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Stancu-Variant of Generalized Baskakov Operators

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Abstract. In the present paper, we introduce Stancu-variant of generalized Baskakov operators and study the rate of convergence using modulus of continuity, order of approximation for the derivative of function f. Direct estimate is proved using K-functional and Ditzian-Totik modulus of smoothness. In the last, we have proved Voronovskaya type theorem.

1. Introduction

In 1998 V. Mihesan[8] constructed an important generalization of the well known Baskakov operators on $[0,\infty)$ with non-negative constant a independent of n,

$$B_n^a(f;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\left(\frac{k}{n}\right), \tag{1}$$

where $f \in C[0,\infty)$ and

$$W_{n,k}^a(x) = e^{-\frac{ax}{1+x}} \frac{p_k(n,a)}{k!} \frac{x^k}{(1+x)^{k+n}}, \tag{2}$$

such that $\sum\limits_{k=0}^{\infty} W_{n,k}^a(x) = 1$ and $p_k(n,a) = \sum\limits_{i=0}^{\infty} \binom{n}{i} (n)_i a^{k-i}$, with $(n)_0 = 1, (n)_i = n(n+1)\dots(n+i-1)$.
In the last decade, many papers were published for generalized Baskakov operators on order of approximation, Voronovskaya type theorem, Kantorovich form, and order of approximation for the derivative of the function([13],[14]).
For $f \in C[0,\infty)$, Stancu[10] introduced the sequence of positive linear operators

$$B_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right),$$

where $p_{n,k}(x) = \binom{n}{i} x^k (1-x)^{n-k}$ and α, β are any two two non negative real numbers such that $0 \le \alpha \le \beta$. If $\alpha = \beta = 0$, it reduces to so called Bernstein operators. Recently, many researchers ([1],[4],[5],[6],[9],[11],[15])have

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introduced Stancu-variant for different linear positive operators. Motivated by the above development, we are giving a Stancu-variant of the operators (1) as:

$$L_{n,a}^{\alpha,\beta}(f;x)=\sum_{k=0}^{\infty}W_{n,k}^{a}(x)f\Big(\frac{k+\alpha}{n+\beta}\Big), \tag{3}$$

where $W_{n,k}^{a}(x)$ is defined in (2) and $0\leq\alpha\leq\beta$. For $\alpha=\beta=0$, we get the operators (1).

2. Approximation Properties of $L_{n,a}^{\alpha,\beta}$

To prove the approximation properties of $L_{n,a}^{\alpha,\beta}$, we need the following lemma[13].

Lemma 2.1. *For $a,x\geq 0, n=1,2,...,$ we have*

$$\begin{aligned}
B_n^a(1;x) &= 1,\\
B_n^a(t;x) &= x+\frac{ax}{n(1+x)},\\
B_n^a(t^2;x) &= \frac{x^2}{n}+\frac{x}{n}+x^2+\frac{a^2x^2}{n^2(1+x)^2}+\frac{2ax^2}{n(1+x)}+\frac{ax}{n^2(1+x)},\\
B_n^a(t^3;x) &= x^3+\frac{3x^2(1+x)}{n}+\frac{x(1+x)(1+2x)}{n^2}+\frac{3ax^3}{n(1+x)}+\frac{1}{n^2}\Big(3ax^2+\frac{3a^2x^3}{(1+x)^2}+\frac{3ax^2}{(1+x)}\Big)\\
&\quad+\frac{1}{n^3}\Big(\frac{ax}{1+x}+\frac{3a^2x^2}{(1+x)^2}+\frac{a^3x^3}{(1+x)^3}\Big),\\
B_n^a(t^4;x) &= x^4+\frac{6x^3(1+x)}{n}+\frac{x^2(1+x)(7+11x)}{n^2}+\frac{x(1+x)(6x^2+6x+1)}{n^3}+\frac{4ax^4}{n(1+x)}\\
&\quad+\frac{1}{n^2}\Big(\frac{12ax^4}{(1+x)+\frac{6a^2x^4}{(1+x)^2}}+\frac{18ax^3}{1+x}\Big)+\frac{1}{n^3}\Big(\frac{8ax^4}{1+x}+\frac{6a^2x^4}{(1+x)^2}+\frac{4a^3x^4}{(1+x)^3}+\frac{18ax^3}{(1+x)}+\frac{18a^2x^3}{(1+x)^2}+\frac{14ax^2}{(1+x)}\Big)\\
&\quad+\frac{1}{n^4}\Big(\frac{ax}{1+x}+\frac{7a^2x^2}{(1+x)^2}+\frac{6a^3x^3}{(1+x)^3}+\frac{a^4x^4}{(1+x)^4}\Big).
\end{aligned}$$

