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Almost Sure Central Limit Theorem for Self-Normalized Partial Sums of Negatively Associated Random Variables

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Abstract. Let $X, X_1, X_2, \ldots$ be a stationary sequence of negatively associated random variables. A universal result in almost sure central limit theorem for the self-normalized partial sums S_n/V_n is established, where: $S_n = \sum_{i=1}^n X_i$, $V_n^2 = \sum_{i=1}^n X_i^2$.

1. Introduction

Starting with Brosamler [1] and Schatte [2], several authors investigated the almost sure central limit theorem (ASCLT) for partial sums S_n/σ_n of random variables in the last two decades. We refer the reader to Brosamler [1], Schatte [2], Lacey and Philipp [3], Ibragimov and Lifshits [4], Berkes and Csáki [5], Hörmann [6], Miao [7] and Wu [8] in this context. If σ_n is replaced by an estimate from the given data, usually denoted by $V_n = \sqrt{\sum_{i=1}^n X_i^2}$, V_n is called a self-normalizer of partial sums. A class of self-normalized random sequences has been proposed and studied in Peligrad and Shao [9], Pena et al. [10] and references therein. The past decade has witnessed a significant development in the field of limit theorems for the self-normalized sum S_n/V_n. We refer to Bentkus and Gótze [11] for the Berry-Esseen bound, Gine et al. [12] for the asymptotic normality, Hu et al. [13] for the Cramer type moderate deviations, Csörgo et al. [14] for the Donsker's theorem, Huang and Pang [15], Zhang and Yang [16] and Wu [17] for the almost sure central limit theorems. In addition, Wu [17] proved the ASCLT for the self-normalized partial sums that reads as follows: Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables in the domain of attraction of the normal law with mean zero. Then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\left\{\frac{S_k}{V_k} \le x\right\} = \Phi(x) \quad \text{a.s.} \quad \text{for} \quad \text{any} \quad x \in \mathbb{R},$$

where: $d_k = \frac{\exp(\ln^\alpha k)}{k}$, $D_n = \sum_{k=1}^n d_k$, $0 \le \alpha < 1/2$, I denotes indicator function, and $\Phi(x)$ stands for the standard normal distribution function.

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Many results concerning the limit theory for the self-normalized partial sums from the NA random sequences have been obtained. However, since the denominator in the formula for the self-normalized partial sums contains random variables, the study of limit theory for self-normalized partial sums of NA random variables is very difficult, and so far, there are very few research results in this field. Thus, this is a challenging, difficult and meaningful research topic.

The purpose of this article is to establish the ASCLT for the self-normalized partial sums of NA random variables. We will show that the ASCLT holds under a fairly general growth condition on d_k, namely if $d_k = k^{-1}\exp(\ln^{\alpha} k)$, $0 \le \alpha < 1/2$.

Definition. Random variables $X_1, X_2, \ldots, X_n, n \ge 2$, are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\mathrm{cov}(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)) \le 0,$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable) functions such that this covariance exists. A sequence of random variables $\{X_i; i \ge 1\}$ is said to be NA if its every finite subfamily is NA.

The concept of negative association was introduced by Alam and Saxena [18] and Joag-Dev and Proschan [19]. Statistical test depends greatly on sampling. The random sampling without replacement from a finite population is NA, but is not independent. Due to the wide applications of NA sampling in multivariate statistical analysis and reliability theory, the limit behaviors of NA random variables have received extensive attention recently. One can refer to: Joag-Dev and Proschan [19] for fundamental properties, Matuła [20] for the three series theorem, Su et al. [21] for the moment inequalities and weak convergence, Shao [22] for the Rosenthal type inequality and the Kolmogorov exponential inequality, Wu and Jiang [23] for the law of the iterated logarithm, Wu [24] for almost sure limit theorems, Wu and Chen [25, 26] for strong representation results of Kaplan-Meier estimator and the Berry-Esseen type bound in kernel density estimation.

