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Convolution of Distributions in Sequential Approach

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In memory of Professor Jan Mikusiński
on the 100th anniversary of His birthday

Abstract. A new class of special upper approximate units together with the known Vladimirov class of special approximate units are used to consider various sequential conditions of convolvability of distributions. The equivalence of these conditions to the known conditions of convolvability given by C. Chevalley and L. Schwartz is proved together with the equivalence of the corresponding definitions of the convolution of distributions.

1. Introduction

The convolution of distributions and other generalized functions is investigated and discussed by many authors in a series of monographs [1, 4, 9, 27, 30, 37–39] and numerous papers, published in an early period of the theory of distributions [7, 8, 31–36, 42] and later [5, 11, 16, 20, 24–26, 29, 40, 41]. The convolution of distributions is often considered in particular cases, e.g. expressed in terms of supports of distributions, but there exist in the literature various general definitions of the convolution of distributions expressed under respective general convolvability conditions imposed on given distributions.

Such convolvability conditions and definitions of the convolution of distributions were introduced, in terms of integrability of functions and distributions, by C. Chevalley in [4] and L. Schwartz in [31, 32] (see also [11, 13, 29]). That the conditions and definitions of Chevalley and Schwartz are equivalent was proved

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by R. Shiraishi in [33] (see Theorem 4.1 in section 4), by means of a parametrix of an iterated Laplacian (cf. [25, 29]). The convolution and tensor product of distributions were also investigated via linear and bilinear maps between various topological spaces of distributions (see [9, 25, 37]) or distribution-valued holomorphic functions (see [2, 10, 12, 27]). Using the first of these approaches N. Ortner formulated in [25] in a suitable generalized form the conditions of convolvability of distributions given by Chevalley and Schwartz. Applying continuity of certain linear maps, he gave a new proof of Shiraishi's result and extended it to other cases.

Another general approach, elementary in the spirit of the book [1] and useful for applications (see [38, 39]), can be called sequential and will be discussed in this paper. Its idea was used already by L. Schwartz (see [31], p. 2) and then by V. S. Vladimirov (see [38], pp. 103–104; see also [39]), though their approaches slightly differed. Their concepts were based on approximations of given distributions (or some functions related to test functions) by sequences or nets of distributions (or test functions) with supports which guarantee the existence of their convolutions (or the related expressions); see the remarks preceding Definitions 5.1-5.3, concerning types (A) and (B) of approximations.

Schwartz and Vladimirov applied two different classes of so-called *approximate units* and *special approximate units*, respectively (see [5]), i.e. suitable sequences of C^∞ functions of compact supports approaching 1 in the space $\mathcal{E}$ and bounded in the space $\mathcal{B}$. We will consider in this paper only the Vladimirov class, denoting it here by Π and its members by $\{\pi_n\}$ (see Definition 2.1 in section 2). The mentioned classes of approximate units correspond to the first of the two simplest known cases of supports for which the convolution of two functions or distributions always exists:

1° at least one of the two supports is compact;

2° both supports are bounded from one side, say: both from below.

Case 1° is standard and used in sequential definitions of the convolution of distributions given by means of approximate units. One can obtain various types of such definitions multiplying either both of given distributions (as indicated by L. Schwartz in [31]; see also [16]) or only one of them (see [16]) by approximate units and passing to the limits in $\mathcal{D}'$. The described definitions for suitable classes of approximate units as well as the sequential definitions of the convolution of Vladimirov's type given in [38, 39] and in [5] appear to be equivalent (see [5], [16], [41] and Theorem 7.1).

However case 2°, investigated for the first time by B. Fisher (see e.g. [6]), is equally natural as case 1°. The counterpart of approximate units in this case are sequences approximating 1 in the space $\mathcal{E}$ of functions with supports bounded from below. Let us remark that Fisher considered only a very specific form of such sequences and his definition of the convolution of distributions depends in a concealed form on the choice of particular sequences of this kind.

Our idea of sequential definitions corresponding to case 2° relies on selecting an appropriate class of sequences of functions with supports subject to this case. The class Γ of *special upper approximate units* $\{\gamma_n\}$, introduced in Definition 2.2, consists of sequences of C^∞ functions on $\mathbb{R}^d$ whose supports are b-bounded, i.e. contained in particular acute cones of the form $[a, \infty)$ for $a \in \mathbb{R}^d$ (the definition of the class Γ admits generalizations). The class Γ corresponds naturally in case 2° to the class Π in case 1°. We use the symbols π_n and γ_n for elements of special approximate units and special upper approximate units, respectively, to recall shapes of the corresponding functions on $\mathbb{R}^1$.

The precise sequential definitions of the convolution of distributions are given in section 5. We recall Vladimirov's definition from [38, 39] and three other related definitions from [16] (of type (B) and type (A), respectively; see Definition 5.1). We give also two new sequential definitions of the convolution of distributions based on the class Γ which correspond to the Schwartz definitios of type (A) and the Vladimirov definition of type (B), respectively (see section 5).

That sequential definitions of the convolution of distributions given by means of the classes Π and Γ are equivalent is not obvious at all. This is a consequence of Theorem 7.1, proved in [16] (see also [5, 41]) and Theorem 7.2. Both theorems are formulated and proved in section 7. It follows from them that each of the sequential versions of Definitions 5.1 and 5.3 is equivalent to any of the mentioned definitions of C. Chevalley and L. Schwartz. The proof requires not only known classical results but also new techniques.

The assertions given in Theorems 7.1 and 7.2 are a part of a statement which is formulated (also for wider classes than Π and Γ considered here) but not entirely proved in [22]. To prove fully both assertions we have to show, in particular, that the conditions in Definition 5.3 imply the conditions in Definition 5.1. These implications were left without a proof in [22] and they follow from Lemma 6.2 presented here (in section 6) with a complete proof. The proof is fully elementary but requires a delicacy in selecting suitable subsequences and a subtle inductive construction of special upper approximate units, satisfying certain conditions, on the base of initial sequences of both classes Π and Γ.

The principal role in the proof of Theorem 7.2 is played by the above mentioned theorem of R. Shiraishi from [33], but we need also Theorem 3.3, formulated in section 3, which is an appropriate extension of characterization of integrable distributions proved in [5] (for a full proof of Theorem 3.3 we refer to [21, 23], sketching some ideas of the proof in Remark 3.4) and two other lemmas.

The presented proofs suitably modified can be used to receive similar results concerning the convolution in other spaces of generalized functions, e.g. the space of tempered distributions or in the spaces of ultradistributions and tempered ultradistributions (cf. results of S. Pilipović and his collaborators in [3, 17, 18, 28]).

2. Preliminaries

We use mostly standard multi-dimensional notation concerning $\mathbb{R}^d$ and $\mathbb{N}_0^d$ and the known spaces of (complex-valued) functions and distributions on $\mathbb{R}^d$: $C^\infty(\mathbb{R}^d)$, $\mathcal{E}(\mathbb{R}^d)$, $\mathcal{B}_0(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$, $\mathcal{D}(\mathbb{R}^d)$, $\mathcal{D}_K(\mathbb{R}^d)$, $\mathcal{D}'(\mathbb{R}^d)$, $\mathcal{D}'_{L^1}(\mathbb{R}^d)$ (cf. [1, 30]).