Next, we prove

Lemma 2.2. *Let $a,x\geq 0$ and $n=1,2,3\ldots$ Then for the operators defined in (3), we have*

$$\begin{aligned}
(i)\quad L_{n,a}^{\alpha,\beta}(1;x) &= 1,\\
(ii)\quad L_{n,a}^{\alpha,\beta}(t;x) &= \frac{n}{n+\beta}x+\frac{a}{n+\beta}\frac{x}{1+x}+\frac{\alpha}{n+\beta},\\
(iii)\quad L_{n,a}^{\alpha,\beta}(t^2;x) &= \frac{n^2+n}{(n+\beta)^2}x^2+\frac{n(1+2\alpha)}{(n+\beta)^2}x+\frac{a^2}{(n+\beta)^2}\frac{x^2}{(1+x)^2}+\frac{2an}{(n+\beta)^2}\frac{x^2}{(1+x)}+\frac{a(1+2\alpha)}{(n+\beta)^2}\frac{x}{1+x}+\frac{\alpha^2}{(n+\beta)^2},\\
(iv)\quad L_{n,a}^{\alpha,\beta}(t^3;x) &= \frac{n^3+3n^2+2n}{(n+\beta)^3}x^3+\frac{n^2(3+3\alpha)+n(3+3\alpha+3a)}{(n+\beta)^3}x^2+\frac{n(1+3\alpha+3\alpha^2)}{(n+\beta)^3}x+\frac{3an^2}{(n+\beta)^3}\frac{x^3}{(1+x)}\\
&\quad+\frac{n}{(n+\beta)^3}\Big(\frac{3a^2x^3}{(1+x)^2}+\frac{3ax^2}{1+x}+\frac{6a\alpha x^2}{1+x}\Big)+\frac{1}{(n+\beta)^3}\Big(\frac{ax}{1+x}+\frac{3a^2x^2}{(1+x)^2}+\frac{a^3x^3}{(1+x)^3}\\
&\quad+\frac{3\alpha a^2x^2}{(1+x)^2}+\frac{3\alpha^2ax}{1+x}+\alpha^3\Big),
\end{aligned}$$

$$
\begin{aligned}
(v)\quad L_{n,a}^{\alpha,\beta}(t^4;x) &= \frac{n^4+6n^3+11n^2+6n}{(n+\beta)^4}x^4+\frac{(6+4\alpha)n^3+(18+12\alpha)n^2+(9+8\alpha)n}{(n+\beta)^4}x^3+\Big(\frac{(7+12\alpha+6\alpha^2)n^2}{(n+\beta)^4}\\
&+\frac{(7+12\alpha+12\alpha a+6\alpha^2)n}{(n+\beta)^4}\Big)x^2+\frac{(1+4\alpha+6\alpha^2+4\alpha^3)n}{(n+\beta)^4}x+\frac{4an^3+12an^2+8an}{(n+\beta)^4}\frac{x^4}{1+x}\\
&+\frac{6a^2n^2+6a^2n}{(n+\beta)^4}\frac{x^4}{(1+x)^2}+\frac{4a^3n}{(n+\beta)^4}\frac{x^4}{(1+x)^3}+\frac{a^4}{(b+\beta)^4}\frac{x^4}{(1+x)^4}+\frac{18an^2+18an}{(n+\beta)^4}\frac{x^3}{(1+x)}\\
&+\frac{(18a^2+12a^2\alpha)n}{(n+\beta)^4}\frac{x^3}{(1+x)^2}+\frac{6a^3+4\alpha a^3}{(n+\beta)^4}\frac{x^3}{(1+x)^3}+\frac{(12a\alpha^2+12a\alpha+14a)n}{(n+\beta)^4}\frac{x^2}{(1+x)}\\
&+\frac{7a^2+12a^2\alpha+6a^2\alpha^2}{(n+\beta)^4}\frac{x^2}{(1+x)^2}+\frac{a+4\alpha a+6\alpha^2a+4\alpha^3a}{(n+\beta)^4}\frac{x}{1+x}+\frac{\alpha^4}{(n+\beta)^4}.
\end{aligned}
$$