In the following, $a_n \sim b_n$ denotes $\lim\limits_{n\to\infty} a_n/b_n = 1$ and the symbol c stands for a generic positive constant which may differ from one place to another. We assume that $\{X, X_n\}_{n\in\mathbb{N}}$ is a stationary sequence of NA random variables. By Newman [27], $\sigma^2 := \mathbb{E}X_1^2 + 2\sum_{k=2}^{\infty}\mathbb{E}X_1X_k$ always exists and $\sigma^2 \in [0, \mathrm{Var}X]$. Furthermore, if $\sigma^2 > 0$, then $\mathrm{Var}S_n \sim n\sigma^2$. For each $n \ge 1$, the symbol S_n/V_n denotes self-normalized partial sums, where: $S_n = \sum_{i=1}^{n} X_i$, $V_n^2 = \sum_{i=1}^{n} X_i^2$.

For every $1 \le i \le n$, let:

$$\bar{X}_{ni} := -\sqrt{n}I(X_i < -\sqrt{n}) + X_iI(|X_i| \le \sqrt{n}) + \sqrt{n}I(X_i > \sqrt{n}),$$

$$\bar{S}_n := \sum_{i=1}^{n}\bar{X}_{ni}, \quad \bar{V}_n^2 := \sum_{i=1}^{n}\bar{X}_{ni}^2, \quad \bar{V}_{n,1}^2 := \sum_{i=1}^{n}\bar{X}_{ni}^2I(\bar{X}_{ni} \ge 0), \quad \bar{V}_{n,2}^2 := \sum_{i=1}^{n}\bar{X}_{ni}^2I(\bar{X}_{ni} < 0),$$

$$\sigma_n^2 := \mathrm{Var}\bar{S}_n, \quad \delta_n^2 := \mathbb{E}\bar{X}_{n1}^2, \quad \delta_{n,1}^2 := \mathbb{E}\bar{X}_{n1}^2I(\bar{X}_{n1} \ge 0), \quad \delta_{n,2}^2 := \mathbb{E}\bar{X}_{n1}^2I(\bar{X}_{n1} < 0).$$

Obviously,

$$\delta_n^2 = \delta_{n,1}^2 + \delta_{n,2}^2, \quad \mathbb{E}\bar{V}_n^2 = n\delta_n^2 = n\delta_{n,1}^2 + n\delta_{n,2}^2.$$

Our theorem is formulated in a more general setting.

Theorem 1.1. *Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a stationary sequence of NA random variables satisfying:*

$$\mathbb{E}X = 0,\ \ 0 < \mathbb{E}X^2 < \infty,\ \ \sigma^2 > 0,\ \ \mathbb{E}X^2 I(X \geq 0) > 0,\ \ \mathbb{E}X^2 I(X < 0) > 0, \tag{1}$$

and

$$\sigma_n^2 \sim \beta^2 \mathbb{E}\bar{V}_n^2 = \beta^2 n\delta_n^2 \ \text{ for some constant } \ \beta > 0. \tag{2}$$

Suppose that $0 \leq \alpha < 1/2$ and set:

$$d_k = \frac{\exp(\ln^\alpha k)}{k},\ \ D_n = \sum_{k=1}^{n} d_k. \tag{3}$$

Then,

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{\frac{S_k}{\beta V_k} \leq x\right\} = \Phi(x) \quad \text{a.s.} \quad \text{for} \quad \text{any} \quad x \in \mathbb{R}. \tag{4}$$

By the terminology of summation procedures (see e.g. Chandrasekharan and Minakshisundaram [28], p. 35), we have the following corollary.

