k}{k!} z^{k+p}.$$

Therefore the function $f_{\mu,p}^{\lambda,\delta}(a,b,c)(z)$ has the following explicit form

$$f_{\mu,p}^{\lambda,\delta}(a,b,c)(z) = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k(c)_k}{[1+(k+p-1)(\mu\delta(k+p)+\mu-\delta)](a)_k(b)_k} z^{k+p} \quad (z \in \mathbb{U}). \tag{1.7}$$

Combining (1.1), (1.4), together with (1.7), we have

$$\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{[1+(k+p-1)(\mu\delta(k+p)+\mu-\delta)](a)_k(b)_k} a_{k+p} z^{k+p} \quad (z \in \mathbb{U}).$$

In particular, we have

$$\mathcal{I}_{0,p}^{\lambda,0}(a,\lambda+p,a)f(z) = f(z) \quad \textit{and} \quad \mathcal{I}_{0,p}^{1,0}(a,p,a)f(z) = \frac{zf'(z)}{p}.$$

It can also be easily shown that

$$z\left[\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)\right]' = (\lambda+p)\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z) - \lambda \mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) \tag{1.8}$$

and

$$z\left[\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)\right]' = a\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) - (a-p)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z). \tag{1.9}$$

By using the operator $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)$, we introduce the following subclasses $\mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$, $\mathcal{K}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$, $C_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi)$ and $QC_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi)$ of the class $\mathcal{A}_p$:

$$\mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi) = \left\{f \in \mathcal{A}_p : \ \mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) \in \mathcal{S}_p^*(\varrho;\phi)\right\},$$

$$\mathcal{K}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi) = \left\{f \in \mathcal{A}_p : \ \mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) \in \mathcal{K}_p(\varrho;\phi)\right\},$$

$$C_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi) = \left\{f \in \mathcal{A}_p : \ \mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) \in C_p(\varrho,\sigma;\phi,\psi)\right\},$$

and

$$QC_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi) = \left\{f \in \mathcal{A}_p : \ \mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) \in QC_p(\varrho,\sigma;\phi,\psi)\right\}.$$

It is easy to verify that

$$f \in \mathcal{K}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi), \tag{1.10}$$

and

$$f \in QC_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi) \Longleftrightarrow \frac{zf'(z)}{p} \in C_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi). \tag{1.11}$$

As a special case, when $p = 1$ and $\delta = 0$, we obtain

$$\mathcal{S}_{\mu,1}^{\lambda,0}(a,b,c;\varrho)(\phi) = \mathcal{S}_{\mu}^{\lambda}(a,b,c;\varrho)(\phi), \ \ \mathcal{K}_{\mu,1}^{\lambda,0}(a,b,c;\varrho)(\phi) = \mathcal{K}_{\mu}^{\lambda}(a,b,c;\varrho)(\phi),$$

$$\mathcal{C}_{\mu,1}^{\lambda,0}(a,b,c;\varrho,\sigma)(\phi,\psi)=\mathcal{C}_{\mu}^{\lambda}(a,b,c;\varrho,\sigma)(\phi,\psi),\ \ \mathcal{QC}_{\mu,1}^{\lambda,0}(a,b,c;\varrho,\sigma)(\phi,\psi)=\mathcal{QC}_{\mu}^{\lambda}(a,b,c;\varrho,\sigma)(\phi,\psi),$$

which were introduced and investigated recently by Wang et al.[21]. Further, for $p=1$ and $\varrho=\delta=\sigma=0$, we have

$$\mathcal{S}_{\mu,1}^{\lambda,0}(a,b,c;0)(\phi)=\mathcal{S}_{\mu}^{\lambda}(a,b,c)(\phi),\ \ \mathcal{K}_{\mu,1}^{\lambda,0}(a,b,c;0)(\phi)=\mathcal{K}_{\mu}^{\lambda}(a,b,c)(\phi),$$

and

$$\mathcal{C}_{\mu,1}^{\lambda,0}(a,b,c;0,0)(\phi,\psi)=\mathcal{C}_{\mu}^{\lambda}(a,b,c)(\phi,\psi),$$

which were introduced and investigated recently by Srivastava et al.[19].

For the sake of convenience, we write

$$\mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)\left(\frac{1+Az}{1+Bz}\right)=\mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho;A,B)\ \ (-1\le B<A\le 1),$$

$$\mathcal{K}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)\left(\frac{1+Az}{1+Bz}\right)=\mathcal{K}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho;A,B)\ \ (-1\le B<A\le 1),$$

$$\mathcal{C}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)\left(\frac{1+Az}{1+Bz};\frac{1+Az}{1+Bz}\right)=\mathcal{C}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma;A,B)\ \ (-1\le B<A\le 1),$$

and

$$\mathcal{QC}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)\left(\frac{1+Az}{1+Bz};\frac{1+Az}{1+Bz}\right)=\mathcal{QC}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma;A,B)\ \ (-1\le B<A\le 1).$$

In the present paper, we aim at proving various inclusion relationships among the classes $\mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$, $\mathcal{K}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$, $\mathcal{C}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi)$ and $\mathcal{QC}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi)$ and argument results of p-valent analytic functions, which are defined by the operator $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)$. Some interesting applications involving these and other families of integral operators are also derived.

2. Inclusion Properties Involving the Operator $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)$

The following lemmas will be required in our investigation.