If $a = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, $x = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and $\phi \in C^\infty(\mathbb{R}^d)$, we denote $[a, \infty) := [\alpha_1, \infty) \times \ldots \times [\alpha_d, \infty) \subset \mathbb{R}^d$, $|x| := (\sum_{\iota=1}^d \xi_\iota^2)^{1/2}$, $|x|_1 := \sum_{\iota=1}^d |\xi_\iota|$ (in particular, $|k|_1 := \sum_{\iota=1}^d \kappa_\iota$ for $k = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}_0^d$) and

$$\phi^{(k)}(x) := \frac{\partial^{\kappa_1+\ldots+\kappa_d}}{\partial\xi_1^{\kappa_1}\ldots\partial\xi_d^{\kappa_d}}\,\phi(\xi_1, \ldots, \xi_d).$$

For a given set $E \subseteq \mathbb{R}^d$ and a function ϕ on $\mathbb{R}^d$, we will use the following convenient notation:

$$E \sqsubset \mathbb{R}^d \quad :\Leftrightarrow \quad E \text{ is a compact subset of } \mathbb{R}^d \tag{2.1}$$

and

$$E^{\triangle} := \{(x,y) \in \mathbb{R}^{2d} \colon\ x+y \in E\}; \qquad \phi^{\triangle}(x,y) := \phi(x+y), \ \ x,y \in \mathbb{R}^d.$$

Clearly, if E is bounded (compact) in $\mathbb{R}^d$, then $E^{\triangle}$ is bounded (compact) in $\mathbb{R}^{2d}$ only in case $E = \emptyset$. If $\phi \in \mathcal{D}(\mathbb{R}^d)$, then $\phi^{\triangle} \in C^{\infty}(\mathbb{R}^{2d})$ but $\phi^{\triangle} \in \mathcal{D}(\mathbb{R}^{2d})$ only in case $\phi = 0$.

Recall that the topology in the space $\mathcal{D}_K(\mathbb{R}^d)$ for a fixed $K \sqsubset \mathbb{R}^d$ is defined by the family $\{q_{i,K} \colon\ i \in \mathbb{N}_0\}$ of seminorms, the topology in the space $\mathcal{E}(\mathbb{R}^d)$ by the family $\{q_{i,K} \colon\ i \in \mathbb{N}_0, K \sqsubset \mathbb{R}^d\}$ of seminorms, and the topologies in the spaces $\mathcal{B}_0(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d)$ by the family $\{q_i \colon\ i \in \mathbb{N}_0\}$ of norms, where

$$q_{i,K}(\phi) := \max_{0 \le |k|_1 \le i} \sup_{x \in K} |\phi^{(k)}(x)|; \qquad q_i(\phi) := \max_{0 \le |k|_1 \le i} \sup_{x \in \mathbb{R}^d} |\phi^{(k)}(x)|$$

for functions ϕ in the respective spaces. We use the symbol $\langle S, \phi \rangle_d$ for the value of $S \in \mathcal{D}'(\mathbb{R}^d)$ on $\phi \in \mathcal{D}(\mathbb{R}^d)$ to mark the dimension of $\mathbb{R}^d$. For $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $S \in \mathcal{D}'(\mathbb{R}^d)$, we define $\check{\phi} \in \mathcal{D}(\mathbb{R}^d)$ and $\check{S} \in \mathcal{D}'(\mathbb{R}^d)$ by $\check{\phi}(x) := \phi(-x)$ for $x \in \mathbb{R}^d$ and $\langle \check{S}, \varphi \rangle_d := \langle S, \check{\varphi} \rangle_d$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. By $\phi \otimes \psi$ for $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$ we mean the function in $\mathcal{D}(\mathbb{R}^{2d})$, given by $(\phi \otimes \psi)(x,y) := \phi(x)\psi(y)$ for $x, y \in \mathbb{R}^d$, and by $S \otimes T$ we traditionally mean the tensor product of $S, T \in \mathcal{D}'(\mathbb{R}^d)$, an element of $\mathcal{D}'(\mathbb{R}^{2d})$ (see [31]).

Beside the usual support (of a function or distribution) we consider another type of support, the *unit support* $s^1(\phi) := \phi^{-1}(\{1\})$ of a smooth function ϕ. Clearly, $s^1(\phi) \subset \operatorname{supp} \phi$. Beside bounded we will also consider b-bounded subsets of $\mathbb{R}^d$: we call a set $B \subset \mathbb{R}^d$ *b-bounded* if $B \subset [a, \infty)$ for some $a \in \mathbb{R}^d$.

Definition 2.1. [see [5]] By a *special approximate unit* on $\mathbb{R}^d$ we mean a sequence $\{\sqcap_n\}$ of smooth functions on $\mathbb{R}^d$ with bounded (i.e. compact) supports such that

(∗) if $A \subset \mathbb{R}^d$ is bounded, then $A \subset s^1(\sqcap_n)$ for sufficiently large $n \in \mathbb{N}$

and

$$M_i := \sup_{n \in \mathbb{N}} q_i(\sqcap_n) < \infty, \qquad i \in \mathbb{N}_0. \tag{2.2}$$

Definition 2.2. By a *special upper approximate unit* on $\mathbb{R}^d$ we mean a sequence $\{\Gamma_n\}$ of smooth functions on $\mathbb{R}^d$ with b-bounded supports such that the following counterpart of condition (∗) is satisfied:

(∗∗) if $B \subset \mathbb{R}^d$ is b-bounded, then $B \subset s^1(\Gamma_n)$ for sufficiently large $n \in \mathbb{N}$

and

$$N_i := \sup_{n \in \mathbb{N}} q_i(\Gamma_n) < \infty, \qquad i \in \mathbb{N}_0. \tag{2.3}$$

We denote the classes of all special approximate units and all special upper approximate units (on $\mathbb{R}^d$) by Π and Γ (by Π_d and Γ_d), respectively.

3. Integrable Distributions

Definition 3.1. We call an $R \in \mathcal{D}'(\mathbb{R}^d)$ *extendible to a function* $\psi \in C^\infty(\mathbb{R}^d)$ if the numerical sequence $\{\langle R, \sqcap_n \psi\rangle_d\}$ is Cauchy for every $\{\sqcap_n\} \in \Pi_d$.

Clearly, if $R \in \mathcal{D}'(\mathbb{R}^d)$ is extendible to each $\psi \in \mathcal{B}(\mathbb{R}^d)$, then the formula

$$\langle \tilde{R}, \psi\rangle_d := \lim_{n\to\infty} \langle R, \sqcap_n \psi\rangle_d, \qquad \psi \in \mathcal{B}(\mathbb{R}^d) \tag{3.1}$$

for any $\{\sqcap_n\} \in \Pi_d$, uniquely defines the mapping $\tilde{R}\colon\ \mathcal{B}(\mathbb{R}^d) \to \mathbb{C}$ which is a linear functional on $\mathcal{B}(\mathbb{R}^d)$ and $\tilde{R}|_{\mathcal{D}(\mathbb{R}^d)} = R$.