Corollary 1.2. *Theorem* 1.1 *remains valid if we replace the weight sequence $\{d_k\}_{k\in\mathbb{N}}$ by any $\{d_k^*\}_{k\in\mathbb{N}}$ such that $0 \leq d_k^* \leq d_k$, $\sum_{k=1}^{\infty} d_k^* = \infty$.*

Remark 1.3. *If $\{X, X_n\}_{n\in\mathbb{N}}$ is a sequence of independent random variables then, (2) holds with $\beta = 1$.*

2. Proofs

The following four lemmas below play an important role in the proof of Theorem 1.1. Lemma 2.1 is due to Joag-Dev and Proschan [19], Lemma 2.2 has been stated by Su et al. [21], Lemma 2.3 has been established by Wu [17], and Lemma 2.4 is of our authorship; due to its length, the proof of Lemma 2.4 is given in Appendix.

Lemma 2.1. (Joag-Dev and Proschan [19]) *If $\{X_i\}_{i\in\mathbb{N}}$ is a sequence of NA random variables and $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of nondecreasing (or nonincreasing) functions, then $\{f_i(X_i)\}_{i\in\mathbb{N}}$ is also a sequence of NA random variables.*

Lemma 2.2. (Su et al. [21]) *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of NA random variables with zero mean and such that $\mathbb{E}|X_i|^p < \infty$, $i = 1, 2, \ldots$ if $p \geq 2$. Then,*

$$\mathbb{E}|S_n|^p \leq c_p \left(\sum_{i=1}^{n} \mathbb{E}|X_i|^p + \left(\sum_{i=1}^{n} \mathbb{E}X_i^2 \right)^{p/2} \right),$$

where $c_p > 0$ only depends on p.

Lemma 2.3. (Wu [17]) *Let $\{\xi, \xi_n\}_{n\in\mathbb{N}}$ be a sequence of uniformly bounded random variables. If there exist constants $c > 0$ and $\delta > 0$, such that*

$$|\mathbb{E}\xi_k\xi_j| \leq c\left(\frac{k}{j}\right)^\delta, \ \ \text{for} \ \ 1 \leq k < j,$$

then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \xi_k = 0 \quad \text{a.s.},$$

where d_k and D_n are defined by (3).

Lemma 2.4. *Suppose that the assumptions of Theorem* 1.1 *hold. Then:*

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\le x\right\}=\Phi(x)\quad\text{a.s.}\quad\text{for}\quad\text{any}\quad x\in\mathbb{R},\tag{5}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\left(f\left(\frac{\bar{V}_{k,l}^2}{k\delta_{k,l}^2}\right)-\mathbb{E}f\left(\frac{\bar{V}_{k,l}^2}{k\delta_{k,l}^2}\right)\right)=0\quad\text{a.s.},\ \ l=1,2,\tag{6}$$

where d_k and D_n are defined by (3) *and f is a bounded function with bounded continuous derivatives.*

Proof of Theorem 1.1. For any given $0<\varepsilon<1$, note that for $x\ge 0$ and $k\le n$,

$$\begin{aligned}\left\{\frac{S_k}{\beta V_k}\le x\right\}&\subseteq\left\{\frac{S_k}{\beta V_k}\le x,\forall 1\le i\le k\ \ |X_i|\le\sqrt{k},\bar{V}_k^2\le(1+\varepsilon)k\delta_k^2\right\}\cup\left\{\bar{V}_k^2>(1+\varepsilon)k\delta_k^2\right\}\cup\left\{\exists 1\le i\le k\ \ |X_i|>\sqrt{k}\right\}\\&\subseteq\left\{\frac{\bar{S}_k}{\beta\delta_k\sqrt{(1+\varepsilon)k}}\le x\right\}\cup\left\{\bar{V}_k^2>(1+\varepsilon)k\delta_k^2\right\}\cup\left\{\bigcup_{i=1}^{k}(|X_i|>\sqrt{k})\right\}.\end{aligned}$$

Hence,

$$I\left(\frac{S_k}{\beta V_k}\le x\right)\le I\left(\frac{\bar{S}_k}{\beta\delta_k\sqrt{(1+\varepsilon)k}}\le x\right)+I\left(\bar{V}_k^2>(1+\varepsilon)k\delta_k^2\right)+I\left(\bigcup_{i=1}^{k}(|X_i|>\sqrt{k})\right),\quad\text{for}\ \ x\ge 0.$$