Lemma 2.1. *Let $f_{\mu,p}^{\lambda_i,\delta}(a,b,c)(z), f_{\mu,p}^{\lambda,\delta}(a_i,b,c)(z), f_{\mu,p}^{\lambda,\delta}(a,b_i,c)(z)$ and $f_{\mu,p}^{\lambda,\delta}(a,b,c_i)(z)$ be defined by (1.7). Then, for $\lambda_i>-p;\ a_i,b_i,c_i\in\mathbb{R}\setminus\mathbb{Z}_0^-$ ($\mathbb{Z}_0^-=\{0,-1,-2,\cdots\}$) $(i=1,2)$ and $\mu,\delta\ge 0$,*

$$f_{\mu,p}^{\lambda_2,\delta}(a,b,c)(z)=f_{\mu,p}^{\lambda_1,\delta}(a,b,c)(z)*\phi_p(\lambda_2+p,\lambda_1+p)(z), \tag{2.1}$$

$$f_{\mu,p}^{\lambda,\delta}(a_1,b,c)(z)=f_{\mu,p}^{\lambda,\delta}(a_2,b,c)(z)*\phi_p(a_2,a_1)(z), \tag{2.2}$$

$$f_{\mu,p}^{\lambda,\delta}(a,b_1,c)(z)=f_{\mu,p}^{\lambda,\delta}(a,b_2,c)(z)*\phi_p(b_2,b_1)(z), \tag{2.3}$$

and

$$f_{\mu,p}^{\lambda,\delta}(a,b,c_1)(z)=f_{\mu,p}^{\lambda,\delta}(a,b,c_2)(z)*\phi_p(c_1,c_2)(z), \tag{2.4}$$

where

$$\phi_p(\alpha,\beta)(z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}z^{k+p}\ \ (z\in\mathbb{U}).$$

Proof. From (1.7), we have

$$\begin{aligned} f_{\mu,p}^{\lambda_2,\delta}(a,b,c)(z) &= \sum_{k=0}^{\infty} \frac{(\lambda_2+p)_k(c)_k}{[1+(k+p-1)(\mu\delta(k+p)+\mu-\delta)](a)_k(b)_k} z^{k+p} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda_1+p)_k(c)_k}{[1+(k+p-1)(\mu\delta(k+p)+\mu-\delta)](a)_k(b)_k} \cdot \frac{(\lambda_2+p)_k}{(\lambda_1+p)_k} z^{k+p} \\ &= f_{\mu,p}^{\lambda_1,\delta}(a,b,c)(z) * \phi_p(\lambda_2+p,\lambda_1+p)(z) \end{aligned}$$

and the assertion (2.1) is proved. The proof of (2.2)-(2.4) is similar to that of (2.1) and the details involved may be omitted. □

Lemma 2.2 (see [13]). *Let $f \in \mathcal{K}$ and $g \in \mathcal{S}^*$. Then, for every analytic function W in $\mathbb{U}$,*

$$\frac{(f * Wg)(\mathbb{U})}{(f * g)(\mathbb{U})} \subset \overline{co}[W(\mathbb{U})],$$

where $\overline{co}[W(\mathbb{U})]$ denotes the closed convex hull of $W(\mathbb{U})$.

Lemma 2.3 (see [18]). *Let $0 < \alpha \le \beta$. If $\beta \ge 2$ or $\alpha + \beta \ge 3$, then the function*

$$\phi_1(\alpha,\beta)(z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} z^{k+1} \quad (z \in \mathbb{U})$$

belongs to the class $\mathcal{K}$ of convex functions.

We begin by proving our first inclusion relationship given by Theorem 2.1 below.

Theorem 2.1. *Let $p \in \mathbb{N}$, $0 \le \varrho < p$, $\mu, \delta \ge 0$ and $\phi \in M$ with*

$$\Re[\phi(z)] > 1 - \frac{1}{p-\varrho} \quad (z \in \mathbb{U}). \tag{2.5}$$

If λ_i and a_i $(i = 1, 2)$ satisfy the following conditions:

$$(i) \ -p < \lambda_2 \le \lambda_1 \ \text{ and } \ \lambda_1 \ge \min\{2-p, 3-2p-\lambda_2\}, \tag{2.6}$$

and

$$(ii) \ 0 < a_2 \le a_1 \ \text{ and } \ a_1 \ge \min\{2, 3-a_2\}, \tag{2.7}$$

then

$$\mathcal{S}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c;\varrho)(\phi) \subset \mathcal{S}_{\mu,p}^{\lambda_2,\delta}(a_2,b,c;\varrho)(\phi) \subset \mathcal{S}_{\mu,p}^{\lambda_2,\delta}(a_1,b,c;\varrho)(\phi).$$

Proof. First of all, we will show that

$$\mathcal{S}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c;\varrho)(\phi) \subset \mathcal{S}_{\mu,p}^{\lambda_2,\delta}(a_2,b,c;\varrho)(\phi). \tag{2.8}$$

Let $f \in \mathcal{S}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c;\varrho)(\phi)$. Then, by the definition of the class $\mathcal{S}_{\mu,p}^{\lambda_1,\delta}(a,b,c;\varrho)(\phi)$, we have

$$\frac{1}{p-\varrho}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z)} - \varrho\right) = \phi(\omega(z)),$$

where ϕ is convex univalent with $\Re[\phi(z)] > 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ with $\omega(0) = 0 = \phi(0) - 1$. Therefore,

$$\frac{z(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z)} = (p-\varrho)\phi(\omega(z)) + \varrho \tag{2.9}$$

and

$$\frac{z[z^{1-p}(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))]'}{z^{1-p}(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))} = (p-\varrho)\phi(\omega(z)) + \varrho - p + 1 \prec \frac{1+z}{1-z}. \tag{2.10}$$