Proof To prove these identities, we use the lemma(2.1) and linearity property

$$
L_{n,a}^{\alpha,\beta}(t;x) = \frac{n}{n+\beta}B_n^a(t;x)+\frac{\alpha}{n+\beta}B_n^a(1;x).
$$

In similar manner, we can prove identities (iii), (iv) and (v).

Lemma 2.3. *Let* $\psi_x^i(t)=(t-x)^i, i=1,2,3,....$ *For* $a,x\geq 0$ *and* $n=1,2,3...,$

$$
\begin{aligned}
L_{n,a}^{\alpha,\beta}(\psi_x^0(t);x) &= 1,\\
L_{n,a}^{\alpha,\beta}(\psi_x^1(t);x) &= \Big(\frac{n}{n+\beta}-1\Big)x+\frac{a}{n+\beta}\frac{x}{1+x}+\frac{\alpha}{n+\beta},\\
L_{n,a}^{\alpha,\beta}(\psi_x^2(t);x) &= \frac{n+\beta^2}{(n+\beta)^2}x^2+\frac{n-2\alpha\beta}{(n+\beta)^2}x+\frac{a^2}{(n+\beta)^2}\frac{x^2}{(1+x)^2}-\frac{2a\beta}{(n+\beta)^2}\frac{x^2}{(1+x)}+\frac{a(1+2\alpha)}{(n+\beta)^2}\frac{x}{1+x}+\frac{\alpha^2}{(n+\beta)^2},\\
L_{n,a}^{\alpha,\beta}(\psi_x^4(t);x) &= \frac{(3-12\beta)n^2+(6+4\beta+2\beta^2+4\beta^3)n+\beta^4}{(n+\beta)^4}x^4+\Big(\frac{(6-12a-12\beta)n^2+(9+8\alpha-12\beta(1+a+\alpha+\alpha\beta))n}{(n+\beta)^4}\\
&+\frac{(6-12a-12\beta-12\alpha\beta^2)}{(n+\beta)^4}\Big)x^3+\frac{3n^2+(7-4\beta+12\alpha a-12\alpha\beta+6\alpha^2)n+6\alpha^2\beta^2}{(n+\beta)^4}x^2\\
&+\frac{(1+4\alpha+6\alpha^2)n-4\alpha^3\beta}{(n+\beta)^4}x+\frac{12an^2+8an-4a\beta^3}{(n+\beta)^4}\frac{x^4}{(1+x)}+\frac{6a^2n+6a^2\beta^2}{(n+\beta)^4}\frac{x^4}{(1+x)^2}-\frac{4a^3\beta}{(n+\beta)^4}\frac{x^4}{(1+x)^3}\\
&+\frac{a^4}{(n+\beta)^4}\frac{x^4}{(1+x)^4}+\frac{12an^2+18an+6a(1+2\alpha)\beta^2}{(n+\beta)^4}\frac{x^3}{1+x}+\frac{6a^2n-(12a^2+12\alpha a^2)\beta}{(n+\beta)^4}\frac{x^3}{(1+x)^2}\\
&+\frac{(6a^3+4\alpha a^3)}{(n+\beta)^4}\frac{x^3}{(1+x)^3}+\frac{(12a\alpha+8a-6a\alpha^2)n-(6a+18\alpha^2a)\beta}{(n+\beta)^4}\frac{x^2}{1+x}+\frac{7a^2+12a^2\alpha+6a^2\alpha^2}{(n+\beta)^4}\frac{x^2}{(1+x)^2}\\
&+\frac{(a)+4\alpha a+6\alpha^2a+4\alpha^3a}{(n+\beta)^4}\frac{x}{1+x}+\frac{\alpha^4}{(n+\beta)^4}.
\end{aligned}
$$

