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A Generalization of Power and Alternating Power Sums to Any Appell Polynomials

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Abstract. The classical power sum and alternating power sum identities can be stated as

$$\sum_{i=0}^{m} s_n(i) = \frac{1}{n+1}\left(B_{n+1}(m+1) - B_{n+1}\right),$$

$$\sum_{i=0}^{m} (-1)^i s_n(i) = \frac{1}{2}\left((-1)^m E_n(m+1) + E_n\right),$$

where $s_n(x) = x^n$ is the simplest possible Appell polynomial for the Sheffer pair $(1, t)$. The impetus for this research starts from the question that what if we replace $s_n(x) = x^n$ by any Appell polynomial. In this paper, we give a generalization of power and alternating power sums to any Appell polynomials.

1. Introduction

As is well known, the Bernoulli polynomials are defined by the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{(see [3–11])}. \tag{1.1}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

The ordinary Euler polynomials are also given by the generating function

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1, 17, 20])}. \tag{1.2}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

There have been numerous investigations on Bernoulli and Euler polynomials and their related ones. Some interesting generalizations of these polynomials were obtained in the recent papers [13, 14, 18]. For

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their connections to many interesting integrals and series associated with zeta functions, one is referred to the detailed and systematic treatment in [19];for explicit computations of the related numbers, one may want to see [15].

Let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}. \tag{1.3}$$

For $\mathbb{P} = \mathbb{C}[x]$, let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L | p(x) \rangle$ denotes the action of the linear functional L on $p(x)$ which satisfies $\langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle$, and $\langle cL | p(x) \rangle = c \langle L | p(x) \rangle$, where c is a complex constant. The linear functional $\left\langle f(t) \middle| \cdot \right\rangle$ on $\mathbb{P}$ is defined by

$$\left\langle f(t) \middle| x^n \right\rangle = a_n, \quad (n \geq 0), \tag{1.4}$$

where $f(t) \in \mathcal{F}$, (see [16]).

From (1.3) and (1.4), we have

$$\left\langle t^k \middle| x^n \right\rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \tag{1.5}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let us set $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{k!} t^k$. Then, by (1.4), we get $\left\langle f_L(t) \middle| x^n \right\rangle = \langle L | x^n \rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra.

The order $o(f(t))$ of a power series $f(t) \neq 0$ is the smallest integer k for which the coefficient of t^k does not vanish (see [16]). If the order of $f(t) (\neq 0)$ is zero, then $f(t)$ is called an invertible series; if the order of $f(t)$ is one, then $f(t)$ is called a delta series. For $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t) f(t)^k \middle| s_n(x) \right\rangle = n! \delta_{n,k}$, for $k \geq 0$. Such a sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$. Also, $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g\left(\overline{f}(t)\right)} e^{y\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k, \quad (y \in \mathbb{C}), \quad (\text{see [16]}), \tag{1.6}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f\left(\overline{f}(t)\right) = t$.

In particular, if $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called the Appell sequence for $g(t)$;if $s_n(x) \sim (1, f(t))$, then $s_n(x)$ is called the associated sequence for $f(t)$ (see [16]).

For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t) \middle| x^k \right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k \middle| p(x) \right\rangle \frac{x^k}{k!}. \tag{1.7}$$

By (1.7), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad e^{yt} p(x) = p(x + y). \tag{1.8}$$

From (1.8), we note that

$$\left\langle e^{yt} \middle| p(x) \right\rangle = p(y). \tag{1.9}$$

The classical power sum and alternating power sum identities can be respectively stated as

$$\sum_{i=0}^{m} s_n(i) = \frac{1}{n+1}\left(B_{n+1}(m+1) - B_{n+1}\right), \tag{1.10}$$

$$\sum_{i=0}^{m} (-1)^i s_n(i) = \frac{1}{2}\left((-1)^m E_n(m+1) + E_n\right), \quad (n, m \in \mathbb{Z}_{\geq 0}), \tag{1.11}$$

where $s_n(x) \sim (1, t)$ is the simplest possible Appell polynomial. Here we would like to mention that, as a refinement of (1.10), some problem related with power sums along arithmetic progressions had been treated in [2].

The impetus for this research starts from the question that what if we replace $s_n(x) = x^n$ by any Appell polynomial.

In this paper, we will introduce Barnes' multiple Bernoulli and Appell mixed-type polynomials, and Barnes' multiple Euler and Appell mixed-type polynomials. Then we will establish one main identity for each of them connecting a sum for the Appell polynomial and that for the mixed-type polynomial. As a consequence, the direct generalization of the identities in (1.10) and (1.11) will follow from these as special simple cases.