Definition 3.2. If $R \in \mathcal{D}'(\mathbb{R}^d)$ is extendible to each function $\psi \in \mathcal{B}(\mathbb{R}^d)$, we call the linear functional $\tilde{R}$ on $\mathcal{B}(\mathbb{R}^d)$ given by (3.1) the *extension* of R to $\mathcal{B}(\mathbb{R}^d)$ whenever it is continuous on $\mathcal{B}(\mathbb{R}^d)$.

Theorem 3.3 (see [5, 21, 23]). *Let $R \in \mathcal{D}'(\mathbb{R}^d)$. The following are equivalent:*

(a) *there exist a $j \in \mathbb{N}_0$ and a $C > 0$ such that*

$$|\langle R, \varphi\rangle_d| \le C q_j(\varphi) \tag{3.2}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$;

(b) *there exists a $j \in \mathbb{N}_0$ such that for every $\varepsilon > 0$ there exists a $K \sqsubset \mathbb{R}^d$ with the property: $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and* supp $\varphi \cap K = \emptyset$ *imply*

$$|\langle R, \varphi\rangle_d| \le \varepsilon q_j(\varphi); \tag{3.3}$$

(c) *for every $\{\sqcap_n\} \in \Pi_d$ the sequence $\{\langle R, \sqcap_n\rangle_d\}$ is Cauchy.*

(d) *R is extendible to all $\psi \in \mathcal{B}(\mathbb{R}^d)$ and $\tilde{R}$ given by (3.1) is the extension of R to $\mathcal{B}(\mathbb{R}^d)$ for which there exist a $j \in \mathbb{N}_0$ and $C > 0$ such that*

$$|\langle \tilde{R}, \psi\rangle_d| \le C q_j(\psi) \tag{3.4}$$

for all $\psi \in \mathcal{B}(\mathbb{R}^d)$;

(e) *R is extendible to all $\psi \in \mathcal{B}(\mathbb{R}^d)$ and $\tilde{R}$ given by (3.1) is the extension of R to $\mathcal{B}(\mathbb{R}^d)$ for which there exists $j \in \mathbb{N}_0$ such that for every $\varepsilon > 0$ there exists a $K \sqsubset \mathbb{R}^d$ with the property: $\psi \in \mathcal{B}(\mathbb{R}^d)$ and* supp $\psi \cap K = \emptyset$ *imply*

$$|\langle \tilde{R}, \psi\rangle_d| \le \varepsilon q_j(\psi). \tag{3.5}$$

Remark 3.4. Conditions (a), (b), (c), whose equivalence is proved in [5], are expressed in terms of a given $R \in \mathcal{D}'(\mathbb{R}^d)$ and test functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Condition ($a$), due to density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{B}_0(\mathbb{R}^d)$ implies that R can be uniquely extended to an *integrable distribution* R°, element of the topological dual $\mathcal{B}_0'(\mathbb{R}^d) =: \mathcal{D}'_{L^1}(\mathbb{R}^d)$

of $\mathcal{B}_0(\mathbb{R}^d)$ (see [30], p. 200), satisfying (3.2) for all $\varphi \in \mathcal{B}_0(\mathbb{R}^d)$. We completed the list of equivalent conditions with (*d*) and (*e*) to show explicitly that each distribution R satisfying any of conditions (*a*)–(*c*) as well as the corresponding integrable distribution R° can be uniquely extended to the linear functional $\tilde{R}$ on $\mathcal{B}(\mathbb{R}^d)$, defined by (3.1); moreover, the estimates given for R in (3.2)–(3.3) are preserved for $\tilde{R}$ in the form of (3.4)–(3.5), so that the continuity of $\tilde{R}$ is assured (for clarity, we use the different symbols to the end of the section, but later we will identify R and $\tilde{R}$). Consequently, $\tilde{R} \in \mathcal{B}'(\mathbb{R}^d)$ and $\langle \tilde{R}, \psi \rangle_d$ is well defined if $\psi \in \mathcal{B}(\mathbb{R}^d)$, in particular if $\psi = 1$ or $\psi = \Gamma_n$ ($n \in \mathbb{N}$) for any $\{\Gamma_n\} \in \Gamma_d$ (see Lemma 3.5).

That (*d*) and (*e*) are indeed equivalent to each of conditions (*a*), (*b*), (*c*) can be shortly justified as follows. Assuming that a distribution R satisfies (*b*), one can easily deduce that R is extendible to an arbitrary $\psi \in \mathcal{B}(\mathbb{R}^d)$. If, in addition, supp $\psi \cap K = \emptyset$ for a given $\psi \in \mathcal{B}(\mathbb{R}^d)$ and a set $K \sqsubset \mathbb{R}^d$ indicated in (*b*), then one can replace φ by the functions $\varphi_n := \Pi_n \psi$ ($n \in \mathbb{N}$) in inequality (3.3) for any $\{\Pi_n\} \in \Pi_d$ and pass to the limit as $n \to \infty$. Hence, due to (3.1), the Leibniz formula and (2.2), the distribution R satisfies condition (*e*). Condition (*d*) easily results from (*e*) and (*a*) is the restriction of (*d*) to the subspace $\mathcal{D}(\mathbb{R}^d)$ of $\mathcal{B}(\mathbb{R}^d)$. For details of the whole proof of the equivalence of conditions (*a*)–(*e*) see [21, 23].

Lemma 3.5. *If* $R \in \mathcal{D}'_{L^1}(\mathbb{R}^d)$, *then*

$$\lim_{n\to\infty} \langle R, \Pi_n \rangle_d = \langle \tilde{R}, 1 \rangle_d = \lim_{n\to\infty} \langle \tilde{R}, \Gamma_n \rangle_d \tag{3.6}$$

for any $\{\Pi_n\} \in \Pi_d$ *and* $\{\Gamma_n\} \in \Gamma_d$, *where* $\tilde{R}$ *is defined in (3.1).*

Proof. The first equality in (3.6) follows directly from (3.1). To prove the second one fix $\chi \in \mathcal{D}(\mathbb{R}^d)$ with $s^1(\chi) \supset [-1,1]^d$ and define $\chi_n \in \mathcal{D}(\mathbb{R}^d)$ by $\chi_n(x) := \chi(x/n)$ for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Clearly,

$$s^1(\chi_n) \supset [-n,n]^d; \quad 1 \le q_i(\chi_n) \le q_i(\chi), \qquad n \in \mathbb{N},\ i \in \mathbb{N}_0. \tag{3.7}$$

