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An Integrodifferential Operator for Meromorphic Functions Associated with the Hurwitz-Lerch Zeta Function

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Abstract. In this paper, we introduce a new integrodifferential operator associated with the Hurwitz Lerch Zeta function in the puncture open disk of the meromorphic functions. We also obtain some properties of the third-order differential subordination and superordination for this integrodifferential operator, by using certain classes of admissible functions.

1. Introduction

Let Σ denote the class of functions $f(z)$ of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the punctured open unit disc $\mathbb{U}^* = \mathbb{U}\backslash\{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The function $f(z)$ has a simple pole at $z = 0$.

We begin by recalling that a general Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ defined by (see, for example, [18, P. 121 et seq.] and [19, P. 194 et seq.])

$$\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s}, \tag{1.2}$$

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($b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, ...\}$, $s \in \mathbb{C}$ when $z \in \mathbb{U}$, $\mathrm{Re}(s) > 1$ when $|z| = 1$).

Several properties of $\Phi(z, s, b)$ can be found in many papers, for example Attiya and Hakami [3], Choi *et al.* [8], Cho *et al.* [7], Ferreira and López [9], Gupta *et al.* [10] and Luo and Srivastava [14]. See, also Kutbi and Attiya([11], [12]), Srivastava and Attiya [17], Srivastava and Gaboury [20], Srivastava *et al.* [21] and Owa and Attiya [16].

Analogous to the operator defined by Srivastava and Attiya [17], we define the following operator associated with the Hurwitz-Lerch Zeta function, as follows:

$$J_{s,b}^{*} : \Sigma \longrightarrow \Sigma$$

the operator defined by:

$$J_{s,b}^{*} f(z) = G(s, b; z) * f(z) \tag{1.3}$$

where the function $G(s, b; z)$ defined by

$$G(s, b; z) = \frac{b^s \Phi(z, s, b)}{z} \tag{1.4}$$

$$\left(z \in \mathbb{U}^*;\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \ s \in \mathbb{C}\right)$$

and $*$ denotes the Hadamard product (or Convolution). Then we can see that

$$J_{s,b}^{*} f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{b}{k+b+1}\right)^s a_k z^k$$

$$\left(z \in \mathbb{U}^*;\ f(z) \in \Sigma;\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \ s \in \mathbb{C}\right).$$

Remark 1. *We note that:*

1. $J_{0,b}^{*} f(z) = f(z)$,
2. $J_{1,\frac{1}{\alpha}-2}^{*} f(z) = \frac{1-2\alpha}{\alpha\, z^{\frac{1}{\alpha}-1}} \int\limits_0^z t^{\frac{1}{\alpha}-2} f(t)\, dt$ $(0 < \alpha < \frac{1}{2})$,
3. $J_{1,b}^{*} f(z) = \frac{b}{z^{b+1}} \int\limits_0^z t^b f(t)\, dt$,
4. $J_{\alpha,\beta}^{*} f(z) = \frac{\beta^\alpha}{\Gamma(\alpha) z^{\beta+1}} \int\limits_0^z t^\beta \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\, dt$ $(\alpha > 0;\ \beta > 0)$,
5. $J_{s,1}^{*} f(z) = \dfrac{1}{z} + \sum\limits_{k=0}^{\infty} \frac{1}{(k+2)^s} a_k z^k$,
6. $J_{-1,1}^{*} f(z) = -z f'(z)$,
7. $J_{-1,-2}^{*} f(z) = \dfrac{f(z) - z f'(z)}{2}$,

8. $J^*_{-n,-1}f(z) = \frac{1}{z} + \sum_{k=0}^{\infty}(-k)^n a_k z^k \; (n \in \mathbb{N}),$

9. $J^*_{-n,1}f(z) = \frac{1}{z} + \sum_{k=0}^{\infty}(k+2)^n a_k z^k \; (n \in \mathbb{N}),$

where $J^*_{1,\frac{1}{\alpha}-2}$ the operator introduced by Cho *et al.* [6], $J^*_{\alpha,\beta}$ the operator introduced by Lashin [13], $J^*_{s,1}$ the operator introduced by Alhindi and Darus [1], $J^*_{1,-n}$ the operator defined by Uralegaddi and Somanatha [23] and $J^*_{1,b}$ is the operator analogous to the generalized Bernardi operator (for Bernardi operator see [5]), when $\mathrm{Re}(b) > 0$, the operator $J^*_{1,b}$ introduced by Bajpai [4].