2. A Generalization of Power and Alternating Power Sums to any Appell Polynomials

Assume that $s_n(x) \sim (g(t), t)$, $r_n(x) \sim (h(t), t)$ are Appell sequences. Then we denote the Appell sequence for $g(t)h(t)$ by $sr_n(x)$ which are called $s_n(x)$ and $r_n(x)$ mixed-type polynomials. That is,

$$sr_n(x) \sim (g(t)h(t), t). \tag{2.1}$$

From (1.6) and (2.1), we have

$$\frac{1}{g(t)h(t)}e^{xt} = \sum_{n=0}^{\infty} sr_n(x)\frac{t^n}{n!}. \tag{2.2}$$

Thus, by (2.2), we get

$$sr_n(x) = \frac{1}{g(t)h(t)}x^n = \frac{1}{g(t)}r_n(x) = \frac{1}{h(t)}s_n(x), \tag{2.3}$$

and

$$tsr_n(x) = nsr_{n-1}(x), \quad (n \geq 1). \tag{2.4}$$

The following facts will be used frequently: for any polynomial $p(x)$, we have

$$\frac{e^{yt}-1}{t}p(x) = \int_x^{x+y} p(u)\,du, \tag{2.5}$$

$$\left\langle \frac{e^{yt}-1}{t}\middle| p(x)\right\rangle = \int_0^y p(u)\,du, \tag{2.6}$$

$$\frac{e^{(m+1)at}-1}{e^{at}-1}p(x) = \sum_{i=0}^{m} p(x+ai), \tag{2.7}$$

$$\left\langle \frac{e^{(m+1)at}-1}{e^{at}-1}\middle| p(x)\right\rangle = \sum_{i=0}^{m} p(ai), \tag{2.8}$$

$$\frac{(-1)^m e^{(m+1)at}+1}{e^{at}+1}p(x)=\sum_{i=0}^{m}(-1)^i p(x+ai), \tag{2.9}$$

and

$$\left\langle \frac{(-1)^m e^{(m+1)at}+1}{e^{at}+1}\middle| p(x)\right\rangle =\sum_{i=0}^{m}(-1)^i p(ai). \tag{2.10}$$

Lemma 1. *Let $s_n(x)$ be the Appell sequence for $g(t)$, namely $s_n(x)\sim(g(t),t)$. For $m_1,m_2,\dots,m_l\in\mathbb{Z}_{\geq 0}$, $a_1,\dots,a_l\in\mathbb{C}$, we have*

$$\left(\frac{e^{(m_l+1)a_l t}-1}{t}\right)\times\cdots\times\left(\frac{e^{(m_1+1)a_1 t}-1}{t}\right)s_n(x) \tag{2.11}$$
$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}s_{n+l,i}(x|(m_1+1)a_1,\dots,(m_l+1)a_l),$$

where

$$(n)_l=n(n-1)\cdots(n-l+1)=\sum_{l=0}^{n}S_1(n,l)x^l$$
$$s_{n,i}(x|x_1,\dots,x_l)=\sum_{\substack{J\subset[1,l]\\|J|=i}}s_n\left(x+x_J\right),\quad(0\leq i\leq l),$$
$$[1,l]=\{1,2,\dots,l\},\quad \text{and } x_J=\sum_{j\in J}x_j.$$

Proof. By induction on l, we prove Lemma 1. For $l=1$, we have

$$\frac{e^{(m_1+1)a_1 t}-1}{t}s_n(x)=\int_x^{x+(m_1+1)a_1}s_n(u)\,du \tag{2.12}$$
$$=\frac{1}{n+1}\left(s_{n+1}(x+(m_1+1)a_1)-s_{n+1}(x)\right)$$
$$=\frac{1}{(n+1)_1}\sum_{i=0}^{1}(-1)^{1-i}s_{n+1,i}(x|(m_1+1)a_1).$$

Assume that, for $l>1$, the following holds:

$$\left(\frac{e^{(m_{l-1}+1)a_{l-1}t}-1}{t}\right)\times\cdots\times\left(\frac{e^{(m_1+1)a_1 t}-1}{t}\right)s_n(x) \tag{2.13}$$
$$=\frac{1}{(n+l-1)_{l-1}}\sum_{i=0}^{l-1}(-1)^{l-1-i}s_{n+l-1,i}(x|(m_1+1)a_1,\dots,(m_{l-1}+1)a_{l-1}).$$