Proof In view of lemma(2.2) and using the equalities,

$$
\begin{aligned}
L_{n,a}^{\alpha,\beta}(\psi_x(t);x) &= L_{n,a}^{\alpha,\beta}(t;x)-xL_{n,a}^{\alpha,\beta}(1;x),\\
L_{n,a}^{\alpha,\beta}(\psi_x^2(t);x) &= L_{n,a}^{\alpha,\beta}(t^2;x)-2xL_{n,a}^{\alpha,\beta}(t;x)+x^2L_{n,a}^{\alpha,\beta}(1;x),\\
L_{n,a}^{\alpha,\beta}(\psi_x^4(t);x) &= L_{n,a}^{\alpha,\beta}(t^4;x)-4xL_{n,a}^{\alpha,\beta}(t^3;x)+6x^2L_{n,a}^{\alpha,\beta}(t^2;x)+4x^3L_{n,a}^{\alpha,\beta}(t;x)+x^4L_{n,a}^{\alpha,\beta}(1;x).
\end{aligned}
$$

we get the proof of this lemma.

Lemma 2.4. *Let $\psi_x^i(t) = (t-x)^i, i = 1,2,3,\ldots$ For $a, x \geq 0$ and $n = 1,2,3\ldots$, we have*

$$\begin{aligned}
\lim_{n\to\infty} nL_{n,a}^{\alpha,\beta}(\psi_x^1(t);x) &= \alpha - \beta x + a\frac{x}{1+x}, \\
\lim_{n\to\infty} nL_{n,a}^{\alpha,\beta}(\psi_x^2(t);x) &= x^2 + x, \\
\lim_{n\to\infty} n^2L_{n,a}^{\alpha,\beta}(\psi_x^4(t);x) &= (3-12\beta)x^4 + (6-12a-12\beta)x^3 + 3x^2 + 12a\frac{x^2}{1+x} + 12a\frac{x^3}{1+x}.
\end{aligned}$$

3. The Degree of Approximation

Theorem 3.1. *If $f \in C[0,\infty)$, $x \in [0,\infty)$ and $\omega(f;\delta)$ is the modulus of continuity, then*

$$|L_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq \left\{1 + \sqrt{\gamma_{n,a}^{\alpha,\beta}(x)}\right\}\omega(f;\delta_{n,\beta}),$$

where $\delta_{n,\beta} = (n+\beta)^{-\frac{1}{2}}$ and

$$\gamma_n^{\alpha,\beta}(x) = \frac{n+\beta^2}{n+\beta}x^2 + \frac{n-2\alpha\beta}{n+\beta}x + \frac{a^2}{n+\beta}\frac{x^2}{(1+x)^2} - \frac{2a\beta}{n+\beta}\frac{x^2}{(1+x)} + \frac{a(1+2\alpha)}{n+\beta}\frac{x}{1+x} + \frac{\alpha^2}{n+\beta}.$$