By the assumption and Theorem 3.3, condition (*e*) holds for a certain $j \in \mathbb{N}_0$. Fix $\varepsilon > 0$ and find a $K \sqsubset \mathbb{R}^d$ such that

$$|\langle \widetilde{R}, \psi \rangle_d| \le \frac{\varepsilon}{a_j}\, q_j(\psi), \tag{3.8}$$

whenever $\psi \in \mathcal{B}$ and supp $\psi \cap K = \emptyset$, where $a_j := 2^{j+1}(1 + q_j(\chi))(1 + N_j)$. Due to (3.7), we may choose an $n_0 \in \mathbb{N}$ such that $s^1(\chi_{n_0}) \supset K$. In view of (3.8), the Leibniz formula, (3.7) and (2.3), we have

$$|\langle \widetilde{R}, 1 \rangle_d - \langle \widetilde{R}, \Gamma_m \rangle_d| \le |\langle \widetilde{R}, (1 - \chi_{n_0})(1 - \Gamma_m) \rangle_d| < \varepsilon$$

for sufficiently large $m \in \mathbb{N}$ (such that $s^1(\Gamma_m) \supset$ supp χ_{n_0}). □

4. Convolution of Distributions

It is well known that the convolution in $\mathcal{D}'(\mathbb{R}^d)$ of distributions $S, T \in \mathcal{D}'(\mathbb{R}^d)$ can be defined as the distribution $S * T$ given by

$$\langle S * T, \varphi \rangle_d := \langle S \otimes T, \varphi^\triangle \rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{4.1}$$

whenever the right hand side is well defined.

In particular, conditions guaranteeing that (4.1) makes sense can be expressed in terms of the supports A of S and B of T (closed sets in $\mathbb{R}^d$). Express the conditions formulated by J. Horváth in [9], p. 383 (see also [10], [12], [11], [5], [24]), with the use of the notation introduced in (2.1), in the following way:

(Σ) $(A \times B) \cap K^\triangle \sqsubset \mathbb{R}^{2d}$ for every $K \sqsubset \mathbb{R}^d$,

(Σ′) $A \cap (K - B) \sqsubset \mathbb{R}^d$ for every $K \sqsubset \mathbb{R}^d$,

which are equivalent for any closed sets $A, B \subseteq \mathbb{R}^d$ (see [9], pp. 383–384). It is clear that if given sets $A, B \subseteq \mathbb{R}^d$ are closed one can equivalently reformulate conditions (Σ) and (Σ′) as follows:

(Δ) $(A \times B) \cap K^\triangle$ is bounded in $\mathbb{R}^{2d}$ for every bounded set K in $\mathbb{R}^d$,

(Δ′) $A \cap (K - B)$ is bounded in $\mathbb{R}^d$ for every bounded set K in $\mathbb{R}^d$.

Independently, Jan Mikusiński introduced in [20] (see also [1], pp. 124–127) a condition which can be formulated in the following sequential form:

(M) if $x_n \in A$ and $y_n \in B$ for $n \in \mathbb{N}$, then $|x_n| + |y_n| \to \infty$ as $n \to \infty$
implies $|x_n + y_n| \to \infty$ as $n \to \infty$.

Sets $A, B \subseteq \mathbb{R}^d$ satisfying (M) are called in [20] and [1] *compatible*. It is worth noting that condition (M) is equivalent to each of conditions (Σ), (Σ′) for closed sets $A, B \subseteq \mathbb{R}^d$ and to each of conditions (Δ), (Δ′) for arbitrary sets A, B in $\mathbb{R}^d$, not necessarily closed.

Recall the two particular cases of subsets A, B of $\mathbb{R}^d$ satisfying the above mentioned equivalent conditions of compatibility:

1° at least one of the sets A, B is bounded;

2° both sets A, B are b-bounded (see section 2).

Case 2° is a specification of a more general situation where A, B are contained in suitable cones (see [1], pp. 129–130, [39], pp. 63–64). It should be noted, however, that there exist compatible sets in $\mathbb{R}^1$ which are unbounded from both sides as well as compatible sets in $\mathbb{R}^d$, unbounded in each direction of $\mathbb{R}^d$ (see [14], [15], [5] and [19]).

Formula (4.1) can be used not only in the above particular cases. A variation of this formula was applied by L. Schwartz in [31] (see also [11, 33]) in his general definition of the convolution of distributions $S, T \in \mathcal{D}'(\mathbb{R}^d)$:

$$\langle S \overset{\mathrm{S}}{*} T, \varphi \rangle_d := \langle (S \otimes T)\, \varphi^\triangle, 1 \rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{4.2}$$

given under the following general integrability condition:

(s) $(S \otimes T)\, \varphi^\triangle \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d})$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, which is satisfied if condition (△) holds; here and further on, a distribution R satisfying any of the conditions in Theorem 3.3 is identified with its extension $\widetilde{R} \in \mathcal{B}'(\mathbb{R}^d)$ (see Remark 3.4).

Earlier, C. Chevalley introduced in [4] the two symmetric general definitions of the convolution of distributions in $\mathcal{D}'(\mathbb{R}^d)$:

$$\langle S \overset{\mathrm{C}_1}{*} T, \varphi\rangle_d \;:=\; \langle S(\check{T} * \varphi), 1\rangle_d, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{4.3}$$

$$\langle S \overset{\mathrm{C}_2}{*} T, \varphi\rangle_d \;:=\; \langle (\check{S} * \varphi)T, 1\rangle_d, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{4.4}$$

under the following conditions of integrability of distributions:

(C$_1$) $\quad S(\check{T} * \varphi) \in \mathcal{D}'_{L^1}(\mathbb{R}^d)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

(C$_2$) $\quad (\check{S} * \varphi)T \in \mathcal{D}'_{L^1}(\mathbb{R}^d)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

respectively (see [4], p. 67). Moreover, Chevalley gave in [4] a third general definition of the convolution of distributions in $\mathcal{D}'(\mathbb{R}^d)$ under a corresponding condition of integrability of functions (see [4], p. 112); let us denote the convolution by $\overset{\mathrm{C}}{*}$ and the condition by (C).

The above definitions (as well as other ones, see e.g. [13, 32]) are equivalent. We recall the following equivalence result of R. Shiraishi (see also [25]):

Theorem 4.1 (see [33]). *Let $S, T \in \mathcal{D}'(\mathbb{R}^d)$. Conditions* (S), (C$_1$), (C$_2$), *and* (C) *are equivalent. If any of the conditions holds, then the convolutions defined in (4.2), (4.3), (4.4), and in [4], p. 112, exist and are equal:*

$$S \overset{\mathrm{S}}{*} T = S \overset{\mathrm{C}_1}{*} T = S \overset{\mathrm{C}_2}{*} T = S \overset{\mathrm{C}}{*} T.$$

5. Sequential Definitions of Convolution

Another way of defining the convolution of distributions in $\mathcal{D}'(\mathbb{R}^d)$ consists in using suitable approximations with supports guaranteeing that the right hand side of formula (4.1) is well defined and then passing to the limit. One may approximate in (4.1) either (A) the distributions S, T in $\mathcal{D}'(\mathbb{R}^d)$ or (B) the functions $\varphi^{\triangle}$ in $\mathcal{E}(\mathbb{R}^{2d})$ and there is a subtle difference between these two possibilities (see Remark 5.4).

The first possibility was indicated by L. Schwartz in [31], p. 2, and the other one was used by V. S. Vladimirov in [38], pp. 103–104 (see also [39], pp. 51–52). To get appropriate approximations they applied two different classes of so-called (see [5]) approximate units, i.e. nets or sequences of functions of the class $\mathcal{D}$, approaching the constant function 1 in $\mathcal{E}$ and bounded in $\mathcal{B}$.