Then the LHS of (2.11) is

$$\frac{1}{(n+l-1)_{l-1}}\sum_{i=0}^{l-1}(-1)^{l-1-i}\frac{e^{(m_l+1)a_l t}-1}{t} \tag{2.14}$$
$$\times s_{n+l-1,i}(x|(m_1+1)a_1,\dots,(m_{l-1}+1)a_{l-1})$$
$$=\frac{1}{(n+l-1)_{l-1}}\sum_{i=0}^{l-1}(-1)^{l-1-i}$$

$$\times \int_x^{x+(m_l+1)a_l} s_{n+l-1,i}\left(u|\left(m_1+1\right)a_1,\ldots,\left(m_{l-1}+1\right)a_{l-1}\right)du$$

$$=\frac{1}{(n+l-1)_{l-1}}\sum_{i=0}^{l-1}(-1)^{l-1-i}$$

$$\times \int_x^{x+(m_l+1)a_l}\sum_{\substack{J\subset[1,l-1]\\|J|=i}} s_{n+l-1}\left(u+\sum_{j\in J}\left(m_j+1\right)a_j\right)du$$

$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l-1}(-1)^{l-1-i}\left\{\sum_{\substack{J\subset[1,l-1]\\|J|=i}} s_{n+l}\left(x+\sum_{j\in J}\left(m_j+1\right)a_j+\left(m_l+1\right)a_l\right)\right.$$

$$\left.-\sum_{\substack{J\subset[1,l-1]\\|J|=i}} s_{n+l}\left(x+\sum_{j\in J}\left(m_j+1\right)a_j\right)\right\}$$

$$=\frac{1}{(n+l)_l}\left\{s_{n+l}\left(x+\left(m_1+1\right)a_1+\cdots+\left(m_l+1\right)a_l\right)\right.$$

$$+\sum_{i=1}^{l-1}(-1)^{l-i}\sum_{\substack{J\subset[1,l-1]\\|J|=i-1}} s_{n+l}\left(x+\sum_{j\in J}\left(m_j+1\right)a_j+\left(m_l+1\right)a_l\right)$$

$$\left.+\sum_{i=1}^{l-1}(-1)^{l-i}\sum_{\substack{J\subset[1,l-1]\\|J|=i}} s_{n+l}\left(x+\sum_{j\in J}\left(m_j+1\right)a_j\right)+(-1)^l s_{n+l}(x)\right\}$$

$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\sum_{\substack{J\subset[1,l]\\|J|=i}} s_{n+l}\left(x+\sum_{j\in J}\left(m_j+1\right)a_j\right)$$

$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\,s_{n+l,i}\left(x|\left(m_1+1\right)a_1,\ldots,\left(m_l+1\right)a_l\right).$$

□

Theorem 1. *Let $s_n(x)$ be the Appell sequence for $g(t)$, namely $s_n(x)\sim(g(t),t)$. Then, for any integers $m_1,\ldots,m_l$ with $m_1,\ldots,m_l\geq 1$, $a_1,\ldots,a_l\neq 0$, we have*

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l} s_n\left(x+a_1i_1+\cdots+a_li_l\right) \tag{2.15}$$

$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\left(B\left(a_1,\ldots,a_l\right)s\right)_{n+l,i}\left(x|\left(m_1+1\right)a_1,\ldots,\left(m_l+1\right)a_l\right).$$

Here $B(a_1,\ldots,a_l)_n(x)=B_n(x|a_1,\ldots,a_l)$ is the Barnes' multiple Bernoulli polynomial whose generating function is given by

$$\frac{t^l}{\prod_{j=1}^{l}\left(e^{a_jt}-1\right)}e^{xt}=\sum_{n=0}^{\infty}B_n\left(x|a_1,\ldots,a_l\right)\frac{t^n}{n!},$$

$(B(a_1,\dots,a_l)s)_n(x)$ is the $B(a_1,\dots,a_l)_n(x)$ and $s_n(x)$ mixed-type polynomial and

$$\begin{aligned}&(B(a_1,\dots,a_l)s)_{n+l,i}(x|x_1,\dots,x_l)\\&=\sum_{\substack{J\subset[1,l]\\|J|=i}}(B(a_1,\dots,a_l)s)_{n+l}\left(x+x_J\right),\quad(0\le i\le l).\end{aligned}$$

Proof. The result follows by computing the following in two different ways:

$$\left(\frac{e^{(m_l+1)a_lt}-1}{e^{a_lt}-1}\right)\times\cdots\times\left(\frac{e^{(m_1+1)a_1t}-1}{e^{a_1t}-1}\right)s_n(x). \tag{2.16}$$