Proof Let $f \in C[0,\infty)$ and $x \geq 0$. Then, using linearity property and monotonicity of the operators defined by (3), we can easily find, for every $\delta > 0$, and $n \in N$,that

$$\begin{aligned}
|L_{n,a}^{\alpha,\beta}(f;x) - f(x)| &\leq \left\{1 + \delta_{n,\beta}^{-1}\sqrt{L_{n,a}^{\alpha,\beta}(\psi_x^2;x)}\right\}\omega(f;\delta_{n,\beta}). \\
&\leq \left\{1 + \sqrt{\frac{n+\beta^2}{n+\beta}x^2 + \frac{n-2\alpha\beta}{n+\beta}x + \frac{a^2}{n+\beta}\frac{x^2}{(1+x)^2} - \frac{2a\beta}{n+\beta}\frac{x^2}{(1+x)} + \frac{a(1+2\alpha)}{n+\beta}\frac{x}{1+x} + \frac{\alpha^2}{n+\beta}}\right\}\omega(f;\delta_{n,\beta}),
\end{aligned}$$

which obtained by using Lemma 2.2 and choosing $\delta_{n,\beta} = (n+\beta)^{-\frac{1}{2}}$. Thus, we arrive at the result.

Remark 3.2. *If we put $\alpha = \beta = 0$, we find the same result given by Mihesan[8]*

$$|B_n^a(f;x) - f(x)| \leq \left\{1 + \sqrt{x(1+x) + \frac{ax}{n(1+x)}\frac{(a+1)x+1}{(1+x)}}\right\}\omega(f;\delta),$$

where $\delta = \frac{1}{\sqrt{n}}$, which shows that $\delta_{n,\beta} \leq \delta$. Therefore, rate of convergence of $L_{n,a}^{\alpha,\beta}$ is better than B_n^a.

Now, we will find the rate of convergence of operators defined by (3) in terms of modulus of continuity of first derivative of function *i.e.* $\omega(f';\delta_{n,\beta}) = \omega_1(f;\delta_{n,\beta})$, which is an improvement over the Theorem 3.1. This type of result was given for Bernstein polynomials by Lorentz ([7],p.p. 21).

Theorem 3.3. *Let $f'(x)$ is the continuous derivative over $[0,\infty)$ and $\omega_1(f;\delta_{n,\beta})$ is the modulus of continuity of $f'(x)$. Then, for $a, x \geq 0$, $0 \leq \alpha \leq \beta$, we have*

$$|L_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq \omega_1\left((n+\beta)^{-1}\right)\sqrt{L_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}\left\{1 + \sqrt{(n+\beta)}\sqrt{L_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}\right\}.$$

Proof For $x_1, x_2 \in [a,b]$, we have

$$\begin{aligned}
f(x_1) - f(x_2) &= (x_1 - x_2)f'(\xi), \\
&= (x_1 - x_2)f'(x_1) + (x_1 - x_2)[f'(\xi) - f'(x_1)], \qquad (4)
\end{aligned}$$

where $x_1 < \xi < x_2$. As we know that

$$|(x_1 - x_2)[f'(\xi) - f'(x_1)]| \leq |x_1 - x_2|(\lambda + 1)\omega_1(\delta), \quad \lambda = \lambda(x_1, x_2; \delta).$$

Next, we get

$$|L_{n,a}^{\alpha,\beta}(f; x) - f(x)| = \left|\sum_{n=0}^{\infty} W_{n,k}^{a}(x)\left\{f\left(\frac{k+\alpha}{n+\beta}\right) - f(x)\right\}\right|. \tag{5}$$

From (4) and (5), we obtained

$$\begin{aligned} |L_{n,a}^{\alpha,\beta}(f; x) - f(x)| &\leq \left|\sum_{n=0}^{\infty} W_{n,k}^{a}(x)\left(\frac{k+\alpha}{n+\beta} - x\right)f'(x)\right| + \omega_1(\delta_{n,\beta})\sum_{k=0}^{\infty}\left|\frac{k+\alpha}{n+\beta} - x\right|(\lambda+1)W_{n,k}^{a}(x), \\ &\leq \omega_1(\delta_{n,\beta})\left\{\sum_{k=0}^{\infty}\left|\frac{k+\alpha}{n+\beta} - x\right|W_{n,k}^{a}(x) + \sum_{\lambda \geq 1}\left|\frac{k+\alpha}{n+\beta} - x\right|\lambda\left(x_1, \frac{k+\alpha}{n+\beta}; \delta\right)W_{n,k}^{a}(x)\right\} \\ &\leq \omega_1(\delta_{n\beta})\left\{\sum_{k=0}^{\infty}\left|\frac{k+\alpha}{n+\beta} - x\right|W_{n,k}^{a}(x) + \delta^{-1}\sum_{k=0}^{\infty}\left(\frac{k+\alpha}{n+\beta} - x\right)^2 W_{n,k}^{a}(x)\right\} \\ &\leq \omega_1(\delta_{n,\beta})\sqrt{L_{n,a}^{\alpha,\beta}(\psi_x^2(t); x)}\left\{1 + \delta_{n,\beta}^{-1}\sqrt{L_{n,a}^{\alpha,\beta}(\psi_x^2(t); x)}\right\}. \end{aligned}$$