Similarly, we have for any given $0<\varepsilon<1$ and $x<0$,

$$I\left(\frac{S_k}{\beta V_k}\le x\right)\le I\left(\frac{\bar{S}_k}{\beta\delta_k\sqrt{(1-\varepsilon)k}}\le x\right)+I\left(\bar{V}_k^2\le(1-\varepsilon)k\delta_k^2\right)+I\left(\bigcup_{i=1}^{k}(|X_i|>\sqrt{k})\right).$$

Furthermore, we get

$$I\left(\frac{S_k}{\beta V_k}\le x\right)\ge I\left(\frac{\bar{S}_k}{\beta\delta_k\sqrt{(1-\varepsilon)k}}\le x\right)-I\left(\bar{V}_k^2\le(1-\varepsilon)k\delta_k^2\right)-I\left(\bigcup_{i=1}^{k}(|X_i|>\sqrt{k})\right),\quad\text{for}\ \ x\ge 0,$$

$$I\left(\frac{S_k}{\beta V_k}\le x\right)\ge I\left(\frac{\bar{S}_k}{\beta\delta_k\sqrt{(1+\varepsilon)k}}\le x\right)-I\left(\bar{V}_k^2>(1+\varepsilon)k\delta_k^2\right)-I\left(\bigcup_{i=1}^{k}(|X_i|>\sqrt{k})\right),\quad\text{for}\ \ x<0.$$

Hence, in order to establish (4), it suffices to prove:

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\frac{\bar{S}_k}{\beta\delta_k\sqrt{k}}\le x\sqrt{1\pm\varepsilon}\right)=\Phi(x\sqrt{1\pm\varepsilon})\ \ \text{a.s.},\tag{7}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI(\bar{V}_k^2>(1+\varepsilon)k\delta_k^2)=0\ \ \text{a.s.},\tag{8}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI(\bar{V}_k^2\le(1-\varepsilon)k\delta_k^2)=0\ \text{ a.s.},\tag{9}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\bigcup_{i=1}^{k}(|X_i|>\sqrt{k})\right)=0\ \text{ a.s.},\tag{10}$$

for any $\varepsilon>0$.

Firstly, we prove (7). By $0=\mathbb{E}X=\mathbb{E}XI(|X|\le\sqrt{k})+\mathbb{E}XI(|X|>\sqrt{k})$ and $\mathbb{E}X^2<\infty$, we have $\mathbb{E}XI(|X|\le\sqrt{k})=-\mathbb{E}XI(|X|>\sqrt{k})$ and $\lim_{x\to\infty}x^2P(|X|>x)=0$, and consequently

$$\begin{aligned}|\mathbb{E}\bar{S}_k| &\le |k\mathbb{E}XI(|X|\le\sqrt{k})|+k^{3/2}P(|X|>\sqrt{k})=|k\mathbb{E}XI(|X|>\sqrt{k})|+o(\sqrt{k})\\ &\le \sqrt{k}\mathbb{E}X^2I(|X|>\sqrt{k})+o(\sqrt{k})=o(\sqrt{k}).\end{aligned}$$

This, and the fact that $\delta_k^2\to\mathbb{E}X^2<\infty$, when $k\to\infty$, imply

$$I\left(\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\le\sqrt{1\pm\varepsilon}x-\alpha\right)\le I\left(\frac{\bar{S}_k}{\beta\delta_k\sqrt{k}}\le\sqrt{1\pm\varepsilon}x\right)\le I\left(\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\le\sqrt{1\pm\varepsilon}x+\alpha\right)\ \text{ for any }\ \alpha>0.$$