On one hand, (2.16) is

$$\begin{aligned}&\left(\frac{e^{(m_l+1)a_lt}-1}{e^{a_lt}-1}\right)\times\cdots\times\left(\frac{e^{(m_2+1)a_2t}-1}{e^{a_2t}-1}\right)\sum_{i_1=0}^{m_1}s_n(x+a_1i_1) \qquad (2.17)\\&=\sum_{i_1=0}^{m_1}\left(\frac{e^{(m_l+1)a_lt}-1}{e^{a_lt}-1}\right)\times\cdots\times\left(\frac{e^{(m_3+1)a_3t}-1}{e^{a_3t}-1}\right)\left(\frac{e^{(m_2+1)a_2t}-1}{e^{a_2t}-1}s_n(x+a_1i_1)\right)\\&=\sum_{i_1=0}^{m_1}\left(\frac{e^{(m_l+1)a_lt}-1}{e^{a_lt}-1}\right)\times\cdots\times\left(\frac{e^{(m_3+1)a_3t}-1}{e^{a_3t}-1}\right)\left(\sum_{i_2=0}^{m_2}s_n(x+a_1i_1+a_2i_2)\right)\\&=\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\left(\frac{e^{(m_l+1)a_lt}-1}{e^{a_lt}-1}\right)\times\cdots\times\left(\frac{e^{(m_3+1)a_3t}-1}{e^{a_3t}-1}\right)s_n(x+a_1i_1+a_2i_2).\end{aligned}$$

Continuing in this fashion, we get the expression on the LHS of (2.15).
On the other hand, (2.16) is

$$\begin{aligned}&\left(\frac{e^{(m_l+1)a_lt}-1}{t}\right)\times\cdots\times\left(\frac{e^{(m_1+1)a_1t}-1}{t}\right)\frac{t^l}{\prod_{j=1}^{l}(e^{a_jt}-1)}s_n(x) \qquad (2.18)\\&=\left(\frac{e^{(m_l+1)a_lt}-1}{t}\right)\times\cdots\times\left(\frac{e^{(m_1+1)a_1t}-1}{t}\right)(B(a_1,\dots,a_l)s)_n(x).\end{aligned}$$

Now, the expression on the RHS of (2.15) can be derived from the Lemma 1. □

Corollary 1. *Let $m_1,m_2,\dots,m_l\in\mathbb{Z}$ with $m_1,m_2,\dots,m_l\ge1$ and $l\in\mathbb{N}$. Then we have*

(a)

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}s_n(a_1i_1+\cdots+a_li_l)\\&=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}(B(a_1,\dots,a_l)s)_{n+l,i}((m_1+1)a_1,\dots,(m_l+1)a_l),\end{aligned}$$

where

$$\begin{aligned}&(B(a_1,\dots,a_l)s)_{n+l,i}(x_1,\dots,x_l)\\&=\sum_{\substack{J\subset[1,l]\\|J|=i}}(B(a_1,\dots,a_l)s)_{n+l}\left(x_J\right),\quad(0\le i\le l).\end{aligned}$$

(b)

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l} s_n\left(x+i_1+\cdots+i_l\right)$$
$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\left(B^{(l)}s\right)_{n+l,i}\left(x|m_1+1,\ldots,m_l+1\right),$$

where $B_n^{(l)}(x)$ are the Bernoulli polynomials of order l given by the generating function

$$\left(\frac{t}{e^t-1}\right)^l e^{xt}=\sum_{n=0}^{\infty}B_n^{(l)}(x)\frac{t^n}{n!}.$$

(c)

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l} s_n\left(i_1+\cdots+i_l\right)$$
$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\left(B^{(l)}s\right)_{n+l,i}\left(m_1+1,\ldots,m_l+1\right).$$

(d)

$$\sum_{i_1,\ldots,i_l=0}^{m} s_n\left(x+i_1+\cdots+i_l\right)$$
$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}\left(B^{(l)}s\right)_{n+l}\left(x+(m+1)i\right).$$

(e)

$$\sum_{i_1,\ldots,i_l=0}^{m} S_n\left(i_1+\cdots+i_l\right)$$
$$=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}\left(B^{(l)}s\right)_{n+l}\left((m+1)i\right).$$

Lemma 2. *Let $s_n(x)$ be the Appell sequence for $g(t)$, namely $s_n(x)\sim(g(t),t)$. Then, for any $m_1,\ldots,m_l\in\mathbb{Z}_{\geq 0}$, $a_1,a_2,\ldots,a_l\in\mathbb{C}$, and $l\in\mathbb{N}$, we have*

$$\left((-1)^{m_l}e^{(m_l+1)a_l t}+1\right)\cdots\left((-1)^{m_1}e^{(m_1+1)a_1 t}+1\right)s_n(x) \tag{2.19}$$
$$=\sum_{i=0}^{l}\tilde{s}_{n,i}\left(x|(m_1+1)a_1,\ldots,(m_l+1)a_l\right).$$

Here, for $l\in\mathbb{N}$,

$$\tilde{s}_{n,i}\left(x|(m_1+1)a_1,\ldots,(m_l+1)a_l\right)=\sum_{\substack{J\subset[1,l]\\|J|=i}}(-1)^{m_J}s_n\left(x+((m+1)a)_J\right), \tag{2.20}$$

and

$$m_J = \sum_{j\in J} m_j, \quad ((m+1)\,a)_J = \sum_{j\in J} \left(m_j+1\right) a_j. \tag{2.21}$$

Proof. We prove Lemma 2 by induction on l.