Taking $\delta_{n,\beta} = (n+\beta)^{-1}$, we get

$$|L_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq \omega_1((n+\beta)^{-1})\sqrt{L_{n,a}^{\alpha,\beta}(\psi_x^2(t); x)}\left\{1 + \sqrt{(n+\beta)}\sqrt{L_{n,a}^{\alpha,\beta}(\psi_x^2(t); x)}\right\},$$

which is the required result.

4. Direct Estimate

Here we introduced the Ditzian-Totik Modulus of smoothness[3] which is defined as:

$$\begin{aligned} \omega_{\varphi^\lambda}^2(f; \delta) &= \sup_{0<h\leq\delta} \| \Delta_{h\varphi(x)}^2 f(x) \|, \\ &= \sup_{0<h\leq\delta} \sup_{x \pm h\varphi^\lambda \in [0,\infty)} |f(x - h\varphi^\lambda(x)) - 2f(x) + f(x + h\varphi^\lambda(x))|, \end{aligned}$$

where $\varphi^2(x) = x(1-x)$. And, Peetre's K-functional is given by

$$K_{\varphi^\lambda}(f, \delta^2) = \inf_g\left(\|f - g\|_{C[0,\infty)} + \delta^2\|\varphi^2\lambda g''\|_{C[0,\infty)}\right), \quad g, g' \in AC_{loc}. \tag{6}$$

The K-functional is equivalent to the modulus of smoothness, i.e.,

$$C^{-1}K_{\varphi^\lambda}(f, \delta^2) \leq \omega_{\varphi^\lambda}^2(f, \delta) \leq CK_{\varphi^\lambda}(f, \delta^2). \tag{7}$$

First result based on Ditziaz-Totik modulus of smoothness was given by Ditzian[2] for the Bernstein polynomials as:

$$|B_n(f; x) - f(x)| \leq C\omega_{\varphi^\lambda}^2(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}).$$

Now, we prove the similar result for the operator $L_{n,a}^{\alpha,\beta}$.

Theorem 4.1. *For $a, x \geq 0$, and $0 \leq \alpha \leq \beta$, we have*

$$|L_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq C\omega_{\varphi^\lambda}^2\Big(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\Big) \ \ for \ \ large \ \ n.$$

Proof Using (6),(7),we can choose $g_n \equiv g_{n,x,\lambda}$ for fixed x and $\lambda + 1$ such that

$$\| \ f - g \ \|_{C[0,\infty)} \leq A\omega_{\varphi^\lambda}^2\Big(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\Big), \tag{8}$$

$$n^{-1}\varphi(x)^{2-2\lambda}\|\varphi^{2\lambda}g''\|_{C[0,\infty)} \leq B\omega_{\varphi^\lambda}^2\Big(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\Big). \tag{9}$$

Next

$$\begin{aligned}
|L_{n,a}^{\alpha,\beta}(f;x) - f(x)| &\leq |L_{n,a}^{\alpha,\beta}(f-g_n;x) - (f-g_n)(x)| + |L_{n,a}^{\alpha,\beta}(g_n;x) - g_n(x)|,\\
&\leq 2 \, \| \ f - g_n \ \|_{C[0,\infty)} + |L_{n,a}^{\alpha,\beta}(g_n;x) - g_n(x)|.
\end{aligned}$$