Applying (1.4), (2.1) and the properties of convolution, we obatin

$$\begin{aligned}\frac{z(\mathcal{I}_{\mu,p}^{\lambda_2,\delta}(a_2,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda_2,\delta}(a_2,b,c)f(z)} &= \frac{z[(f_{\mu,p}^{\lambda_2,\delta}(a_2,b,c)*f)(z)]'}{(f_{\mu,p}^{\lambda_2,\delta}(a_2,b,c)*f)(z)}\\
&= \frac{z[(f_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)*\phi_p(\lambda_2+p,\lambda_1+p)*f)(z)]'}{(f_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)*\phi_p(\lambda_2+p,\lambda_1+p)*f)(z)}\\
&= \frac{\phi_p(\lambda_2+p,\lambda_1+p)(z)*z[(f_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)*f)(z)]'}{\phi_p(\lambda_2+p,\lambda_1+p)(z)*(f_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)*f)(z)}\\
&= \frac{\phi_p(\lambda_2+p,\lambda_1+p)(z)*z(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))'}{\phi_p(\lambda_2+p,\lambda_1+p)(z)*\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z)}.\end{aligned} \tag{2.11}$$

Thus, by using (2.9) and (2.11), we get

$$\begin{aligned}\frac{1}{p-\varrho}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda_2,\delta}(a_2,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda_2,\delta}(a_2,b,c)f(z)} - \varrho\right) &= \frac{1}{p-\varrho}\left(\frac{\phi_p(\lambda_2+p,\lambda_1+p)(z)*z(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))'}{\phi_p(\lambda_2+p,\lambda_1+p)(z)*\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z)} - \varrho\right)\\
&= \frac{1}{p-\varrho}\left(\frac{\phi_p(\lambda_2+p,\lambda_1+p)(z)*[(p-\varrho)\phi(\omega(z))+\varrho]\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z)}{\phi_p(\lambda_2+p,\lambda_1+p)(z)*\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z)} - \varrho\right).\end{aligned} \tag{2.12}$$

It follows from (2.5) and (2.10) that $z^{1-p}\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z) \in \mathcal{S}^*$. Also, by Lemma 2.3, we see that $z^{1-p}\phi_p(\lambda_2 + p,\lambda_1+p)(z) \in \mathcal{K}$.

Let us define

$$s(\omega(z)) = (p-\varrho)\phi(\omega(z)) + \varrho. \tag{2.13}$$

Then, in view of (2.13) and Lemma 2.2, we have

$$\frac{\{[z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)]*s(\omega)z^{1-p}\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f\}(\mathbb{U})}{\{[z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)]*z^{1-p}\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f\}(\mathbb{U})} \subset \overline{co}s[\omega(\mathbb{U})], \tag{2.14}$$

because s is convex univalent function.

Combining (2.12) and (2.14), we conclude that

$$\frac{1}{p-\varrho}\left(\frac{[\phi_p(\lambda_2+p,\lambda_1+p)*s(\omega)\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f](\mathbb{U})}{[\phi_p(\lambda_2+p,\lambda_1+p)*\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f](\mathbb{U})} - \varrho\right) \subseteq \phi(\mathbb{U}),$$

and hence (2.12) is subordinate to ϕ in $\mathbb{U}$, and that is $f \in \mathcal{S}_{\mu,p}^{\lambda_2,\delta}(a_2,b,c;\varrho)(\phi)$. Thus, the assertion (2.8) of Theorem 2.1 holds true. Moreover, by using the arguments similar to those detailed above with (2.2), we can prove the second part of Theorem 2.1 also holds true. The proof of Theorem 2.1 is evidently completed. □

Theorem 2.2. *Let* $0 \le \varrho < p,\ \lambda > -p,\ \mu,\delta \ge 0$ *and* $\phi \in M$ *with (2.5) holds. If* b_i *and* c_i $(i = 1,2)$ *satisfy the following conditions:*

$$(i)\ 0 < b_2 \le b_1 \ \ and\ \ b_1 \ge \min\{2, 3 - b_2\}, \tag{2.15}$$

and

$$(ii)\ 0 < c_1 \le c_2 \ \ and\ \ c_2 \ge \min\{2, 3 - c_1\}, \tag{2.16}$$

then

$$\mathcal{S}^{\lambda,\delta}_{\mu,p}(a, b_2, c_2; \varrho)(\phi) \subset \mathcal{S}^{\lambda,\delta}_{\mu,p}(a, b_1, c_2; \varrho)(\phi) \subset \mathcal{S}^{\lambda,\delta}_{\mu,p}(a, b_1, c_1; \varrho)(\phi).$$

Proof. Applying the same techniques as in the proof of Theorem 2.1, and using (2.3), (2.4), in conjunction with Lemmas 2.2 and 2.3, we obtain the result asserted by Theorem 2.2. □

Theorem 2.3. *Let* $0 \le \varrho < p,\ \mu,\delta \ge 0$ *and* $\phi \in M$ *with (2.5) holds. If* λ_i *satisfies (2.6), and* a_i *satisfies (2.7),* $i = 1,2$, *then*

$$\mathcal{K}^{\lambda_1,\delta}_{\mu,p}(a_2, b, c; \varrho)(\phi) \subset \mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_2, b, c; \varrho)(\phi) \subset \mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_1, b, c; \varrho)(\phi).$$