Thus, by (5), we have as $n\to\infty$,

$$\begin{aligned}\Phi(\sqrt{1\pm\varepsilon}x-\alpha) &\leftarrow \frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\le\sqrt{1\pm\varepsilon}x-\alpha\right)\\ &\le \frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\frac{\bar{S}_k}{\beta\delta_k\sqrt{k}}\le\sqrt{1\pm\varepsilon}x\right)\\ &\le \frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\le\sqrt{1\pm\varepsilon}x+\alpha\right)\\ &\to \Phi(\sqrt{1\pm\varepsilon}x+\alpha)\ \text{ a.s.}\end{aligned}\tag{11}$$

Letting $\alpha\to0$ in (11), we obtain that (7) holds.

Now, we prove (8). Since $\mathbb{E}\bar{V}_k^2=k\delta_k^2$, $\bar{V}_k^2=\bar{V}_{k,1}^2+\bar{V}_{k,2}^2$, $\mathbb{E}\bar{V}_{k,l}^2=k\delta_{k,l}^2$, and $\delta_{k,l}^2\le\delta_k^2$, $l=1,2$, it follows that

$$\begin{aligned}I(\bar{V}_k^2>(1+\varepsilon)k\delta_k^2) &= I(\bar{V}_k^2-\mathbb{E}\bar{V}_k^2>\varepsilon k\delta_k^2)\le I(\bar{V}_{k,1}^2-\mathbb{E}\bar{V}_{k,1}^2>\varepsilon k\delta_k^2/2)+I(\bar{V}_{k,2}^2-\mathbb{E}\bar{V}_{k,2}^2>\varepsilon k\delta_k^2/2)\\ &\le I(\bar{V}_{k,1}^2>(1+\varepsilon/2)k\delta_{k,1}^2)+I(\bar{V}_{k,2}^2>(1+\varepsilon/2)k\delta_{k,2}^2).\end{aligned}$$

Therefore, by the arbitrariness of $\varepsilon>0$, in order to prove (8), it suffices to show that, for $l=1,2$,

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI(\bar{V}_{k,l}^2>(1+\varepsilon)k\delta_{k,l}^2)=0\ \text{ a.s.}\tag{12}$$

For a given $\varepsilon > 0$, let f denote a bounded function with bounded continuous derivatives, such that

$$I(x > 1 + \varepsilon) \le f(x) \le I(x > 1 + \varepsilon/2).$$

Obviously, $\bar{X}_{ni}^2 I(\bar{X}_{ni} \ge 0)$ is monotonic on X_i, and thus, by Lemma 2.1, $\{\bar{X}_{ni}^2 I(\bar{X}_{ni} \ge 0)\}_{n\ge1, i\le n}$ is also a sequence of NA random variables. From Lemma 2.2, the Markov inequality and the facts that $\mathbb{E}\bar{V}_{k,1}^2 = k\delta_{k,1}^2$ and $\delta_{k,1}^2 \to \mathbb{E}X^2 I(X \ge 0) > 0$, we get

$$\begin{aligned}
\mathbb{P}\left(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2\right) &= \mathbb{P}\left(\bar{V}_{k,1}^2 - \mathbb{E}\bar{V}_{k,1}^2 > \varepsilon k\delta_{k,1}^2/2\right) \\
&\le c\frac{\mathbb{E}(\bar{V}_{k,1}^2 - \mathbb{E}\bar{V}_{k,1}^2)^2}{k^2} \le c\frac{\mathbb{E}\bar{X}_{k1}^4 I(\bar{X}_{k1} \ge 0)}{k} \\
&\le c\frac{\mathbb{E}X^4 I(0 \le X \le \sqrt{k}) + k^2 P(|X| > \sqrt{k})}{k}. \qquad (13)
\end{aligned}$$