We denote by $H[a,n]$, the class of analytic functions in $\mathbb{U}$ in the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k \quad (a \in \mathbb{C};\; n \in \mathbb{N} = \{1, 2, \cdots\})$$

and $H = H[1,1]$.

In our investigation we need the following definitions and theorem:

Definition 1.1. *Let $f(z)$ and $F(z)$ be analytic functions. The function $f(z)$ is said to be subordinate to $F(z)$, written $f(z) \prec F(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.*

Definition 1.2. *[2, P. 441] Let $\mathbb{D}$ be the set of analytic functions $q(z)$ and univalent on $\overline{\mathbb{U}}\backslash E(q)$, where*

$$E(q) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

is such that $\min\left|q'(\zeta)\right| = \rho > 0$ *for $\zeta \in \partial\mathbb{U}\backslash E(q)$. Further, let $\mathbb{D}(a) = \{q(z) \in \mathbb{D} : q(0) = a\}$ and $\mathbb{D}_1 = \mathbb{D}(1)$.*

Definition 1.3. *[2, P. 440] Let $\Psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the third-order differential subordination:*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{1.5}$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying(1.5). A dominant $\widetilde{q}(z)$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants of (1.5) is called the best dominant of (1.5).

Definition 1.4. *[2, P. 440] Let Ω be a set in $\mathbb{C}$, $q \in \mathbb{D}$ and $n \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition :*

$$\psi(r, s, t, u; z) \notin \Omega$$

whenever

$$r = q(\zeta),\; s = k\zeta q'(\zeta),$$

$$\mathrm{Re}\left(\frac{t}{s} + 1\right) \geq k\,\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$

and

$$\mathrm{Re}\left(\frac{u}{s}\right) \geq k^2\,\mathrm{Re}\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}$; $\zeta \in \partial\mathbb{U}\backslash E(q)$ and $k \geq n$.

Analogous to the second order differential superordinations introduced by Miller and Mocanu [15], Tang *et al.* [22] defined the differential superordinations as follows:

Definition 1.5. *[22, P. 3] Let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and the function $h(z)$ be analytic in $\mathbb{U}$. If the functions $p(z)$ and*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z))$$

are univalent in $\mathbb{U}$ and satisfy the following third-order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z)), \tag{1.6}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for $p(z)$ satisfying (1.6). A univalent subordinant $\widetilde{q}(z)$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all supordinants $q(z)$ of (1.6) is said to be the best superordinant.

Definition 1.6. *[22, P. 4] Let Ω be a set in $\mathbb{C}$, $q \in H[a, n]$ and $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition :*

$$\psi(r, s, t, u; \zeta) \in \Omega,$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m},$$

$$\mathrm{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\,\mathrm{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$\mathrm{Re}\left(\frac{u}{s}\right) \leq \frac{1}{m^2}\,\mathrm{Re}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}$ and $m \geq n \geq 2$.

Also, we need the following theorems in our investigations:

Theorem 1.1. *[2, p. 449] Let $p(z) \in H[a, n]$ with $n \in \mathbb{N}\backslash\{1\}$. Also, let $q(z) \in \mathbb{D}(a)$ and satisfy the following conditions:*