Proof. In view of (1.10) and Theorem 2.1, we observe that

$$\begin{aligned} f \in \mathcal{K}^{\lambda_1,\delta}_{\mu,p}(a_2, b, c; \varrho)(\phi) &\Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^{\lambda_1,\delta}_{\mu,p}(a_2, b, c; \varrho)(\phi) \\ &\Longrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^{\lambda_2,\delta}_{\mu,p}(a_2, b, c; \varrho)(\phi) \\ &\Longleftrightarrow f \in \mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_2, b, c; \varrho)(\phi) \end{aligned}$$

and

$$\begin{aligned} f \in \mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_2, b, c; \varrho)(\phi) &\Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^{\lambda_2,\delta}_{\mu,p}(a_2, b, c; \varrho)(\phi) \\ &\Longrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^{\lambda_2,\delta}_{\mu,p}(a_1, b, c; \varrho)(\phi) \\ &\Longleftrightarrow f \in \mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_1, b, c; \varrho)(\phi), \end{aligned}$$

which evidently proves Theorem 2.3. □

Similarly, we can derive Theorem 2.4 below by applying (1.10) and Theorem 2.2.

Theorem 2.4. *Let* $0 \le \varrho < p,\ \lambda > -p,\ \mu,\delta \ge 0$ *and* $\phi \in M$ *with (2.5) holds. If* b_i *satisfies (2.15), and* c_i *satisfies (2.16),* $i = 1,2$, *then*

$$\mathcal{K}^{\lambda,\delta}_{\mu,p}(a, b_2, c_2; \varrho)(\phi) \subset \mathcal{K}^{\lambda,\delta}_{\mu,p}(a, b_1, c_2; \varrho)(\phi) \subset \mathcal{K}^{\lambda,\delta}_{\mu,p}(a, b_1, c_1; \varrho)(\phi).$$

Upon setting

$$\phi(z)=\frac{1+Az}{1+Bz}\quad(-1\le B<A\le 1;\ z\in\mathbb{U})$$

in Theorems 2.1-2.4, we have the following result.

Corollary 2.1. *Let* $p\in\mathbb{N},\ 0\le\varrho<p,\ \mu,\delta\ge 0$ *and*

$$\Re\left(\frac{1+Az}{1+Bz}\right)>1-\frac{1}{p-\varrho}\quad(-1\le B<A\le 1;\ z\in\mathbb{U}).$$

If $\lambda_i,\ a_i,\ b_i$ *and* c_i $(i=1,2)$ *satisfy (2.6), (2.7), (2.15) and (2.16), respectively, then*

$$\mathcal{S}^{\lambda_1,\delta}_{\mu,p}(a_2,b_2,c_2;\varrho;A,B)\subset\mathcal{S}^{\lambda_2,\delta}_{\mu,p}(a_2,b_2,c_2;\varrho;A,B)\subset\mathcal{S}^{\lambda_2,\delta}_{\mu,p}(a_1,b_2,c_2;\varrho;A,B)$$
$$\subset\mathcal{S}^{\lambda_2,\delta}_{\mu,p}(a_1,b_1,c_2;\varrho;A,B)\subset\mathcal{S}^{\lambda_2,\delta}_{\mu,p}(a_1,b_1,c_1;\varrho;A,B)$$

and

$$\mathcal{K}^{\lambda_1,\delta}_{\mu,p}(a_2,b_2,c_2;\varrho;A,B)\subset\mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_2,b_2,c_2;\varrho;A,B)\subset\mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_1,b_2,c_2;\varrho;A,B)$$
$$\subset\mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_1,b_1,c_2;\varrho;A,B)\subset\mathcal{K}^{\lambda_2,\delta}_{\mu,p}(a_1,b_1,c_1;\varrho;A,B).$$

To prove next theorems, we will use the following lemma.

Lemma 2.4. *Let* $p\in\mathbb{N},\ 0\le\varrho<p$ *and* $\phi\in M$ *with (2.5) holds. If* $f\in\mathcal{K}$ *and* $q\in\mathcal{S}^*_p(\varrho;\phi)$*, then* $(z^{p-1}f)*q\in\mathcal{S}^*_p(\varrho;\phi)$.

Proof. If $q\in\mathcal{S}^*_p(\varrho;\phi)$, then, from (2.13) and the definition of the class $\mathcal{S}^*_p(\varrho;\phi)$, we know that

$$zq'(z)=q(z)[(p-\varrho)\phi(\omega(z))+\varrho]=q(z)s(\omega(z)),$$

where ω is a Schwarz function. Thus,

$$\frac{1}{p-\varrho}\left(\frac{z[(z^{p-1}f(z))*q(z)]'}{(z^{p-1}f(z))*q(z)}-\varrho\right)=\frac{1}{p-\varrho}\left(\frac{(z^{p-1}f(z))*zq'(z)}{(z^{p-1}f(z))*q(z)}-\varrho\right)$$
$$=\frac{1}{p-\varrho}\left(\frac{f(z)*z^{1-p}q(z)s(\omega(z))}{f(z)*z^{1-p}q(z)}-\varrho\right).\tag{2.17}$$

By using similar method to those in the proof of Theorem 2.1, we deduce that (2.17) is subordinate to ϕ in $\mathbb{U}$, and hence $(z^{p-1}f)*q\in\mathcal{S}^*_p(\varrho;\phi)$. □

Lemma 4 in [17] is a special case of the above Lemma 2.4.