In the definition below we use only the class of all special approximate units, described in Definition 2.1 in section 2 and applied by V. S. Vladimirov in his definition of the convolution of distributions.

Definition 5.1. Let $S, T \in \mathcal{D}'(\mathbb{R}^d)$. We give the following four definitions of the *convolution of distributions S* and T in $\mathcal{D}'(\mathbb{R}^d)$:

$$\langle S \overset{\Pi_0}{*} T, \varphi\rangle_d := \lim_{n\to\infty} \langle(\pi_n^1 S)\otimes(\pi_n^2 T), \varphi^\triangle\rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{5.1}$$

$$\langle S \overset{\Pi_1}{*} T, \varphi\rangle_d := \lim_{n\to\infty} \langle(\pi_n^1 S)\otimes T, \varphi^\triangle\rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{5.2}$$

$$\langle S \overset{\Pi_2}{*} T, \varphi\rangle_d := \lim_{n\to\infty} \langle S\otimes(\pi_n^2 T), \varphi^\triangle\rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{5.3}$$

$$\langle S \overset{\Pi}{*} T, \varphi\rangle_d := \lim_{n\to\infty} \langle S\otimes T, \pi_n \varphi^\triangle\rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{5.4}$$

for any $\{\pi_n^1\},\{\pi_n^2\} \in \Pi_d$ and $\{\pi_n\} \in \Pi_{2d}$, under the corresponding convolvability conditions imposed on the distributions S and T:

(π_0) $\{\langle(\pi_n^1 S)\otimes(\pi_n^2 T), \varphi^\triangle\rangle_{2d}\}$ is a Cauchy sequence,
(π_1) $\{\langle(\pi_n^1 S)\otimes T, \varphi^\triangle\rangle_{2d}\}$ is a Cauchy sequence,
(π_2) $\{\langle S\otimes(\pi_n^2 T), \varphi^\triangle\rangle_{2d}\}$ is a Cauchy sequence,
(π) $\{\langle S\otimes T, \pi_n \varphi^\triangle\rangle_{2d}\}$ is a Cauchy sequence,

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\{\pi_n^1\},\{\pi_n^2\} \in \Pi_d$ and $\{\pi_n\} \in \Pi_{2d}$, respectively.

All expressions under the limit sign in (5.1)–(5.4) are well defined, since the supports of the respective distributions in (5.1)–(5.3) satisfy case 1° of compatibility and $\pi_n\varphi^\triangle \in \mathcal{D}(\mathbb{R}^{2d})$ in case of formula (5.4). Moreover, the convolvability conditions guarantee that the limits exist and do not depend on the choice of special approximate units, i.e. definitions (5.1)–(5.4) are consistent.

Remark 5.2. Let us recall that the definition (5.1) was discussed in [31] for a wider class, namely for the class of all *approximate units* on $\mathbb{R}^d$ (this name is used in [5]), than the above class Π_d of all special approximate units. Definitions (5.2) and (5.3) were considered in [16] for both classes (see also [41]). Definition (5.4) was given in [38] for the class Π_{2d} and then extended in [5] to the class of all approximate units on $\mathbb{R}^{2d}$. Each of the above definitions (considered for both classes) is equivalent to each of the definitions of the convolution mentioned in Theorem 4.1 (see [5, 16, 41]).

In the following definition, alternative to Definition 5.1, we will use special upper approximate units (see Definition 2.2) instead of special approximate units.

Definition 5.3. Let $S, T \in \mathcal{D}'(\mathbb{R}^d)$. We give the following two definitions of the *convolution of distributions S* and T in $\mathcal{D}'(\mathbb{R}^d)$:

$$\langle S \overset{\Gamma_0}{*} T, \varphi\rangle_d := \lim_{n\to\infty} \langle(\gamma_n^1 S)\otimes(\gamma_n^2 T), \varphi^\triangle\rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{5.5}$$

$$\langle S \overset{\Gamma}{*} T, \varphi\rangle_{2d} := \lim_{n\to\infty} \langle S\otimes T, \gamma_n\varphi^\triangle\rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{5.6}$$

for any $\{\gamma_n^1\},\{\gamma_n^2\} \in \Gamma_d$ and $\{\gamma_n\} \in \Gamma_{2d}$, under the corresponding convolvability conditions imposed on the distributions S and T:

(γ_0) $\{\langle(\gamma_n^1 S)\otimes(\gamma_n^2 T), \varphi^\triangle\rangle_{2d}\}$ is a Cauchy sequence,
(γ) $\{\langle S\otimes T, \gamma_n\varphi^\triangle\rangle_{2d}\}$ is a Cauchy sequence,

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\{\text{r}_n^1\}$,$\{\text{r}_n^2\} \in \Gamma_d$ and $\{\text{r}_n\} \in \Gamma_{2d}$, respectively.

The expressions under the limit sign in (5.5)–(5.6) are well defined, since the supports of $\{\text{r}_n^1\}$ and $\{\text{r}_n^2\}$ in (5.5) satisfy case 2° of compatibility and $\text{r}_n\varphi^\triangle \in \mathcal{D}(\mathbb{R}^{2d})$ in case of formula (5.6). Moreover, the convolvability conditions guarantee that the limits exist and do not depend on the choice of special upper approximate units, i.e. definitions (5.5)–(5.6) are consistent.

Remark 5.4. Due to Lemma 6.1 below, the definitions (5.1), (5.2), (5.3) can be written in the form of (5.4) with $\sqcap_n$ replaced by the functions of the form $\sqcap_n^1 \otimes \sqcap_n^2$, $\sqcap_n^1 \otimes 1$, $1 \otimes \sqcap_n^2$, respectively, where $\{\sqcap_n^1\}, \{\sqcap_n^2\} \in \Pi_d$ and 1 denotes the constant function on $\mathbb{R}^d$. In particular, the definition (5.1), which is of the type discussed in [31] (though reduced here to the class Π_d), can be expressed in the form of the definition (5.4) introduced in [38], but the first one is considered for the narrower class of all $\{\sqcap_n\} \in \Pi_{2d}$ with $\sqcap_n$ of the form $\sqcap_n^1 \otimes \sqcap_n^2$, where $\{\sqcap_n^1\}, \{\sqcap_n^2\} \in \Pi_d$. Similarly, the definition (5.5) can be expressed in the form (5.6), but it is considered for the narrower class of $\{\text{r}_n\} \in \Gamma_{2d}$ reduced to the sequences of functions r_n of the form $\text{r}_n^1 \otimes \text{r}_n^2$, where $\{\text{r}_n^1\}, \{\text{r}_n^2\} \in \Gamma_d$, so the equivalence of the definitions is not obvious. Nevertheless, the definitions (5.5) and (5.6) are equivalent to any of the definitions (5.1)–(5.4) (see Theorem 7.2).