For $l = 1$, we have

$$\begin{aligned}\left((-1)^{m_1} e^{(m_1+1)a_1 t}+1\right) s_n(x) &= (-1)^{m_1} s_n\left(x+(m_1+1)\,a_1\right) + s_n(x)\\ &= \sum_{i=0}^{1} \tilde{s}_{n,i}\left(x|\,(m_1+1)\,a_1\right).\end{aligned}$$

Assume that, for $l > 1$, the following holds:

$$\begin{aligned}&\left((-1)^{m_{l-1}} e^{(m_{l-1}+1)a_{l-1}t}+1\right)\cdots\left((-1)^{m_1} e^{(m_1+1)a_1 t}+1\right) s_n(x) \\ &= \sum_{i=0}^{l-1} \tilde{s}_{n,i}\left(x|\,(m_1+1)\,a_1,\ldots,(m_{l-1}+1)\,a_{l-1}\right).\end{aligned} \tag{2.22}$$

Then the LHS of (2.19) is

$$\begin{aligned}&\sum_{i=0}^{l-1}\left((-1)^{m_l} e^{(m_l+1)a_l t}+1\right)\tilde{s}_{n,i}\left(x|\,(m_1+1)\,a_1,\ldots,(m_{l-1}+1)\,a_{l-1}\right)\\ &= \sum_{i=0}^{l-1}\left((-1)^{m_l} e^{(m_l+1)a_l t}+1\right)\sum_{\substack{J\subset[1,l-1]\\|J|=i}} (-1)^{m_J} s_n\left(x+((m+1)\,a)_J\right)\\ &= \sum_{i=0}^{l-1}\left\{\sum_{\substack{J\subset[1,l-1]\\|J|=i}} (-1)^{m_J+m_l} s_n\left(x+((m+1)\,a)_J+(m_l+1)\,a_l\right)\right.\\ &\quad\left.+\sum_{\substack{J\subset[1,l-1]\\|J|=i}} (-1)^{m_J} s_n\left(x+((m+1)\,a)_J\right)\right\}\\ &= (-1)^{m_{[1,l]}} s_n\left(x+((m+1)\,a)_{[1,l]}\right)\\ &\quad+\sum_{i=1}^{l-1}\sum_{\substack{J\subset[1,l-1]\\|J|=i-1}} (-1)^{m_j+m_l} s_n\left(x+((m+1)\,a)_J+(m_l+1)\,a_l\right)\\ &\quad+\sum_{i=1}^{l-1}\sum_{\substack{J\subset[1,l-1]\\|J|=i}} (-1)^{m_J} s_n\left(x+((m+1)\,a)_J\right)+s_n(x)\\ &= (-1)^{m_{[1,l]}} s_n\left(x+((m+1)\,a)_{[1,l]}\right)\\ &\quad+\sum_{i=1}^{l-1}\sum_{\substack{J\subset[1,l]\\|J|=i}} (-1)^{m_J} s_n\left(x+((m+1)\,a)_J\right)+s_n(x)\end{aligned}$$

$$= \sum_{i=0}^{l} \sum_{\substack{J\subset[1,l] \\ |J|=i}} (-1)^{m_J} s_n\left(x + ((m+1)a)_J\right)$$

$$= \sum_{i=0}^{l} \tilde{s}_{n,i}\left(x|\,(m_1+1)a_1, \ldots, (m_l+1)a_l\right).$$

□

Theorem 2. *Let $s_n(x)$ be the Appell sequence for $g(t)$, namely $s_n(x) \sim (g(t), t)$. Then, for any integers $m_1, \ldots, m_l$ with $m_1, m_2, \ldots, m_l \geq 1$, $a_1, \ldots, a_l \neq 0$, we have*

$$\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} s_n(x + a_1 i_1 + \cdots + a_l i_l) \tag{2.23}$$

$$= 2^{-l} \sum_{i=0}^{l} \left(\tilde{E}(a_1, \ldots, a_l)\, s\right)_{n,i} \left(x|\,(m_1+1)a_1, \ldots, (m_l+1)a_l\right),$$

where $E(a_1, \ldots, a_l)_n(x) = E_n(x|a_1, \ldots, a_l)$ is the Barnes' multiple Euler polynomial whose generating function is given by

$$\prod_{j=1}^{l} \left(\frac{2}{e^{a_j t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x|a_1, \ldots, a_l) \frac{t^l}{l!}, \quad (\textit{see [12]}),$$