Theorem 2.5. *Let* $0\le\varrho,\sigma<p,\ \mu,\delta\ge 0$ *and* $\phi,\psi\in M$*, and let* ϕ,ψ *satisfy (2.5). If* λ_i *satisfies (2.6), and* a_i *satisfies (2.7),* $i=1,2$*, then*

$$C^{\lambda_1,\delta}_{\mu,p}(a_2,b,c;\varrho,\sigma)(\phi,\psi)\subset C^{\lambda_2,\delta}_{\mu,p}(a_2,b,c;\varrho,\sigma)(\phi,\psi)\subset C^{\lambda_2,\delta}_{\mu,p}(a_1,b,c;\varrho,\sigma)(\phi,\psi).$$

Proof. We begin by proving that

$$C^{\lambda_1,\delta}_{\mu,p}(a_2,b,c;\varrho,\sigma)(\phi,\psi)\subset C^{\lambda_2,\delta}_{\mu,p}(a_2,b,c;\varrho,\sigma)(\phi,\psi).\tag{2.18}$$

Let $f \in \mathcal{C}_{\mu,p}^{\lambda_1,\delta}(a_2, b, c; \varrho, \sigma)(\phi, \psi)$. Then, by the definition, we have

$$\frac{1}{p-\sigma}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))'}{q_1(z)} - \sigma\right) \prec \psi(z) \quad (z \in \mathbb{U})$$

with $q_1 \in \mathcal{S}_p^*(\varrho;\phi)$. We thus get that

$$z(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))' = q_1(z)[(p-\sigma)\psi(\omega(z)) + \sigma],$$

where ω is a Schwarz function.

From Lemma 2.4, we know that

$$q_2(z) = \phi_p(\lambda_2 + p, \lambda_1 + p)(z) * q_1(z) \in \mathcal{S}_p^*(\varrho;\phi).$$

Making use of methods of earlier proofs, we conclude that

$$\begin{aligned}
&\frac{1}{p-\sigma}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda_2,\delta}(a_2,b,c)f(z))'}{q_2(z)} - \sigma\right)\\
&\quad = \frac{1}{p-\sigma}\left(\frac{\phi_p(\lambda_2+p,\lambda_1+p)(z) * z(\mathcal{I}_{\mu,p}^{\lambda_1,\delta}(a_2,b,c)f(z))'}{\phi_p(\lambda_2+p,\lambda_1+p)(z) * q_1(z)} - \sigma\right)\\
&\quad = \frac{1}{p-\sigma}\left(\frac{z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)(z) * z^{1-p}q_1(z)[(1-\sigma)\psi(\omega(z))+\sigma]}{z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)(z) * z^{1-p}q_1(z)} - \sigma\right)\\
&\prec \psi(z) \quad (z \in \mathbb{U}).
\end{aligned}$$

Therefore $f \in \mathcal{C}_{\mu,p}^{\lambda_2,\delta}(a_2, b, c; \varrho, \sigma)(\phi, \psi)$, which implies that the assertion (2.18) of Theorem 2.5 holds true. The proof of the second inclusion is similar to that of (2.18). □

In view of (2.3), (2.4), together with Lemma 2.4, and by similarly applying the method of the proof of Theorem 2.5, we can obtain the following result.

Theorem 2.6. *Let $0 \le \varrho, \sigma < p$, $\lambda > -p$, $\mu, \delta \ge 0$ and $\phi, \psi \in M$, and let ϕ, ψ satisfy (2.5). If b_i satisfies (2.15), and c_i satisfies (2.16), $i = 1, 2$, then*

$$\mathcal{C}_{\mu,p}^{\lambda,\delta}(a, b_2, c_2; \varrho, \sigma)(\phi, \psi) \subset \mathcal{C}_{\mu,p}^{\lambda,\delta}(a, b_1, c_2; \varrho, \sigma)(\phi, \sigma) \subset \mathcal{C}_{\mu,p}^{\lambda,\delta}(a, b_1, c_1; \varrho, \sigma)(\phi, \psi).$$

By means of (1.11), and using the similar methods of the proofs of Theorems 2.3 and 2.4, respectively, we get the following results. Here, we choose to omit the details involved.

Theorem 2.7. *Let $0 \le \varrho, \sigma < p$, $\mu, \delta \ge 0$ and $\phi, \psi \in M$, and let ϕ, ψ satisfy (2.5). If λ_i satisfies (2.6), and a_i satisfies (2.7), $i = 1, 2$, then*

$$\mathcal{QC}_{\mu,p}^{\lambda_1,\delta}(a_2, b, c; \varrho, \sigma)(\phi, \psi) \subset \mathcal{QC}_{\mu,p}^{\lambda_2,\delta}(a_2, b, c; \varrho, \sigma)(\phi, \psi) \subset \mathcal{QC}_{\mu,p}^{\lambda_2,\delta}(a_1, b, c; \varrho, \sigma)(\phi, \psi).$$

Theorem 2.8. *Let $0 \le \varrho, \sigma < p,\ \lambda > -p,\ \mu, \delta \ge 0$ and $\phi, \psi \in M$, and let ϕ, ψ satisfy (2.5). If b_i satisfies (2.15), and c_i satisfies (2.16), $i = 1, 2$, then*

$$\mathcal{QC}^{\lambda,\delta}_{\mu,p}(a, b_2, c_2; \varrho; \sigma)(\phi, \psi) \subset \mathcal{QC}^{\lambda,\delta}_{\mu,p}(a, b_1, c_2; \varrho; \sigma)(\phi, \sigma) \subset \mathcal{QC}^{\lambda,\delta}_{\mu,p}(a, b_1, c_1; \varrho; \sigma)(\phi, \psi).$$

By taking

$$\phi(z) = \psi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \le B < A \le 1;\ z \in \mathbb{U})$$

in Theorems 2.5-2.8, we get the following result.