Using $\mathbb{E}X^2 < \infty$, we have $x^2 P(|X| > x) = o(1)$, as $x \to \infty$. Hence,

$$\mathbb{E}X^4 I(0 \le X \le \sqrt{k}) = \int_0^\infty \mathbb{P}\left(|X| I(0 \le X \le \sqrt{k}) > t\right) 4t^3 \mathrm{d}t \le c\int_0^{\sqrt{k}} \mathbb{P}(|X| > t) t^3 \mathrm{d}t = \int_0^{\sqrt{k}} o(1) t \mathrm{d}t = o(1)k.$$

This and (13) imply

$$\mathbb{P}\left(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2\right) \to 0.$$

Therefore, it follows from (6) and the Toeplitz lemma that

$$\begin{aligned}
0 &\le \frac{1}{D_n}\sum_{k=1}^n d_k I\left(\bar{V}_{k,1}^2 > (1+\varepsilon)k\delta_{k,1}^2\right) \le \frac{1}{D_n}\sum_{k=1}^n d_k f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) \\
&\sim \frac{1}{D_n}\sum_{k=1}^n d_k \mathbb{E} f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) \le \frac{1}{D_n}\sum_{k=1}^n d_k \mathbb{E} I\left(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2\right) \\
&= \frac{1}{D_n}\sum_{k=1}^n d_k \mathbb{P}(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2) \\
&\to 0 \quad \text{a.s.}
\end{aligned}$$

Hence, (12) holds for $l = 1$. Using similar methods to those used in the proof of (12) for $l = 1$, we can prove (12) for $l = 2$. Consequently, (8) holds. Moreover, applying identical methods to those used in the proof of (8), we can prove (9).

Finally, we shall prove (10). Note that

$$\mathbb{E}\left|I\left(\bigcup_{i=1}^k (|X_i| > \sqrt{k})\right)\right| \le \sum_{i=1}^k \mathbb{P}(|X_i| > \sqrt{k}) = k\mathbb{P}(|X| > \sqrt{k}) \to 0, \quad \text{as} \quad k \to \infty.$$

Therefore,

$$I\left(\bigcup_{i=1}^k (|X_i| > \sqrt{k})\right) \to 0 \quad \text{a.s.}, \quad \text{as} \quad k \to \infty.$$

Thus, by the Toeplitz lemma,

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\bigcup_{i=1}^{k}(|X_i| > \sqrt{k})\right) \to 0 \quad \text{a.s.}$$

Hence, (10) holds. This completes the proof of Theorem 1.1.

3. Appendix

As it has been mentioned, we give the proof of Lemma 2.4 in this part of our paper.

Proof of Lemma 2.4. Obviously, $\bar{X}_{ni}$ is monotonic on X_i, and thus, by Lemma 2.1, $\{\bar{X}_{ni}\}_{n\geq 1, i\leq n}$ is also a sequence of NA random variables. By the cental limit theorem for NA random variables and the properties that: $\sigma_n^2 \sim \beta^2 n\delta_n^2$, $\delta_n^2 \to \mathbb{E}X^2 > 0$ as $n \to \infty$ (see conditions (1) and (2)), we get

$$\frac{\bar{S}_n - \mathbb{E}\bar{S}_n}{\beta\delta_n\sqrt{n}} \xrightarrow{d} \mathcal{N},$$

where $\xrightarrow{d}$ denotes the convergence in distribution and $\mathcal{N}$ denotes the standard normal random variable. This implies that for any function $g(x)$, which is bounded and has bounded continuous derivatives

$$\mathbb{E}g\left(\frac{\bar{S}_n - \mathbb{E}\bar{S}_n}{\beta\delta_n\sqrt{n}}\right) \longrightarrow \mathbb{E}g(\mathcal{N}), \quad \text{as} \quad n \to \infty,$$

Hence, by the Toeplitz lemma, we obtain

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E}g\left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right) = \mathbb{E}g(\mathcal{N}).$$

On the other hand, it follows from Theorem 7.1 of Billingsley [29] and Section 2 of Peligrad and Shao [9] that (5) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k g\left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right) = \mathbb{E}g(\mathcal{N}) \quad \text{a.s.}$$

Hence, in order to prove (5), it suffices to show that

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k\left(g\left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right) - \mathbb{E}g\left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right)\right) = 0 \quad \text{a.s.}, \tag{14}$$

for any g from the class of bounded functions having bounded continuous derivatives.