$(E(a_1, \ldots, a_l)\, s)_n(x)$ is the $E(a_1, \ldots, a_l)_n(x)$ and $s_n(x)$ mixed-type polynomial and

$$(E(a_1, \ldots, a_l)\, s)_{n,i}\left(x|\,(m_1+1)a_1, \ldots, (m_l+1)a_l\right)$$

$$= \sum_{\substack{J\subset[1,l] \\ |J|=i}} (-1)^{m_J} (E(a_1, \ldots, a_l)\, s)_n \left(x + ((m+1)a)_J\right).$$

Proof. Now, we compute the following in two different ways:

$$\left(\frac{(-1)^{m_l} e^{(m_l+1)a_l t}+1}{e^{a_l t}+1}\right) \times \cdots \times \left(\frac{(-1)^{m_1} e^{(m_1+1)a_1 t}+1}{e^{a_1 t}+1}\right) s_n(x). \tag{2.24}$$

On one hand, (2.24) is

$$\left(\frac{(-1)^{m_l} e^{(m_l+1)a_l t}+1}{e^{a_l t}+1}\right) \times \cdots \times \left(\frac{(-1)^{m_2} e^{(m_2+1)a_2 t}+1}{e^{a_2 t}+1}\right) \tag{2.25}$$

$$\times \sum_{i_1=0}^{m_1} (-1)^{i_1} s_n(x + a_1 i_1)$$

$$= \sum_{i_1=0}^{m_1} (-1)^{i_1} \left(\frac{(-1)^{m_l} e^{(m_l+1)a_l t}+1}{e^{a_l t}+1}\right) \times \cdots \times \left(\frac{(-1)^{m_3} e^{(m_3+1)a_3 t}+1}{e^{a_3 t}+1}\right)$$

$$\times \left(\frac{(-1)^{m_2} e^{(m_2+1)a_2 t}+1}{e^{a_2 t}+1} s_n(x + a_1 i_1)\right)$$

$$= \sum_{i_1=0}^{m_1} (-1)^{i_1} \left(\frac{(-1)^{m_l} e^{(m_l+1)a_l t}+1}{e^{a_l t}+1}\right) \times \cdots \times \left(\frac{(-1)^{m_3} e^{(m_3+1)a_3 t}+1}{e^{a_3 t}+1}\right)$$

$$\times \sum_{i_2=0}^{m_2} (-1)^{i_2} s_n (x + a_1 i_1 + a_2 i_2).$$

Continuing in this manner, we obtain the expression on the LHS of (2.23).
On the other hand, (2.24) is

$$\begin{aligned} &2^{-l} \left((-1)^{m_l} e^{(m_l+1)a_l t} + 1\right) \cdots \left((-1)^{m_1} e^{(m_1+1)a_1 t} + 1\right) \prod_{j=1}^{l} \left(\frac{2}{e^{a_j t} + 1}\right) s_n (x) \\ =&2^{-l} \left((-1)^{m_l} e^{(m_l+1)a_l t} + 1\right) \cdots \left((-1)^{m_1} e^{(m_1+1)a_1 t} + 1\right) (E (a_1, \ldots, a_l) s)_n (x). \end{aligned}$$

Now, the expression on the RHS of (2.23) follows from Lemma 2. □

Corollary 2. *For $m_1, \ldots, m_l \in \mathbb{Z}$ with $m_1, m_2, \ldots, m_l \geq 1$, $l \in \mathbb{N}$ and $a_1, a_2, \ldots, a_l \neq 0$, we have*

(a)

$$\begin{aligned} &\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} s_n (a_1 i_1 + \cdots + a_l i_l) \\ =&2^{-l} \sum_{i=0}^{l} \left(\tilde{E} (a_1, \ldots, a_l) s\right)_{n,i} ((m_1 + 1) a_1, \ldots, (m_l + 1) a_l), \end{aligned}$$

where

$$\begin{aligned} &\left(\tilde{E} (a_1, \ldots, a_l) s\right)_{n,i} ((m_1 + 1) a_1, \ldots, (m_l + 1) a_l) \\ =&\sum_{\substack{J \subset [1,l] \\ |J| = i}} (-1)^{m_J} (E (a_1, \ldots, a_l) s)_n \left(((m + 1) a)_J\right), \quad (0 \leq i \leq l). \end{aligned}$$