Corollary 2.2. *Under the conditions of Corollary 2.1, we have*

$$\mathcal{C}^{\lambda_1,\delta}_{\mu,p}(a_2, b_2, c_2; \varrho, \sigma; A, B) \subset \mathcal{C}^{\lambda_2,\delta}_{\mu,p}(a_2, b_2, c_2; \varrho, \sigma; A, B) \subset \mathcal{C}^{\lambda_2,\delta}_{\mu,p}(a_1, b_2, c_2; \varrho, \sigma; A, B)$$
$$\subset \mathcal{C}^{\lambda_2,\delta}_{\mu,p}(a_1, b_1, c_2; \varrho, \sigma; A, B) \subset \mathcal{C}^{\lambda_2,\delta}_{\mu,p}(a_1, b_1, c_1; \varrho, \sigma; A, B)$$

and

$$\mathcal{QC}^{\lambda_1,\delta}_{\mu,p}(a_2, b_2, c_2; \varrho, \sigma; A, B) \subset \mathcal{QC}^{\lambda_2,\delta}_{\mu,p}(a_2, b_2, c_2; \varrho, \sigma; A, B) \subset \mathcal{QC}^{\lambda_2,\delta}_{\mu,p}(a_1, b_2, c_2; \varrho, \sigma; A, B)$$
$$\subset \mathcal{QC}^{\lambda_2,\delta}_{\mu,p}(a_1, b_1, c_2; \varrho, \sigma; A, B) \subset \mathcal{QC}^{\lambda_2,\delta}_{\mu,p}(a_1, b_1, c_1; \varrho, \sigma; A, B).$$

Remark 2.1. *(i) By putting $p = 1,\ \delta = 0,\ \lambda = \lambda_2 = \lambda_1 - 1\ (\lambda \ge 0)$ and $a = a_2 = a_1 - 1\ (a \ge 1)$ in Theorems 2.1, 2.3, 2.5 and 2.7, respectively, we have the results obtained by Wang et al.[21, Theorems 1-4, respectively].*

(ii) By taking $p = 1,\ \varrho = \delta = \sigma = 0,\ \lambda = \lambda_2 = \lambda_1 - 1\ (\lambda \ge 0)$ and $a = a_2 = a_1 - 1\ (a \ge 1)$ in Theorems 2.1, 2.3 and 2.5, respectively, we get the results obtained by Srivastava et al.[19, Theorems 1-2, Corollary 3 and Theorems 4-5, respectively].

Remark 2.2. *We note that, in [19,21] there are no results concerning inclusion relationships among the function classes with respect to the parameters b and c. However, in this paper, we obtain some inclusion relationships with respect to the parameters b and c, see, for details, the above Theorems 2.2, 2.4, 2.6 and 2.8.*

3. Inclusion Properties by Convolution

In this section, we will show that the function classes $\mathcal{S}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho)(\phi)$, $\mathcal{K}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho)(\phi)$, $\mathcal{C}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho, \sigma)(\phi, \psi)$ and $\mathcal{QC}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho, \sigma)(\phi, \psi)$ are preserved under convolution with convex functions.

Theorem 3.1. *Let $p \in \mathbb{N},\ 0 \le \varrho, \sigma < p,\ \lambda > -p,\ \mu, \delta \ge 0,\ g \in \mathcal{K}$ and $\phi, \psi \in M$, and let ϕ, ψ satisfy (2.5). Then*

$$(i)\ f \in \mathcal{S}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho)(\phi) \Longrightarrow (z^{p-1}g) * f \in \mathcal{S}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho)(\phi),$$

$$(ii)\ f \in \mathcal{K}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho)(\phi) \Longrightarrow (z^{p-1}g) * f \in \mathcal{K}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho)(\phi),$$

$$(iii)\ f \in \mathcal{C}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho, \sigma)(\phi, \psi) \Longrightarrow (z^{p-1}g) * f \in \mathcal{C}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho, \sigma)(\phi, \psi),$$

$$(iv)\ f \in \mathcal{QC}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho, \sigma)(\phi, \psi) \Longrightarrow (z^{p-1}g) * f \in \mathcal{QC}^{\lambda,\delta}_{\mu,p}(a, b, c; \varrho, \sigma)(\phi, \psi).$$

Proof.

(i) Let $f \in \mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$ and $g \in \mathcal{K}$. Based on the same concept as the proof of Theorem 2.1, we have

$$\frac{1}{p-\varrho}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)((z^{p-1}g)*f)(z))'}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)((z^{p-1}g)*f)(z)}-\varrho\right)=\frac{1}{p-\varrho}\left(\frac{(z^{p-1}g(z))*z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z))'}{(z^{p-1}g(z))*\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}-\varrho\right)$$

$$=\frac{1}{p-\varrho}\left(\frac{(z^{p-1}g(z))*s(\omega)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{(z^{p-1}g(z))*\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}-\varrho\right)$$

$$=\frac{1}{p-\varrho}\left(\frac{g(z)*s(\omega)z^{1-p}\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{g(z)*z^{1-p}\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}-\varrho\right)\prec\phi(z)\ \ (z\in\mathbb{U}),$$

and we obtain $(z^{p-1}g)*f \in \mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$. □

(ii) Let $f \in \mathcal{K}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$ and $g \in \mathcal{K}$. Then, by (1.10), we know $\frac{zf'(z)}{p} \in \mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$, and from (i), we have $(z^{p-1}g(z))*\frac{zf'(z)}{p} \in \mathcal{S}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$. On the other hand, $(z^{p-1}g(z))*\frac{zf'(z)}{p}=\frac{z((z^{p-1}g)*f)'(z)}{p}$. Also, by applying (1.10), we get $(z^{p-1}g)*f \in \mathcal{K}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho)(\phi)$. □