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) > 0, \qquad \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq k,$$

where $z \in \mathbb{U}; \zeta \in \partial\mathbb{U}\backslash E(q)$ and $k \geq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

then

$$p(z) \prec q(z).$$

Theorem 1.2. *[22, p. 4] Let $q(z) \in H[a,n]$ and $\psi \in \Psi'_n[\Omega, q]$.If*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$$

is univalent in $\mathbb{U}$ and $p(z) \in \mathbb{D}(a)$ satisfy the following conditions:

$$\mathrm{Re}\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \quad \left|\frac{\zeta p'(\zeta)}{q'(z)}\right| \leq m,$$

$$(z \in \mathbb{U}; \zeta \in \partial\mathbb{U}; m \geq n \geq 2)$$

and

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{U}\right\},$$

implies that

$$q(z) \prec p(z).$$

In this paper, by using the third-order differential subordination and superordination results by Antonino and Miller [2] and Tang *et al.* [22], we define certain classes of admissible functions and investigate some subordination and superordination properties of meromorphic functions associated with the integrodifferential operator $J^*_{s,b}$ defined by (1.3). Furthermore, new differential sandwich-type theorems are obtained.

2. Third Order Differential Subordination with $J^*_{s,b}$

Definition 2.1. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathbb{D}$. The class of admissible functions $\Phi_\Gamma[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\phi(a_1, a_2, a_3, a_4; z) \notin \Omega,$$

whenever

$$a_1 = q(\varsigma), \quad a_2 = \frac{k\,\zeta q'(\zeta) + b\,q(\zeta)}{b},$$

$$\mathrm{Re}\left(\frac{b\,(a_3 - a_1)}{(a_2 - a_1)} - 2b\right) \geq k\ \mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$$\mathrm{Re}\left(\frac{b^2\,(a_4 - a_1) - 3b\,(b+1)\,(a_3 - a_1)}{(a_2 - a_1)} + 3b^2 + 6b + 2\right) \geq k^2\,\mathrm{Re}\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}$, $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}$, $\zeta \in \partial\mathbb{U}\backslash E(q)$ and $k \in \mathbb{N}\backslash\{1\}$.

Theorem 2.1. *Let $\phi \in \Phi_\Gamma[\Omega, q]$. If $f(z) \in \Sigma$ and $q(z) \in \mathbb{D}_1$ satisfy the following conditions:*

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \quad \left|\frac{z\left(J^*_{s-1,b}f(z) - J^*_{s,b}f(z)\right)}{q'(\zeta)}\right| \leq \frac{k}{|b|} \tag{2.1}$$

and

$$\left\{\phi(zJ^*_{s,b}f(z), zJ^*_{s-1,b}f(z), zJ^*_{s-2,b}f(z), zJ^*_{s-3,b}f(z); z) : z \in \mathbb{U}\right\} \subset \Omega, \tag{2.2}$$

then

$$zJ^*_{s,b}f(z) \prec q(z). \tag{2.3}$$

Proof. Let us define the analytic function $p(z)$ as:

$$p(z) = zJ^{*}_{s,b}f(z) \quad (z \in \mathbb{U}). \tag{2.4}$$

Using the definition of $J^{*}_{s,b}f(z)$, we can prove that

$$z\left(J^{*}_{s,b}\, f(z)\right)' = bJ^{*}_{s-1,b}f(z) - (b+1)J^{*}_{s,b}f(z), \tag{2.5}$$

then we get

$$zJ^{*}_{s-1,b}f(z) = \frac{zp'(z) + bp(z)}{b}, \tag{2.6}$$

which implies

$$zJ^{*}_{s-2,b}f(z) = \frac{z^2p''(z) + (2b+1)zp'(z) + b^2p(z)}{b^2}. \tag{2.7}$$

Also, we can see that

$$zJ^{*}_{s-3,b}f(z) = \frac{z^3p'''(z) + 3\,(b+1)\,z^2p''(z) + (3b^2+3b+1)zp'(z) + b^3p(z)}{b^3}. \tag{2.8}$$