(b)

$$\begin{aligned} &\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} s_n (x + i_1 + \cdots + i_l) \\ =&2^{-l} \sum_{i=0}^{l} \left(\tilde{E}^{(l)} s\right)_{n,i} (x | m_1 + 1, \ldots, m_l + 1), \end{aligned}$$

where $E_n^{(l)}(x)$ are the Euler polynomials of order l given by the generating funciton

$$\left(\frac{2}{e^t + 1}\right)^l e^{xt} = \sum_{n=0}^{\infty} E_n^{(l)} (x) \frac{t^n}{n!}.$$

(c)

$$\begin{aligned} &\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} s_n (i_1 + \cdots + i_l) \\ =&2^{-l} \sum_{i=0}^{l} \left(\tilde{E}^{(l)} s\right)_{n,i} (m_1 + 1, \ldots, m_l + 1). \end{aligned}$$

(d)

$$\begin{aligned}&\sum_{i_1,\ldots,i_l=0}^{m}(-1)^{i_1+\cdots+i_l}s_n\left(x+i_1+\cdots+i_l\right)\\&=2^{-l}\sum_{i=0}^{l}\binom{l}{i}(-1)^{mi}\left(E^{(l)}s\right)_n\left(x+(m+1)i\right).\end{aligned}$$

(e)

$$\begin{aligned}&\sum_{i_1,\ldots,i_l=0}^{m}(-1)^{i_1+\cdots+i_l}s_n\left(i_1+\cdots+i_l\right)\\&=2^{-l}\sum_{i=0}^{l}\binom{l}{i}(-1)^{mi}\left(E^{(l)}s\right)_n\left((m+1)i\right).\end{aligned}$$

3. Examples on Theorem 1

1. From Corollary 1 (e) with $s_n(x)=x^n\sim(1,t)$ and $l=1$, we can derive the classical power sum identity as follows:

$$\sum_{i=0}^{m}i^n=\frac{1}{n+1}\left(B_{n+1}(m+1)-B_{n+1}\right).$$

However, it is instructive to repeat our proof for this simplest possible case:

$$\begin{aligned}&\sum_{i=0}^{m}i^n\\&=\left\langle\frac{e^{(m+1)t}-1}{e^t-1}\middle|x^n\right\rangle\\&=\left\langle\frac{e^{(m+1)t}-1}{t}\middle|\frac{t}{e^t-1}x^n\right\rangle\\&=\left\langle\frac{e^{(m+1)t}-1}{t}\middle|B_n(x)\right\rangle\\&=\int_0^{m+1}B_n(u)\,du\\&=\frac{1}{n+1}\left(B_{n+1}(m+1)-B_{n+1}\right).\end{aligned}$$

More generally, from Corollary 1 (c), we have

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}\left(i_1+\cdots+i_l\right)^n\\&=\sum_{i=0}^{m_1+\cdots+m_l}p_i(m_1,\ldots,m_l)\,i^n\\&=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}B_{n+l,i}^{(l)}(m_1+1,\ldots,m_l+1),\end{aligned}$$

where

$$\begin{aligned}
&p_i(m_1,\dots,m_l)\\
&=\#\left\{(i_1,\dots,i_l)\in\mathbb{Z}^l \,\middle|\, 0\le i_j\le m_j\ (1\le j\le l),\ i_1+\cdots+i_l=i\right\}\\
&=\text{the coefficient of } t^i \text{ in the expansion of}\\
&\left(\frac{1-t^{m_1+1}}{1-t}\right)\times\cdots\times\left(\frac{1-t^{m_l+1}}{1-t}\right).
\end{aligned}$$

2. Apply Theorem 1 to $s_n(x)=B_n(x|b_1,\dots,b_k)=B(b_1,\dots,b_k)_n(x)$. Then we have

$$\begin{aligned}
&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}B_n(x+a_1i_1+\cdots+a_li_l|b_1,\dots,b_k)\\
&=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}B(a_1,\dots,a_l,b_1,\dots,b_k)_{n+l,i}(x|(m_1+1)a_1,\dots,(m_l+1)a_l)\\
&=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\sum_{\substack{J\subset[1,l]\\|J|=i}}B(a_1,\dots,a_l,b_1,\dots,b_k)_{n+l}\left(x+((m+1)a)_J\right)\\
&=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\sum_{\substack{J\subset[1,l]\\|J|=i}}B_{n+l}\left(x+((m+1)a)_J|a_1,\dots,a_l,b_1,\dots,b_k\right),
\end{aligned}$$