Let for $k \geq 1$,

$$\xi_k = g\left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right) - \mathbb{E}g\left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right).$$

Observe that, for any $1 \leq k < j$, we get,

$$\begin{aligned}
|\mathbb{E}\xi_k\xi_j| &= \left|\mathrm{Cov}\left(g\left(\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right), g\left(\frac{\bar{S}_j-\mathbb{E}\bar{S}_j}{\beta\delta_j\sqrt{j}}\right)\right)\right| \\
&= \left|\mathrm{Cov}\left(g\left(\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right), g\left(\frac{\bar{S}_j-\mathbb{E}\bar{S}_j}{\beta\delta_j\sqrt{j}}\right)-g\left(\frac{\sum_{i=k+1}^{j}(\bar{X}_{ji}-\mathbb{E}\bar{X}_{ji})}{\beta\delta_j\sqrt{j}}\right)\right)\right| \\
&\quad + \left|\mathrm{Cov}\left(g\left(\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}\right), g\left(\frac{\sum_{i=k+1}^{j}(\bar{X}_{ji}-\mathbb{E}\bar{X}_{ji})}{\beta\delta_j\sqrt{j}}\right)\right)\right| \\
&=: I_1+I_2.
\end{aligned}$$

Clearly, since g is a bounded Lipschitz function, there exists a constant $c>0$ such that $|g(x)|\le c, |g(x)-g(y)|\le c|x-y|$, for any $x,y\in\mathbb{R}$. As $\{\bar{X}_{ji}\}_{j\ge1,i\le j}$ is a sequence of NA random variables, as well as Lemma 2.2 and condition $\delta_n^2\to\mathbb{E}X^2<\infty$ hold, we obtain that

$$I_1\le c\frac{\mathbb{E}\left|\sum_{i=1}^{k}(\bar{X}_{ji}-\mathbb{E}\bar{X}_{ji})\right|}{\sqrt{j}}\le c\frac{\sqrt{\mathbb{E}\left(\sum_{i=1}^{k}(\bar{X}_{ji}-\mathbb{E}\bar{X}_{ji})\right)^2}}{\sqrt{j}}\le c\frac{\sqrt{k\mathbb{E}X^2}}{\sqrt{j}}\le c\left(\frac{k}{j}\right)^{1/2}.$$

It follows from the definition of negative association that $\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\beta\delta_k\sqrt{k}}$, $\frac{\sum_{i=k+1}^{j}(\bar{X}_{ji}-\mathbb{E}\bar{X}_{ji})}{\beta\delta_j\sqrt{j}}$ are NA and the assumption that g is a bounded function with bounded continuous derivatives. Thus, Lemma 2.3 of Zhang [30] is applied with: $f(x):=-\sqrt{k}I(x<-\sqrt{k})+xI(|x|\le\sqrt{k})+\sqrt{k}I(x>\sqrt{k})$, $g(y):=-\sqrt{j}I(y<-\sqrt{j})+yI(|y|\le\sqrt{j})+\sqrt{j}I(y>\sqrt{j})$; the stationarity of $\{X_i\}$, and the facts that $\mathbb{E}X^2<\infty$ and $\sigma^2>0$ imply $\sum_{m=2}^{\infty}|\mathrm{Cov}(X_1,X_m)|<\infty$. Therefore,