Let us define the parameters a_1, a_2, a_3 and a_4 as:

$$a_1 = r,\ a_2 = \frac{s+br}{b},\ a_3 = \frac{t + (1+2b)s + b^2r}{b^2}$$

and

$$a_4 = \frac{u + 3\,(b+1)\,t + (3b^2+3b+1)s + b^3r}{b^3}.$$

Now, we define the transformation

$$\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$$

$$\psi(r,s,t,u;z) = \phi(a_1,a_2,a_3,a_4;z). \tag{2.9}$$

By using the relations from (2.4) to (2.8), we have

$$\begin{aligned} &\psi(\,p(z),\, z\,p'(z),\, z^{\,2}p''(z), z^3p'''(z)\,;\, z) \\ &= \phi\left(zJ^{*}_{s,b}f(z), zJ^{*}_{s-1,b}f(z), zJ^{*}_{s-2,b}f(z); zJ^{*}_{s-3,b}f(z); z\right). \end{aligned} \tag{2.10}$$

Therefore, we can rewrite (2.2) as

$$\psi(\,p(z),\, z\,p'(z),\, z^{\,2}p''(z), z^3p'''(z)\,;\, z) \in \Omega.$$

Then the proof is completed by showing that the admissibility condition for $\phi \in \Phi_{\Gamma}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition (1.3), since

$$\frac{t}{s} + 1 = \frac{b(a_3 - a_1)}{a_2 - a_1} - 2b \tag{2.11}$$

and

$$\frac{u}{s} = \frac{b^2\,(a_4 - a_1) - 3b\,(b+1)\,(a_3 - a_1)}{(a_2 - a_1)} + 3b^2 + 6b + 2\,.$$

We also note that

$$\left|\frac{zp'(z)}{q'(\zeta)}\right| = \left|\frac{bz\left(J^*_{s-1,b}f(z) - J^*_{s,b}f(z)\right)}{q'(\zeta)}\right|$$
$$\leq k\,.$$

Therefore, $\psi \in \Psi_2[\Omega, q]$ and hence by Theorem 1.1, $p(z) \prec q(z)$. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $\Phi_\Gamma[h(\mathbb{U}), q]$ is written as $\Phi_\Gamma[h, q]$.

The following theorem is a directly consequence of Theorem 2.1 .

Theorem 2.2. *Let $\phi \in \Phi_\Gamma[h, q]$. If $f(z) \in \Sigma$ and $q(z) \in \mathbb{D}_1$ satisfy the following conditions:*

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \qquad \left|\frac{z\left(J^*_{s-1,b}f(z) - J^*_{s,b}f(z)\right)}{q'(\zeta)}\right| \leq \frac{k}{|b|} \tag{2.12}$$

and

$$\phi(zJ^*_{s,b}f(z),\, zJ^*_{s-1,b}f(z), zJ^*_{s-2,b}f(z), zJ^*_{s-3,b}f(z)\,;\, z) \prec h(z), \tag{2.13}$$

then

$$zJ^*_{s,b}f(z) \prec q(z).$$

The next corollary is an extension of Theorem 2.1 to the case where the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 2.1. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in $\mathbb{U}$, $q(0) = 1$. Let $\phi \in \Phi_\Gamma[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in \Sigma$ satisfies*

$$\mathrm{Re}\left(\frac{\zeta q''_\rho(\zeta)}{q'_\rho(\zeta)}\right) \geq 0, \qquad \left|\frac{z\left(J^*_{s-1,b}f(z) - J^*_{s,b}f(z)\right)}{q'_\rho(\zeta)}\right| \leq \frac{k}{|b|} \tag{2.14}$$

and

$$\phi(zJ^*_{s,b}f(z),\, zJ^*_{s-1,b}f(z), zJ^*_{s-2,b}f(z), zJ^*_{s-3,b}f(z)\,;\, z) \in \Omega, \tag{2.15}$$

then

$$zJ^*_{s,b}f(z) \prec q(z),$$

where $z \in \mathbb{U}$ and $\zeta \in \partial\mathbb{U}\backslash E(q_\rho)$.