(iii) Let $f \in \mathcal{C}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi)$ and $g \in \mathcal{K}$. Then there exists a function $q \in \mathcal{S}_p^*(\varrho;\phi)$ such that

$$\frac{1}{p-\sigma}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z))'}{q(z)}-\sigma\right)\prec\psi(z)\ \ (z\in\mathbb{U}),$$

that is, that

$$z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z))'=[(p-\sigma)\psi(\omega(z))+\sigma]q(z)$$

where ω is a Schwarz function. By Lemma 2.4, we have that $(z^{p-1}g)*q \in \mathcal{S}_p^*(\varrho;\phi)$.

Since

$$\frac{1}{p-\sigma}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)((z^{p-1}g)*f)(z))'}{((z^{p-1}g)*q)(z)}-\sigma\right)=\frac{1}{p-\sigma}\left(\frac{(z^{p-1}g(z))*z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z))'}{(z^{p-1}g(z))*q(z)}-\sigma\right)$$

$$=\frac{1}{p-\sigma}\left(\frac{g(z)*[(p-\sigma)\psi(\omega(z))+\sigma]z^{1-p}q(z)}{g(z)*z^{1-p}q(z)}-\sigma\right)\prec\psi(z)\ \ (z\in\mathbb{U}),$$

and so that $(z^{p-1}g)*f \in \mathcal{C}_{\mu,p}^{\lambda,\delta}(a,b,c;\varrho,\sigma)(\phi,\psi)$. □

(iv) The proof of (iv) follows from (1.11) and (iii). □

Corollary 3.1. *Let $p \in \mathbb{N}$, $0 \le \varrho,\sigma < p$, $\lambda > -p$, $\mu,\delta \ge 0$ and $\phi,\psi \in M$, and let ϕ,ψ satisfy (2.5). Suppose also that*

$$h_1(z)=\sum_{k=1}^{\infty}\left(\frac{1+\xi}{k+\xi}\right)z^k\ \ (\xi>-1;\ z\in\mathbb{U}),\tag{3.1}$$

$$h_2(z)=\frac{1}{1-\varepsilon}\log\left[\frac{1-\varepsilon z}{1-z}\right]\ \ (\log 1=0;\ |\varepsilon|\le 1\ (\varepsilon\ne 1);\ z\in\mathbb{U}),\tag{3.2}$$

and

$$h_3(z)=\sum_{k=1}^{\infty}\frac{z^k}{k}=-\log(1-z).\tag{3.3}$$

Then, for $j = 1, 2, 3$, we have

$$(i)\ f \in \mathcal{S}^{\lambda,\delta}_{\mu,p}(a,b,c;\varrho)(\phi) \Longrightarrow (z^{p-1}h_j) * f \in \mathcal{S}^{\lambda,\delta}_{\mu,p}(a,b,c;\varrho)(\phi),$$

$$(ii)\ f \in \mathcal{K}^{\lambda,\delta}_{\mu,p}(a,b,c;\varrho)(\phi) \Longrightarrow (z^{p-1}h_j) * f \in \mathcal{K}^{\lambda,\delta}_{\mu,p}(a,b,c;\varrho)(\phi),$$

$$(iii)\ f \in C^{\lambda,\delta}_{\mu,p}(a,b,c;\varrho,\sigma)(\phi,\psi) \Longrightarrow (z^{p-1}h_j) * f \in C^{\lambda,\delta}_{\mu,p}(a,b,c;\varrho,\sigma)(\phi,\psi),$$

$$(iv)\ f \in QC^{\lambda,\delta}_{\mu,p}(a,b,c;\varrho,\sigma)(\phi,\psi) \Longrightarrow (z^{p-1}h_j) * f \in QC^{\lambda,\delta}_{\mu,p}(a,b,c;\varrho,\sigma)(\phi,\psi).$$

Proof. The function h_1 was shown to be convex by Ruschewyh [14], while h_2 and h_3 are well known to be convex in $\mathbb{U}$. Thus, the assertions (i)-(iv) follow from Theorem 3.1. □

Remark 3.1. *(i) By setting $p = 1$, $\varrho = \delta = \sigma = 0$ and $j = 1, 2$ in the assertions (i)-(iii) of Theorem 3.1 and Corollary 3.1, respectively, we immediately derive the results obtained by Srivastava et al.[19, Theorem 7 and Corollary 6, respectively].*

(ii) By taking $\mu = \delta = 0$, $p = b = 1$, $\varrho = \eta$, $\sigma = \beta$ and $j = 1, 2$ in the assertions (i)-(iii) of Theorem 3.1 and Corollary 3.1, respectively, we have the results obtained by Cho and Yoon [2, Theorem 3.1 and Corollary 3.1, respectively].

Remark 3.2. *From (3.1), (3.2) and (3.3), we easily notice that, for $f \in \mathcal{A}$,*

$$F(z) = (f * h_1)(z) = \frac{1+\xi}{z^{\xi}} \int_0^z t^{\xi-1} f(t)dt \quad (\xi > -1)$$

(*generalized Libera integral operator* [8]),

$$G(z) = (f * h_2)(z) = \int_0^z \frac{f(t) - f(\varepsilon t)}{t - zt} dt,$$

and

$$H(z) = (f * h_3)(z) = \int_0^z \frac{f(t)}{t} dt$$

are well-known operators. Also, for $p = j = 1$ and $\varrho = \delta = \sigma = 0$, the applications of the assertions (i)-(iii) of Corollary 3.1 can be found in Srivastava et al.[19, Theorem 3, Corollary 5 and Theorem 6, respectively].