From (8), we get

$$|L_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq 2A\omega_{\varphi^\lambda}^2\Big(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\Big) + |L_{n,a}^{\alpha,\beta}(g_n;x) - g_n(x)|. \tag{10}$$

Now, the last term can be calculated by using Taylor's formula

$$\begin{aligned}
|L_{n,a}^{\alpha,\beta}(g_n(t) - g_n(x);x)| &\leq |g_n'(x)L_{n,a}^{\alpha,\beta}((t-x);x)| + \Big|L_{n,a}^{\alpha,\beta}\Big(\int_t^x (x-u)g_n''(u)du;x\Big)\Big|\\
&\leq L_{n,a}^{\alpha,\beta}\Big(\frac{|x-\frac{k}{n}|}{\varphi^{2\lambda}(x)}\int_{\frac{k}{n}}^x \varphi^{2\lambda}(u)|g_n''(u)du|;x\Big)\\
&\leq \| \ \varphi^{2\lambda}g_n'' \ \|_{C[0,\infty)} \frac{1}{\varphi^{2\lambda}(x)} L_{n,a}^{\alpha,\beta}((t-x)^2;x)\\
&\leq \| \ \varphi^{2\lambda}g_n'' \ \|_{C[0,\infty)} \frac{1}{\varphi^{2\lambda}(x)}\Big[\frac{n+\beta^2}{(n+\beta)^2}x^2 + \frac{n-2\alpha\beta}{(n+\beta)^2}x + \frac{a^2}{(n+\beta)^2}\frac{x^2}{(1+x)^2} - \frac{2a\beta}{(n+\beta)^2}\frac{x^2}{(1+x)}\\
&\quad + \frac{a(1+2\alpha)}{(n+\beta)^2}\frac{x}{1+x} + \frac{\alpha^2}{(n+\beta)^2}\Big]\\
&\leq \| \ \varphi^{2\lambda}g_n'' \ \|_{C[0,\infty)} \frac{x(1+x)(n+\beta)^{-1}}{\varphi^{2\lambda}(x)}\Big[\frac{n+\beta^2}{(n+\beta)}\frac{x}{1+x} + \frac{n-2\alpha\beta}{(n+\beta)}\frac{1}{1+x} + \frac{a^2}{(n+\beta)}\frac{x}{(1+x)^3}\\
&\quad - \frac{2a\beta}{(n+\beta)}\frac{x}{(1+x)^2} + \frac{a(1+2\alpha)}{(n+\beta)}\frac{1}{(1+x)^2} + \frac{\alpha^2}{(n+\beta)x(1+x)}\Big]\\
&\leq \| \ \varphi^{2\lambda}g_n'' \ \|_{C[0,\infty)} \varphi^{2-2\lambda}(x)(n+\beta)^{-1}\Big[\frac{n+\beta^2}{(n+\beta)}\frac{x}{1+x} + \frac{n-2\alpha\beta}{(n+\beta)}\frac{1}{1+x} + \frac{a^2}{(n+\beta)}\frac{x}{(1+x)^3}\\
&\quad - \frac{2a\beta}{(n+\beta)}\frac{x}{(1+x)^2} + \frac{a(1+2\alpha)}{(n+\beta)}\frac{1}{(1+x)^2} + \frac{\alpha^2}{(n+\beta)x(1+x)}\Big].
\end{aligned}$$

From (9), we have

$$\begin{aligned}
|L_{n,a}^{\alpha,\beta}(g_n(t) - g_n(x);x)| &\leq B\omega_{\varphi^\lambda}^2\Big(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\Big[\frac{n+\beta^2}{(n+\beta)}\frac{x}{1+x} + \frac{n-2\alpha\beta}{(n+\beta)}\frac{1}{1+x} + \frac{a^2}{(n+\beta)}\frac{x}{(1+x)^3}\\
&\quad - \frac{2a\beta}{(n+\beta)}\frac{x}{(1+x)^2} + \frac{a(1+2\alpha)}{(n+\beta)}\frac{1}{(1+x)^2} + \frac{\alpha^2}{(n+\beta)x(1+x)}\Big].
\end{aligned} \tag{11}$$