The definition of the convolution in $\mathcal{D}'(\mathbb{R}^d)$ given by means of formula (5.5) was inspired by one of B. Fisher's definitions of the convolution of distributions on $\mathbb{R}^1$ (see [6], Definition 6), based on special upper approximate units of a very particular form. Notice that the specific form of these sequences does not guarantee the consistency of the definition of the convolution which may depend on the choice of the sequences.

6. Lemmas

In the proof of Theorem 7.2 we need the following two lemmas.

Lemma 6.1. *Let $S, T \in \mathcal{D}'(\mathbb{R}^d)$. Let $\sigma, \tau \in \mathcal{E}(\mathbb{R}^d)$ be functions whose supports $A :=$ supp σ, $B :=$ supp τ satisfy case 1° or case 2° of compatibility, indicated above, and let $\omega \in \mathcal{E}(\mathbb{R}^{2d})$ be a function such that* supp $\omega \subset I^\triangle$ *for some compact set $I \sqsubset \mathbb{R}^d$. Then*

$$\langle(\sigma S) \otimes (\tau T), \omega\rangle_{2d} = \langle S \otimes T, (\sigma \otimes \tau)\omega\rangle_{2d}. \tag{6.1}$$

Proof. That (6.1) holds for $\omega = \phi \otimes \psi$ with $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$ and so for all $\omega \in \mathcal{D}(\mathbb{R}^{2d})$, follows from the definition of the tensor product of distributions.

In the general case, fix $\omega \in \mathcal{E}(\mathbb{R}^{2d})$ with supp $\omega \subset I^\triangle$ and $\varepsilon > 0$, and choose $\chi_1, \chi_2 \in \mathcal{E}(\mathbb{R}^{2d})$ such that

$$A \times B \subset s^1(\chi_1) \subset \text{supp } \chi_1 \subset A_\varepsilon \times B_\varepsilon; \quad I^\triangle \subset s^1(\chi_2) \subset \text{supp } \chi_2 \subset I_\varepsilon^\triangle,$$

where $A_\varepsilon, B_\varepsilon, I_\varepsilon$ are $\varepsilon-$neighborhoods of A, B, I, respectively.

Denote $\rho := \sigma \otimes \tau$ and $R := (\sigma S) \otimes (\tau T)$. The above inclusions imply that $\chi_1\rho = \rho$, $\chi_1 R = R$, $\chi_2\omega = \omega$ and supp $(\chi_1\chi_2)$ is a subset of $(A_\varepsilon \times B_\varepsilon) \cap I_\varepsilon^\triangle$. But the latter is a bounded set in $\mathbb{R}^{2d}$ in both cases 1° and 2° (see conditions (Σ) and (Δ)). Hence $\chi_1\chi_2 \in \mathcal{D}(\mathbb{R}^{2d})$ and we can reduce our general case to the case considered at the beginning:

$$\langle R, \omega\rangle_{2d} = \langle \chi_1 R, \chi_2\omega\rangle_{2d} = \langle S\otimes T, \rho\chi_1\chi_2\omega\rangle_{2d} = \langle S\otimes T, (\sigma\otimes\tau)\omega\rangle_{2d}.$$

The assertion of the lemma is thus proved. □

Lemma 6.2. *Let* $R \in \mathcal{D}'(\mathbb{R}^{2d})$. *Assume that* supp $R \subset I^{\triangle}$ *for a certain compact set* $I \sqsubset \mathbb{R}^d$ *and there is an* $\alpha \in \mathbb{C}$ *such that*

$$\lim_{n\to\infty}\langle R, \gamma_n^1 \otimes \gamma_n^2\rangle_{2d} = \alpha \tag{6.2}$$

for arbitrary special upper approximate units $\{\gamma_n^1\}, \{\gamma_n^2\} \in \Gamma_d$. *Then*

$$\lim_{n\to\infty}\langle R, \pi_n^1 \otimes \pi_n^2\rangle_{2d} = \alpha \tag{6.3}$$

for arbitrary special approximate units $\{\pi_n^1\}, \{\pi_n^2\} \in \Pi_d$.

Proof. We begin with choosing $\chi \in \mathcal{E}(\mathbb{R}^{2d})$ and a set $J \sqsubset \mathbb{R}^d$ such that

$$I^{\triangle} \subset s^1(\chi) \subset \operatorname{supp}\chi \subset J^{\triangle}. \tag{6.4}$$

Obviously, $R = \chi R$ on $\mathbb{R}^{2d}$.

Fix $\{\pi_n^1\}, \{\pi_n^2\} \in \Pi_d$ and denote $\alpha_n := \langle R, \pi_n^1\otimes\pi_n^2\rangle_{2d}$ for $n \in \mathbb{N}$. We have to select from every subsequence $\{\beta_n\}$ of $\{\alpha_n\}$ a subsequence $\{\gamma_n\}$ such that $\lim_{n\to\infty}\gamma_n = \alpha$. Let us fix $\{\beta_n\}$, i.e. fix an increasing sequence of $m_n \in \mathbb{N}$ (in symbols: $m_n \uparrow \infty$) such that $\beta_n = \alpha_{m_n}$, and denote shortly $\overline{\pi}_n^{\iota} := \pi_{m_n}^{\iota}$ for $n \in \mathbb{N}$ and $\iota \in \{1,2\}$. We have to find a sequence of indices $r_n \in \mathbb{N}$, $r_n \uparrow \infty$, such that $\gamma_n \to \alpha$ as $n \to \infty$, where $\gamma_n := \beta_{r_n} = \langle R, \overline{\pi}_{r_n}^1 \otimes \overline{\pi}_{r_n}^2\rangle_{2d}$ for $n \in \mathbb{N}$.

In addition, fix $\{\gamma_n^1\}, \{\gamma_n^2\} \in \Gamma_d$ and denote

$$\vartriangle_{p,r}^{\iota} := \gamma_p^{\iota} - \gamma_p^{\iota}\overline{\pi}_r^{\iota}, \qquad p, r \in \mathbb{N},\ \iota \in \{1,2\}. \tag{6.5}$$

Clearly, we have

$$\lim_{r\to\infty} \vartriangle_{p,r}^{\iota} = 0 \quad \text{in } \mathcal{E}(\mathbb{R}^d), \qquad p \in \mathbb{N},\ \iota \in \{1,2\}. \tag{6.6}$$

Starting from the fixed pair $\{\gamma_n^1\}, \{\gamma_n^2\} \in \Gamma_d$ of special upper approximate units and the fixed pair $\{\overline{\pi}_n^1\}, \{\overline{\pi}_n^2\} \in \Pi_d$ of subsequences of the given pair $\{\pi_n^1\}, \{\pi_n^2\} \in \Pi_d$ of special approximate units, we are going to select such pairs of respective subsequences $\{\gamma_{p_n}^1\}, \{\gamma_{p_n}^2\}$ of $\{\gamma_n^1\}, \{\gamma_n^2\}$ and $\{\overline{\pi}_{r_n}^1\}, \{\overline{\pi}_{r_n}^2\}$ of $\{\overline{\pi}_n^1\}, \{\overline{\pi}_n^2\}$ that the sequences $\{\widetilde{\gamma}_n^1\}, \{\widetilde{\gamma}_n^2\}$ of the functions defined by means of formula (6.7) below form a new pair of special upper approximate units satisfying certain additional conditions.