Proof. By using Theorem 2.1, we have $J^*_{s,b}f(z) \prec q_\rho(z)$. Then we obtain the result from $q_\rho(z) \prec q(z)$. □

Corollary 2.2. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in $\mathbb{U}$, $q(0) = 1$. Let $\phi \in \Phi_\Gamma[h, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in \Sigma$ satisfies*

$$\mathrm{Re}\left(\frac{\zeta q''_\rho(\zeta)}{q'_\rho(\zeta)}\right) \geq 0, \qquad \left|\frac{z\left(J^*_{s-1,b}f(z) - J^*_{s,b}f(z)\right)}{q'_\rho(\zeta)}\right| \leq \frac{k}{|b|} \tag{2.16}$$

and

$$\phi(zJ^*_{s,b}f(z), zJ^*_{s-1,b}f(z), zJ^*_{s-2,b}f(z), zJ^*_{s-3,b}f(z); z) \prec h(z), \tag{2.17}$$

then

$$zJ^*_{s,b}f(z) \prec q(z),$$

where $z \in \mathbb{U}$ *and* $\zeta \in \partial\mathbb{U}\backslash E(q_\rho)$.

Theorem 2.3. *Let* $h(z)$ *be univalent in* $\mathbb{U}$. *Let* $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$. *Suppose that the differential equation:*

$$\phi\left(q(z), \frac{zq'(z) + bq(z)}{b}, \frac{z^2q''(z) + (2b+1)zq'(z) + b^2q(z)}{b^2}, \frac{z^3q'''(z) + 3(b+1)z^2q''(z) + (3b^2+3b+1)zq'(z) + b^3q(z)}{b^3}; z\right) = h(z), \tag{2.18}$$

has a solution $q(z)$ *with* $q(0) = 1$ *which satisfies (2.1). If* $f(z) \in \Sigma$ *satisfies (2.17) and*

$$\phi(zJ^*_{s,b}f(z), zJ^*_{s-1,b}f(z), zJ^*_{s-2,b}f(z), zJ^*_{s-3,b}f(z); z)$$

is analytic in $\mathbb{U}$, *then*

$$zJ^*_{s,b}f(z) \prec q(z) \tag{2.19}$$

and $q(z)$ *is the best dominant of (2.19).*

Proof. By using Theorem 2.1 that $q(z)$ is a dominant of (2.17). Since $q(z)$ satisfies (2.18), it is also a solution of (2.17) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. □

In the case $q(z) = 1 + Mz$ $(M > 0)$ and in view of the Definition 2.1, the class of admissible functions $\Phi_\Gamma[\Omega, q]$ denoted by $\Phi_\Gamma[\Omega, M]$ is defined below.

Definition 2.2. *Let* Ω *be a set in* $\mathbb{C}$ *and* $M > 0$. *The class of admissible functions* $\Phi_\Gamma[\Omega, M]$ *consists of those functions* $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ *that satisfy the admissibility condition*

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{(b+k)Me^{i\theta}}{b}, 1 + \frac{L + \left(b^2 + k(2b+1)\right)Me^{i\theta}}{b^2}, 1 + \frac{N + 3(b+1)L + \left(b^3 + k\left(3b^2 + 3b + 1\right)\right)Me^{i\theta}}{b^3}; z\right) \notin \Omega, \tag{2.20}$$

where $z \in \mathbb{U}$, $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ *and* $\mathrm{Re}(Ne^{-i\theta}) \geq 0$ *for all real* θ *and* $k \in \mathbb{N}\backslash\{1\}$.

Corollary 2.3. *Let* $\phi \in \Phi_\Gamma[\Omega, M]$. *If* $f(z) \in \Sigma$ *satisfies the following conditions:*

$$\left|z\left(J^*_{s-1,b}f(z) - J^*_{s,b}f(z)\right)\right| \leq \frac{kM}{|b|} \tag{2.21}$$

and

$$\phi(zJ^*_{s,b}f(z), zJ^*_{s-1,b}f(z), zJ^*_{s-2,b}f(z), zJ^*_{s-3,b}f(z); z) \in \Omega, \tag{2.22}$$

then

$$\left|zJ^*_{s,b}(z) - 1\right| < M.$$

In the case $\Omega = q(\mathbb{U}) = \{\omega : \ |w-1| < M \ \ (M>0)\}$, for simplification we denote by $\Phi_\Gamma[M]$ to the class $\Phi_\Gamma[\Omega, M]$.