Using (10) and (11), we get

$$\begin{aligned}|L_{n,a}^{\alpha,\beta}(f(t)-f(x);x)| &\leq M\omega_{\lambda}^{2}\Big(f,(n+\beta)^{\frac{-1}{2}}\varphi(x)^{1-\lambda}\Big)\Big[\frac{n+\beta^{2}}{(n+\beta)}\frac{x}{1+x}+\frac{n-2\alpha\beta}{(n+\beta)}\frac{1}{1+x}+\frac{a^{2}}{(n+\beta)}\frac{x}{(1+x)^{3}} \\ &\quad -\frac{2a\beta}{(n+\beta)}\frac{x}{(1+x)^{2}}+\frac{a(1+2\alpha)}{(n+\beta)}\frac{1}{(1+x)^{2}}+\frac{\alpha^{2}}{(n+\beta)x(1+x)}\Big]\end{aligned}$$

where $M = max(2A, B)$. For a large value of n

$$|L_{n,a}^{\alpha,\beta}(f(t)-f(x);x)| \leq M\omega_{\lambda}^{2}\Big(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\Big).$$

Asymptotic relation is the study of rate of convergence for at least two times differentiable functions which was given by Voronovskaya [12]. Here, we prove a similar result.

Theorem 4.2. *Let $a, x \geq 0$, $0 \leq \alpha \leq \beta$ and $n \in N$. For $f \in C^2[0,\infty)$, we have*

$$\lim_{n\to\infty} n\{L_{n,a}^{\alpha,\beta}(f;x)-f(x)\} = \Big(\alpha-\beta x+\frac{ax}{1+x}\Big)f'(x)+\frac{x^{2}+x}{2}f''(x).$$

Proof Let $x, t \in [0,\infty)$, $f \in C^2[0,\infty)$. By Taylor's formula, we have

$$f(t) = f(x)+(t-x)f'(x)+\frac{(t-x)^{2}}{2}f''(x)+\eta(t,x)(t-x)^{2},$$

where the function $\eta(t,x) \in C[0,\infty)$ and $\lim_{t\to x}\eta(t,x) = 0$. Multiplying both sides by $W_{n,k}^{a}(x)$ and summing over k, we get

$$L_{n,a}^{\alpha,\beta}(f;x) = f(x)L_{n,a}^{\alpha,\beta}(1;x)+f'(x)L_{n,a}^{\alpha,\beta}(t-x;x)+\frac{f''(x)}{2}L_{n,a}^{\alpha,\beta}((t-x)^{2};x)+L_{n,a}^{\alpha,\beta}(\eta(t,x)(t-x);x).$$

Using lemma(2.2), we obtain

$$\lim_{n\to\infty} n\{L_{n,a}^{\alpha,\beta}(f;x)-f(x)\} = \Big(\alpha-\beta x+\frac{ax}{1+x}\Big)f'(x)+\frac{x^{2}+x}{2}f''(x)+\lim_{n\to\infty} nL_{n,a}^{\alpha,\beta}(\eta(t;x)(t-x)^{2};x). \tag{12}$$

Now, the last term can be obtained using Holder's inequality and lemma 2.4

$$nL_{n,a}^{\alpha,\beta}(\eta(t;x)(t-x)^{2};x) \leq n^{2}L_{n,a}^{\alpha,\beta}((t-x)^{4};x)L_{n,a}^{\alpha,\beta}(\eta(t;x)^{2};x),$$

Let $\varphi(t;x) = \eta^2(t;x)$. Then, $\lim_{t\to x}\varphi(t;x) = 0$. Therefore,

$$\lim_{n\to\infty} nL_{n,a}^{\alpha,\beta}(\eta(t;x)(t-x)^{2};x) = 0.$$

On substituting this value in equation (12) , we get the desired result.

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