We have to construct in a suitable manner two increasing sequences $\{p_n\}$ and $\{r_n\}$ of positive integers; the second one is going to be just the required sequence of indices. We will define p_n and r_n inductively in such a way that if the functions $\widetilde{\gamma}_n^1, \widetilde{\gamma}_n^2$ of the class $\mathcal{E}(\mathbb{R}^d)$ are defined by the formula:

$$\widetilde{\gamma}_n^{\iota} := \gamma_{p_n}^{\iota} + \overline{\pi}_{r_n}^{\iota} - \gamma_{p_n}^{\iota}\overline{\pi}_{r_n}^{\iota} = \vartriangle_{p_n,r_n}^{\iota} + \overline{\pi}_{r_n}^{\iota}, \qquad n \in \mathbb{N},\ \iota \in \{1,2\} \tag{6.7}$$

and if the functions $\theta_n^1, \theta_n^2, \theta_n^3$ of the class $\mathcal{E}(\mathbb{R}^{2d})$ are defined by the formulae:

$$\theta_n^1 := \Delta^1_{p_n,r_n} \otimes \overline{\Pi}^2_{r_n}, \quad \theta_n^2 := \overline{\Pi}^1_{r_n} \otimes \Delta^2_{p_n,r_n}, \quad \theta_n^3 := \Delta^1_{p_n,r_n} \otimes \Delta^2_{p_n,r_n} \tag{6.8}$$

for all $n \in \mathbb{N}$, then the following two conditions are satisfied:

$$\operatorname{supp} \widetilde{\Gamma}^\iota_n \subset s^1(\Gamma^\iota_{p_{n+1}}), \qquad n \in \mathbb{N},\ \iota \in \{1,2\}; \tag{6.9}$$

and

$$|\langle R, \theta_n^\kappa \rangle_{2d}| < 1/n, \qquad n \in \mathbb{N},\ \kappa \in \{1,2,3\}. \tag{6.10}$$

We will need the following two properties of supports of the functions $\widetilde{\Gamma}^\iota_n$ and of the unit supports of $\widetilde{\Gamma}^\iota_n$ and $\overline{\Pi}^\iota_n$:

$$\operatorname{supp} \widetilde{\Gamma}^\iota_n \quad \text{are b-bounded,} \qquad \iota \in \{1,2\},\ n \in \mathbb{N}; \tag{6.11}$$

and

$$s^1(\Gamma^\iota_{p_n}) \cup s^1(\overline{\Pi}^\iota_{r_n}) \subseteq s^1(\widetilde{\Gamma}^\iota_n), \qquad \iota \in \{1,2\},\ n \in \mathbb{N}, \tag{6.12}$$

both resulting easily from (6.7). Notice that the second property is a consequence of the specific form of the functions $\widetilde{\Gamma}^\iota_n$. Namely, the following implication holds for any $x \in \mathbb{R}^d$: if $\Gamma^\iota_{p_n}(x) = 1$ or $\overline{\Pi}^\iota_{r_n}(x) = 1$, then $\widetilde{\Gamma}^\iota_n(x) = 1$, by the first equality in (6.7). It is important to notice that properties (6.11) and (6.12) follow directly from (6.7) considered for arbitrary $p_n, r_n \in \mathbb{N}$ and thus they are independent of the inductive construction.

Put $p_1 := 1$ and fix $n \in \mathbb{N}$. Assume that the indices $p_1 < \ldots < p_n$ and, in case $n > 1$, the indices $r_1 < \ldots < r_{n-1}$ are already chosen. We may select an index r_n, with $r_n > r_{n-1}$ in case $n > 1$, such that the functions θ_n^κ and $\chi\theta_n^\kappa$ are small enough in $\mathcal{E}(\mathbb{R}^{2d})$ and in $\mathcal{D}_{K_n}(\mathbb{R}^{2d})$ for a certain $K_n \Subset \mathbb{R}^{2d}$, respectively, to fulfil the inequalities in (6.10) for $\kappa \in \{1,2,3\}$. In fact, such a possibility follows from (6.8), (6.6), (2.2) and continuity of R restricted to $\mathcal{D}_{K_n}(\mathbb{R}^d)$. Namely $R = \chi R$, the supports of θ_n^κ are, by (6.8) and (6.5), of the form $A_n^\kappa \times B_n^\kappa$, where A_n^κ and B_n^κ are b-bounded sets in $\mathbb{R}^d$ and $\operatorname{supp} \chi \subset J^\Delta$, in view of (6.4). Consequently, $(A_n^\kappa \times B_n^\kappa) \cap J^\Delta \Subset \mathbb{R}^{2d}$, due to condition (Σ) fulfilled in case 2° of compatibility of the sets A_n^κ and B_n^κ for $\kappa \in \{1,2,3\}$. If the indices $p_1 < \ldots < p_n$ and $r_1 < \ldots < r_n$ are already constructed we may choose an index $p_{n+1} > p_n$ such that (6.9) holds, by (6.11) and $(**)$. The inductive construction of sequences $\{p_n\}$ and $\{r_n\}$ satisfying conditions (6.9) and (6.10) is thus completed.

We will show that the sequences $\{\widetilde{\Gamma}^1_n\}$ and $\{\widetilde{\Gamma}^2_n\}$, given by (6.7) for $\{p_n\}$ and $\{r_n\}$ just constructed, satisfy besides (6.11) also the remaining conditions of Definition 2.2. By (6.12) and (6.9), we have the inclusions:

$$s^1(\Gamma^\iota_{p_n}) \subseteq s^1(\widetilde{\Gamma}^\iota_n) \subset \operatorname{supp} \widetilde{\Gamma}^\iota_n \subset s^1(\Gamma^\iota_{p_{n+1}}), \qquad \iota \in \{1,2\},\ n \in \mathbb{N},$$

which imply that condition $(**)$ is satisfied by $\{\widetilde{\Gamma}^\iota_n\}$, because it is fulfilled by $\{\Gamma^\iota_{p_n}\}$ for $\iota \in \{1,2\}$. Since the sequences $\{\overline{\Pi}^\iota_{r_n}\}$ and $\{\Gamma^\iota_{p_n}\}$ fulfil conditions (2.2) and (2.3), respectively, we deduce from (6.7) and the Leibniz formula that condition (2.3) is satisfied by the sequence $\{\widetilde{\Gamma}^\iota_n\}$ for $\iota \in \{1,2\}$. Consequently, $\{\widetilde{\Gamma}^1_n\}, \{\widetilde{\Gamma}^2_n\} \in \Gamma_d$.