4. Argument Properties for the Operator $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a,b,c)$

Now, we will obtain some argument results involving the operator $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a,b,c)$. Unless otherwise mentioned, we shall assume throughout this section that $p \in \mathbb{N}$, $\theta > 0$, $\gamma > 0$, $\tau \geq 0$ and $z \in \mathbb{U}$.

Lemma 4.1 (see [7]). *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$. Further suppose that*

$$\left|\arg(p(z) + \eta z p'(z))\right| < \frac{\pi}{2}(\theta + \frac{2}{\pi}\arctan(\eta\theta)) \ \ (\eta, \theta > 0),$$

then

$$\left|\arg p(z)\right| < \frac{\pi}{2}\theta.$$

Theorem 4.1. *Let $f \in \mathcal{A}_p$ and $\lambda > -p$. If*

$$\left|\arg\left[(1-\tau)\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{z^p}\right)^{\gamma} + \tau\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{z^p}\right)^{\gamma}\left(\frac{\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z)}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}\right)\right]\right|$$

$$< \frac{\pi}{2}\left(\theta + \frac{2}{\pi}\arctan\left[\frac{\tau}{\gamma(\lambda+p)}\theta\right]\right),$$

then

$$\left|\arg\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{z^p}\right)^{\gamma}\right| < \frac{\pi}{2}\theta.$$

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{z^p}\right)^{\gamma}, \tag{4.1}$$

where $p(z) = 1 + c_1 z + \cdots$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) \neq 0$.

Differentiating both sides of (4.1) logarithmically with respect to z and multiplying by z, we have

$$p + \frac{zp'(z)}{\gamma p(z)} = \frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}. \tag{4.2}$$

Using (1.8) in (4.2), we obtain

$$\begin{aligned} p(z) &+ \frac{\tau}{\gamma(\lambda+p)}zp'(z) \\ &= (1-\tau)\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{z^p}\right)^{\gamma} + \tau\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{z^p}\right)^{\gamma}\left(\frac{\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z)}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}\right). \end{aligned} \tag{4.3}$$

Thus, by applying Lemma 4.1 to (4.3) with $\eta = \frac{\tau}{\gamma(\lambda+p)}$, we readily get the assertion of Theorem 4.1. □

In view of (1.9), and using the similar method of proof of Theorem 4.1, we can get the following result.

Theorem 4.2. *Let $f \in \mathcal{A}_p$ and $a > p$. If*

$$\left|\arg\left[(1-\tau)\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)}{z^p}\right)^{\gamma} + \tau\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)}{z^p}\right)^{\gamma}\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)}\right)\right]\right|$$

$$< \frac{\pi}{2}\left(\theta + \frac{2}{\pi}\arctan\left[\frac{\tau}{\gamma(a-p)}\theta\right]\right),$$

then

$$\left|\arg\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)}{z^p}\right)^{\gamma}\right| < \frac{\pi}{2}\theta.$$

If we set $p = \gamma = 1$ in Theorems 4.1 and 4.2, respectively, we obtain the following corollaries.

Corollary 4.1. *Let $f \in \mathcal{A}$ and $\lambda > -1$. If*

$$\left|\arg\left(\frac{(1-\tau)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)+\tau\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z)}{z}\right)\right| < \frac{\pi}{2}\left(\theta+\frac{2}{\pi}\arctan\left[\frac{\tau}{\lambda+1}\theta\right]\right),$$

then

$$\left|\arg\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{z}\right)\right| < \frac{\pi}{2}\theta.$$

Corollary 4.2. *Let $f \in \mathcal{A}$ and $a > 1$. If*

$$\left|\arg\left(\frac{(1-\tau)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)+\tau\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{z}\right)\right| < \frac{\pi}{2}\left(\theta+\frac{2}{\pi}\arctan\left[\frac{\tau}{a-1}\theta\right]\right),$$

then

$$\left|\arg\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)}{z}\right)\right| < \frac{\pi}{2}\theta.$$

Theorem 4.3. *Let $f \in \mathcal{A}_p$ and $c > -p$. If*

$$\left|\arg\left[(1-\tau)\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)}{z^p}\right)^{\gamma}+\tau\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)}{z^p}\right)^{\gamma}\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)}\right)\right]\right|$$
$$< \frac{\pi}{2}\left(\theta+\frac{2}{\pi}\arctan\left[\frac{\tau}{\gamma(c+p)}\theta\right]\right),$$

then

$$\left|\arg\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)}{z^p}\right)^{\gamma}\right| < \frac{\pi}{2}\theta,$$

where the function $F_{p,c}(z)$ is defined by

$$F_{p,c}(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{4.4}$$

Proof. Firstly, we find from the definition (4.4) that

$$z\left(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)\right)'=(c+p)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)-c\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z). \tag{4.5}$$

Let

$$p(z)=\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)}{z^p}\right)^{\gamma}.$$

Then, by using (4.5), we easily get

$$p(z)+\frac{\tau}{\gamma(c+p)}zp'(z)$$
$$=(1-\tau)\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)}{z^p}\right)^{\gamma}+\tau\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)}{z^p}\right)^{\gamma}\left(\frac{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)F_{p,c}(z)}\right).$$

Finally, by applying Lemma 4.1 with $\eta=\frac{\tau}{\gamma(c+p)}$, we immediately obtain the required result. □

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