Corollary 2.4. *Let* $\phi \in \Phi_\Gamma[M]$. *If* $f(z) \in \Sigma$ *satisfies the condition (2.21) and*

$$\left|\phi(zJ^*_{s,b}f(z), zJ^*_{s-1,b}f(z), zJ^*_{s-2,b}f(z), zJ^*_{s-3,b}f(z); z) - 1\right| < M, \tag{2.23}$$

then

$$\left|zJ^*_{s,b}(z) - 1\right| < M.$$

Putting $\phi(a_1, a_2, a_3, a_4; z) = a_2 = 1 + \frac{(b+k)Me^{i\theta}}{b}$ in Corollary 2.4, we have the following corollary:

Corollary 2.5. *Let* $M > 0$ *and* $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ *with* $\mathrm{Re}(b) < \dfrac{-k}{2}$ $(k \in \mathbb{N}\backslash\{1\})$. *If* $f(z) \in \Sigma$ *satisfies the condition (2.21) and*

$$\left|zJ^*_{s-1,b}f(z) - 1\right| < M,$$

then

$$\left|zJ^*_{s,b}(z) - 1\right| < M.$$

Corollary 2.6. *Let* $k \in \mathbb{N}\backslash\{1\}$, $M > 0$ *and* $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$. *If* $f(z) \in \Sigma$ *satisfies the condition*

$$\left|z\left(J^*_{s-1,b}f(z) - J^*_{s,b}f(z)\right)\right| < \frac{kM}{|b|}, \tag{2.24}$$

(2.21)then

$$\left|zJ^*_{s,b}(z) - 1\right| < M.$$

Proof. Let

$$\phi(a_1, a_2, a_3, a_4; z) = a_2 - a_1.$$

Using Corollary 2.3 with $\Omega = h(\mathbb{U})$ and

$$h(z) = \frac{kM}{|b|}z \quad (z \in \mathbb{U}).$$

Now we show that $\phi \in \Phi_\Gamma[\Omega, M]$.

Since the condition (2.21) is satisfied from the condition (2.24) and

$$\left|\phi\left(1 + Me^{i\theta}, 1 + \frac{(b+k)Me^{i\theta}}{b}, 1 + \frac{L + \left(b^2 + k(2b+1)\right)Me^{i\theta}}{b^2}\right.\right.$$
$$\left.\left.1 + \frac{N + 3(b+1)L + \left(b^3 + k\left(3b^2 + 3b + 1\right)\right)Me^{i\theta}}{b^3}; z\right)\right|$$
$$= \left|\frac{kMe^{i\theta}}{b}\right|$$
$$= \frac{kM}{|b|},$$

then we have Corollary 2.6. □

Corollary 2.7. *Let $k \in \mathbb{N}\backslash\{1\}$, $M > 0$ and $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$. If $f(z) \in \Sigma$ satisfies the condition (2.21) and*

$$\left|z\left(J^*_{s-3,b}f(z) - J^*_{s-2,b}f(z)\right)\right| < \frac{2\left(|b+1|^2 + |2b+3|\right)M}{|b|^3}, \tag{2.25}$$

then

$$\left|zJ^*_{s,b}(z) - 1\right| < M.$$

Proof. We define

$$\phi(a_1, a_2, a_3, a_4; z) = a_4 - a_3.$$

Using Corollary 2.3 with $\Omega = h(\mathbb{U})$ and

$$h(z) = \frac{2\left(|b+1|^2 + |2b+3|\right)M}{|b|^3} z \quad (z \in \mathbb{U}).$$

Now we show that $\phi \in \Phi_\Gamma[\Omega, M]$.