To complete the proof notice that, by (6.7), we have the identity:

$$\widetilde{\Gamma}_n^1 \otimes \widetilde{\Gamma}_n^2 - \overline{\Pi}_{r_n}^1 \otimes \overline{\Pi}_{r_n}^2 = \theta_n^1 + \theta_n^2 + \theta_n^3,$$

which yields

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \langle R, \widetilde{\Gamma}_n^1 \otimes \widetilde{\Gamma}_n^2 \rangle_{2d} = \alpha,$$

by (6.10) and (6.2), the assumption of the lemma. Hence $\lim_{n\to\infty} \alpha_n = \alpha$. Consequently, (6.3) holds and the assertion of the lemma is proved. □

7. Equivalence of Definitions

Theorem 7.1 (see [16]). *Let $S, T \in \mathcal{D}'(\mathbb{R}^d)$. Conditions* (Π_0), (Π_1), (Π_2), (Π) *are equivalent to each of the conditions listed in Theorem 4.1. If any of these conditions is satisfied, then the convolutions defined above exist and the following equalities hold:*

$$S \overset{\Pi}{*} T = S \overset{\Pi_0}{*} T = S \overset{\Pi_1}{*} T = S \overset{\Pi_2}{*} T = S \overset{S}{*} T. \tag{7.1}$$

Theorem 7.2. *Let $S, T \in \mathcal{D}'(\mathbb{R}^d)$. Conditions* (Γ) *and* (Γ_0) *are equivalent to each of the conditions listed in Theorem 4.1 and Theorem 7.1. If any of these conditions is satisfied, then the convolutions defined above exist and are equal:*

$$S \overset{\Gamma}{*} T = S \overset{\Gamma_0}{*} T = S \overset{S}{*} T. \tag{7.2}$$

Proof. We will prove the equivalence of convolvability conditions given in the Theorems 7.1 and 7.2 according to the following scheme of implications:

$$\begin{array}{ccccccccccc}
 & & (\Pi_1) & \longrightarrow & (C_1) & & & & (\Pi) & \longrightarrow & (\Pi_0) \\
 & \nearrow & & & & \searrow & & \nearrow & & & \\
(\Pi_0) & & & & & & (S) & & & & \uparrow \\
 & \searrow & & & & \nearrow & & \searrow & & & \\
 & & (\Pi_2) & \longrightarrow & (C_2) & & & & (\Gamma) & \longrightarrow & (\Gamma_0)
\end{array}$$

Assume condition (Π_0) and put $R := (S \otimes T)\, \varphi^{\triangle}$ for a fixed $\varphi \in \mathcal{D}(\mathbb{R}^d)$. By (4.1) and Lemma 6.1, we deduce that (6.3) holds for all $\{\Pi_n^1\}, \{\Pi_n^2\} \in \Pi_d$ for a certain $\alpha \in \mathbb{C}$ which does not depend on the sequences $\{\Pi_n^1\}$ and $\{\Pi_n^2\}$. Hence the double limit in the following equality exists and the equality holds:

$$\lim_{n,m\to\infty} \langle R, \Pi_n^1 \otimes \Pi_m^2 \rangle_{2d} = \alpha$$

for all $\{\Pi_n^1\}, \{\Pi_n^2\} \in \Pi_d$. Since the supports of the functions Π_n^ι and 1 satisfy case 1° of compatibility for $\iota \in \{1, 2\}$, the functions $(\Pi_n^1 \otimes 1)\varphi^{\triangle}$ and $(1 \otimes \Pi_n^2)\varphi^{\triangle}$ belong to $\mathcal{D}(\mathbb{R}^{2d})$ for $n \in \mathbb{N}$, where 1 means the constant function on $\mathbb{R}^d$. Therefore

$$\lim_{m\to\infty} \langle R, \Pi_n^1 \otimes \Pi_m^2 \rangle_{2d} = \langle (\Pi_n^1 S) \otimes T, \varphi^{\triangle} \rangle_{2d} = \langle (\Pi_n^1 S) * T, \varphi \rangle_d,$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \langle R, \sqcap_n^1 \otimes \sqcap_m^2 \rangle_{2d} = \langle S \otimes (\sqcap_m^2 T), \varphi^{\triangle} \rangle_{2d} = \langle S * (\sqcap_m^2 T), \varphi \rangle_d,$$

for all $m \in \mathbb{N}$, by the continuity of $S \otimes T$, Lemma 6.1 and (4.1). Hence

$$\lim_{n\to\infty} \langle (\sqcap_n^1 S) \otimes T, \varphi^{\triangle} \rangle_{2d} = \alpha = \lim_{n\to\infty} \langle (\sqcap_n^1 S) * T, \varphi \rangle_d; \tag{7.3}$$

$$\lim_{n\to\infty} \langle S \otimes (\sqcap_n^2 T), \varphi^{\triangle} \rangle_{2d} = \alpha = \lim_{n\to\infty} \langle S * (\sqcap_n^2 T), \varphi \rangle_d. \tag{7.4}$$

By the first equalities in (7.3) and (7.4), condition (Π_0) implies conditions (Π_1) and (Π_2) as well as the equalities $S \overset{\Pi_0}{*} T = S \overset{\Pi_1}{*} T = S \overset{\Pi_2}{*} T$.

Assume condition (Π_1) and fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$. By (4.1), the limits in the equalities in (7.3) exist and the equalities hold for all $\{\sqcap_n^1\} \in \Pi_d$ and a certain $\alpha \in \mathbb{C}$ not depending on $\{\sqcap_n^1\}$. Consequently, conditions (C_1) and (S) as well as the equalities $S \overset{\Pi_1}{*} T = S \overset{C_1}{*} T = S \overset{S}{*} T$ hold true, by (7.3) and the known identity $\langle (\sqcap_n^1 S) * T, \varphi \rangle_d = \langle S(\check{T} * \varphi), \sqcap_n^1 \rangle_d$ (see e.g. [23]), in view of Theorems 3.3 and 4.1. Analogously, condition (Π_2) implies conditions (C_2), (S) and the equalities $S \overset{\Pi_2}{*} T = S \overset{C_2}{*} T = S \overset{S}{*} T$.

That (S) implies (Π) and (Γ) and the identities $S \overset{S}{*} T = S \overset{\Pi}{*} T = S \overset{\Gamma}{*} T$ follows from Lemma 3.5 applied to $R := (S \otimes T)\,\varphi^{\triangle} \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d})$ for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Implications (Π) $\Rightarrow$ (Π_0) and (Γ) $\Rightarrow$ (Γ_0) as well as the equalities $S \overset{\Pi}{*} T = S \overset{\Pi_0}{*} T$ and $S \overset{\Gamma}{*} T = S \overset{\Gamma_0}{*} T$, respectively, follow from Lemma 6.1, because $\{\sqcap_n^1 \otimes \sqcap_n^2\} \in \Pi_{2d}$ and $\{\Gamma_n^1 \otimes \Gamma_n^2\} \in \Gamma_{2d}$ for any $\{\sqcap_n^1\}, \{\sqcap_n^2\} \in \Pi_d$ and $\{\Gamma_n^1\}, \{\Gamma_n^2\} \in \Gamma_d$, respectively.

Since implication (Γ_0) $\Rightarrow$ (Π_0) and the equality $S \overset{\Gamma_0}{*} T = S \overset{\Pi_0}{*} T$ follow from Lemma 6.2, the proof of the equivalence of all considered conditions and of all equalities in (7.1) and (7.2) is completed. □

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