$$\begin{aligned}
I_2 &\le -c\mathrm{Cov}\left(\frac{\bar{S}_k-\mathbb{E}\bar{S}_k}{\sqrt{k}}, \frac{\sum_{i=k+1}^{j}(\bar{X}_{ji}-\mathbb{E}\bar{X}_{ji})}{\sqrt{j}}\right) \\
&= \frac{-c}{\sqrt{kj}}\mathrm{Cov}\left(\sum_{l=1}^{k}(\bar{X}_{kl}-\mathbb{E}\bar{X}_{kl}), \sum_{i=k+1}^{j}(\bar{X}_{ji}-\mathbb{E}\bar{X}_{ji})\right) \\
&\le \frac{-c}{\sqrt{kj}}\sum_{l=1}^{k}\sum_{i=k+1}^{j}\mathrm{Cov}(X_l,X_i) \\
&\le \frac{-c}{\sqrt{kj}}\sum_{l=1}^{k}\sum_{m=k-l+2}^{j-l+1}\mathrm{Cov}(X_1,X_m) \\
&\le \frac{ck}{\sqrt{kj}}\sum_{m=2}^{\infty}|\mathrm{Cov}(X_1,X_m)| \\
&\le c\left(\frac{k}{j}\right)^{1/2}.
\end{aligned}$$

Hence, by Lemma 2.3, (14) holds.

Now, we prove (6). Let

$$\eta_k=f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right)-\mathbb{E}f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right)\quad\text{for any}\quad 1\le k<j.$$

Since $\{\bar{X}_{ni}^2 I(\bar{X}_{ni} \geq 0)\}_{n\geq 1, i\leq n}$ is a sequence of NA random variables, we have $\delta_{j,1}^2 \to \mathbb{E}X^2 I(X \geq 0)$ and $0 < \mathbb{E}X^2 I(X \geq 0) < \infty$. Thus, using Lemma 2.3 of Zhang [30] twice is used with: $f(x) := x^2 I(0 \leq x \leq \sqrt{k}) + kI(x > \sqrt{k})$, $g(y) := y^2 I(0 \leq y \leq \sqrt{j}) + jI(y > \sqrt{j})$, we have for $1 \leq k < j$,

$$
\begin{aligned}
|\mathbb{E}\eta_k\eta_j| &= \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\bar{V}_{j,1}^2}{j\delta_{j,1}^2}\right)\right)\right| \\
&= \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\bar{V}_{j,1}^2}{j\delta_{j,1}^2}\right) - f\left(\frac{\bar{V}_{j,1}^2 - \sum_{i=1}^k \bar{X}_{ji}^2 I(\bar{X}_{ji} \geq 0)}{j\delta_{j,1}^2}\right)\right)\right| \\
&\quad + \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\bar{V}_{j,1}^2 - \sum_{i=1}^k \bar{X}_{ji}^2 I(\bar{X}_{ji} \geq 0)}{j\delta_{j,1}^2}\right)\right)\right| \\
&\leq c\frac{\mathbb{E}\left(\sum_{i=1}^k \bar{X}_{ji}^2 I(\bar{X}_{ji} \geq 0)\right)}{j} - c\mathrm{Cov}\left(\frac{\bar{V}_{k,1}^2}{k}, \frac{\bar{V}_{j,1}^2 - \sum_{i=1}^k \bar{X}_{ji}^2 I(\bar{X}_{ji} \geq 0)}{j}\right) \\
&\leq c\frac{k}{j} - \frac{c}{kj}\sum_{l=1}^{k}\sum_{i=k+1}^{j} \mathrm{Cov}\left(\bar{X}_{kl}^2 I(\bar{X}_{kl} \geq 0), \bar{X}_{ji}^2 I(\bar{X}_{ji} \geq 0)\right) \\
&\leq c\frac{k}{j} - \frac{c}{\sqrt{kj}}\sum_{l=1}^{k}\sum_{i=k+1}^{j} \mathrm{Cov}(X_l, X_i) \\
&\leq c\frac{k}{j} + c\left(\frac{k}{j}\right)^{1/2} \\
&\leq c\left(\frac{k}{j}\right)^{1/2}.
\end{aligned}
$$

By Lemma 2.3, (6) holds. This completes the proof of Lemma 2.4.

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