Since

$$\begin{aligned}
&\left|\phi\left(1 + Me^{i\theta}, 1 + \frac{(b+k)Me^{i\theta}}{b}, 1 + \frac{L + \left(b^2 + k(2b+1)\right)Me^{i\theta}}{b^2}\right.\right. \\
&\left.\left. 1 + \frac{N + 3(b+1)L + \left(b^3 + k\left(3b^2 + 3b + 1\right)\right)Me^{i\theta}}{b^3}; z\right)\right| \\
&= \left|\frac{N + (2b+3)L + k(b+1)^2 Me^{i\theta}}{b^3}\right| \\
&= \left|\frac{Ne^{-i\theta} + (2b+3)Le^{-i\theta} + k(b+1)^2 M}{b^3 e^{-i\theta}}\right| \\
&\geq \frac{\operatorname{Re}\left(Ne^{-i\theta}\right) + |2b+3| \operatorname{Re}\left(Le^{-i\theta}\right) + k|b+1|^2 M}{|b|^3} \\
&\geq \frac{(k-1)kM|2b+3| + k|b+1|^2 M}{|b|^3} \\
&\geq \frac{2\left(|b+1|^2 + |2b+3|\right)M}{|b|^3},
\end{aligned}$$

we completes the proof of Corollary 2.7. □

3. Third Order Differential Superordination with $J^*_{s,b}$

Definition 3.1. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in H$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_\Gamma[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\phi(a_1, a_2, a_3, a_4; \zeta) \in \Omega,$$

whenever

$$a_1 = q(z), \quad a_2 = \frac{\zeta q'(z) + b\,q(z)}{mb},$$

$$\mathrm{Re}\left(\frac{b\,(a_3-a_1)}{(a_2-a_1)}-2b\right)\leq\frac{1}{m}\,\mathrm{Re}\left(\frac{\zeta q''(z)}{q'(z)}+1\right),$$

$$\mathrm{Re}\left(\frac{b^2\,(a_4-a_1)-3b\,(b+1)\,(a_3-a_1)}{(a_2-a_1)}+3b^2+6b+2\right)\leq\frac{1}{m^2}\,\mathrm{Re}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z\in\mathbb{U},\ b\in\mathbb{C}\backslash\mathbb{Z}_0^-,\ s\in\mathbb{C}\ \zeta\in\partial\mathbb{U}$ *and* $m\in\mathbb{N}\backslash\{1\}$.

Theorem 3.1. *Let* $\phi\in\Phi'_\Gamma[\Omega,q]$. *If* $f(z)\in\Sigma$ *and* $zJ^*_{s,b}f(z)\in\mathbb{D}_1$ *satisfy the following conditions:*

$$\mathrm{Re}\left(\frac{zq''(z)}{q'(z)}\right)\geq 0,\qquad\left|\frac{z\left(J^*_{s-1,b}f(z)-J^*_{s,b}f(z)\right)}{q'(z)}\right|\leq\frac{m}{|b|},\tag{3.1}$$

$$\left\{\phi(zJ^*_{s,b}f(z),\,zJ^*_{s-1,b}f(z),zJ^*_{s-2,b}f(z),zJ^*_{s-3,b}f(z)\,;\,z):z\in\mathbb{U}\right\}$$

is univalent, and

$$\Omega\subset\left\{\phi(zJ^*_{s,b}f(z),\,zJ^*_{s-1,b}f(z),zJ^*_{s-2,b}f(z),zJ^*_{s-3,b}f(z)\,;\,z):z\in\mathbb{U}\right\},\tag{3.2}$$

then

$$q(z)\prec zJ^*_{s,b}f(z).$$

Proof. *Let the functions* $p(z)$ *and* ψ*be defined by (2.4) and (2.9). Since* $\phi\in\Phi'_\Gamma[\Omega,q]$.*Therefore (2.10) and (3.2) imply*

$$\Omega\subset\psi(p(z),\,zp'(z),\,z^2p''(z),z^3p'''(z)\,;\,z).$$

The admissible condition for $\phi\in\Phi'_\Gamma[\Omega,q]$ *is equivalent to the admissible condition for* ψ *in Definition 1.6 with* $n=2$. *Therefore,* $\psi\in\Psi'_2[\Omega,q]$, *and by using (3.1) and Theorem 1.2, we have*

$$q(z)\prec p(z)$$

which yields

$$q(z)\prec zJ^*_{s,b}f(z).$$

Therefore we completes the proof of Theorem 3.1. □

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $\Phi'_\Gamma[h(\mathbb{U}),q]$ is written as $\Phi'_\Gamma[h,q]$.

The following theorem is a directly consequence of Theorem 2.1 .

Theorem 3.2. *Let* $\phi\in\Phi'_\Gamma[h,q]$. *Also, let* $h(z)$ *be analytic in* $\mathbb{U}$. *If* $f(z)\in\Sigma$ *and* $zJ^*_{s,b}f(z)\in\mathbb{D}_1$ *satisfies the condition (3.1),*

$$\left\{\phi(zJ^*_{s,b}f(z),\,zJ^*_{s-1,b}f(z),zJ^*_{s-2,b}f(z),zJ^*_{s-3,b}f(z)\,;\,z):z\in\mathbb{U}\right\}$$

is univalent in $\mathbb{U}$, *and*

$$h(z)\prec\phi(zJ^*_{s,b}f(z),\,zJ^*_{s-1,b}f(z),zJ^*_{s-2,b}f(z),zJ^*_{s-3,b}f(z)\,;\,z),\tag{3.3}$$

then

$$q(z)\prec zJ^*_{s,b}f(z).$$

Theorem 3.3. *Let $h(z)$ be analytic in $\mathbb{U}$, also, let $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ and ψ be given by (2.9). Suppose that the differential equation (2.18) has a solution $q(z) \in \mathbb{D}_1$. If $f(z) \in \Sigma$ satisfies the condition (3.1),*

$$\left\{\phi(zJ^*_{s,b}f(z),\, zJ^*_{s-1,b}f(z), z\, J^*_{s-2,b}f(z), z\, J^*_{s-3,b}f(z)\, ;\, z) : z \in \mathbb{U}\right\}$$

is univalent in $\mathbb{U}$, and

$$h(z) \prec \phi(zJ^*_{s,b}f(z),\, zJ^*_{s-1,b}f(z), z\, J^*_{s-2,b}f(z), z\, J^*_{s-3,b}f(z)\, ;\, z),$$

then

$$q(z) \prec zJ^*_{s,b}f(z). \tag{3.4}$$

and $q(z)$ is the best subordinant of (3.3).

Proof. The proof is similar to that of Theorem 2.3 and it is being omitted here. □

By combining Theorem 2.2 and Theorem 3.2 we obtain the following sandwich type result.

Corollary 3.1. *Let $h_1(z)$ and $q_1(z)$ be analytic in $\mathbb{U}$.Also, let $h_2(z)$ be univalent in $\mathbb{U}$, $q_2(z) \in \mathbb{D}_1$with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_\Gamma[h, q] \cap \Phi'_\Gamma[h, q]$. If $f(z) \in \Sigma$, $zJ^*_{s,b}f(z) \in \mathbb{D}_1 \cap H$,*

$$\left\{\phi(zJ^*_{s,b}f(z),\, zJ^*_{s-1,b}f(z), z\, J^*_{s-2,b}f(z), z\, J^*_{s-3,b}f(z)\, ;\, z) : z \in \mathbb{U}\right\}$$

is univalent in $\mathbb{U}$,and the conditions (2.12) and (3.1) are satisfied, Also, let

$$h_1(z) \prec \phi(zJ^*_{s,b}f(z),\, zJ^*_{s-1,b}f(z), z\, J^*_{s-2,b}f(z), z\, J^*_{s-3,b}f(z)\, ;\, z) \prec h_2(z), \tag{3.5}$$

then $q_1(z) \prec zJ^*_{s,b}f(z) \prec q_2(z)$.

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