where

$$((m+1)a)_J=\sum_{j\in J}\left(m_j+1\right)a_j.$$

3. Applying Theorem 1 to the poly-Bernoulli polynomial of index k

$$s_n(x)=\mathbb{B}_n^{(k)}(x)\sim\left(\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})},t\right),$$

we obtain

$$\begin{aligned}
&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}\mathbb{B}_n^{(k)}(x+a_1i_1+\cdots+a_li_l)\\
&=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\left(B(a_1,\dots,a_l)\,\mathbb{B}^{(k)}\right)_{n+l,i}(x|(m_1+1)a_1,\dots,(m_l+1)a_l)\\
&=\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\sum_{\substack{J\subset[1,l]\\|J|=i}}S_{n+l}^{(l,k)}\left(x+((m+1)a)_J|a_1,\dots,a_l\right).
\end{aligned}$$

Here $S_n^{(l,k)}(x|a_1,\dots,a_l)$ are the Barnes' multiple Bernoulli and poly-Bernoulli mixed-type polynomials studied in [4] whose generating function is given by

$$\left(\frac{t^l}{\prod_{j=1}^{l}(e^{a_jt}-1)}\right)\times\left(\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\right)e^{xt}=\sum_{n=0}^{\infty}S_n^{(l,k)}(x|a_1,\dots,a_l)\frac{t^n}{n!}.$$

In the special case of $a_1 = a_2 = \cdots = a_l = 1$, and $m_1 = \cdots = m_l = m$, we get

$$\begin{aligned}&\sum_{i_1,\ldots,i_l=0}^{m} \mathbb{B}_n^{(k)}(x + i_1 + \cdots + i_l)\\ =&\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}S_{n+1}^{(l,k)}(x+(m+1)i),\end{aligned}$$

where $S_n^{(l,k)}(x)$ is the higher-order Bernoulli and poly-Bernoulli mixed-type polynomials studied in [7] whose generating function is given by

$$\left(\frac{t}{e^t-1}\right)^l \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt} = \sum_{n=0}^{\infty} S_n^{(l,k)}(x)\frac{t^n}{n!}.$$

For $l = 1$ and $x = 0$, we have

$$\sum_{i=0}^{m}\mathbb{B}_n^{(k)}(i) = \frac{1}{n+1}\left\{S_{n+1}^{(1,k)}(m+1) - S_{n+1}^{(1,k)}\right\}.$$

In [9], it is known that

$$\mathbb{B}_n^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}\left\{\sum_{j=0}^{n-l}\frac{(-1)^{n-j-l}\,j!}{(j+1)^k}S_2(n-l,j)\right\}x^l,$$

where $S_2(n,l)$ is the Stirling number of the second kind. Thus, we get

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}\left\{\sum_{j=0}^{n-l}\frac{(-1)^{n-l-j}\,j!}{(j+1)!}S_2(n-l,j)\right\}\frac{1}{l+1}\left(B_{l+1}(m+1) - B_{l+1}\right)\\ =&\frac{1}{n+1}\left\{S_{n+1}^{(1,k)}(m+1) - S_{n+1}^{(1,k)}\right\}.\end{aligned}$$

4. We now apply Theorem 1 to the generalized Barnes' multiple Frobenius-Euler polynomial

$$\begin{aligned}s_n(x) &= H_n(x|b;\lambda;\mu)\\ &= H_n(x|b_1,\ldots,b_s;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\ &\sim\left(\prod_{j=1}^{s}\left(\frac{e^{b_j t}-\lambda_j}{1-\lambda_j}\right)^{\mu_j},t\right).\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}H_n(x+a_1i_1+\cdots+a_li_l|b;\lambda;\mu)\\ =&\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\left(BH(a;b;\lambda;\mu)\right)_{n+l,i}(x|(m_1+1)a_1,\ldots,(m_l+1)a_l)\\ =&\frac{1}{(n+l)_l}\sum_{i=0}^{l}(-1)^{l-i}\sum_{\substack{J\subset[1,l]\\|J|=i}}BH_{n+l}\left(x+((m+1)a)_J|a;b;\lambda;\mu\right),\end{aligned}$$

where

$$\begin{aligned}&BH\left(a;b;\lambda;\mu\right)_n\left(x\right)\\=&BH_n\left(x|a;b;\lambda;\mu\right)\\=&BH_n\left(x|a_1,\ldots,a_l;b_1,\ldots,b_s;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s\right)\end{aligned}$$

are given by the generating function

$$\begin{aligned}&\prod_{j=1}^{l}\left(\frac{t}{e^{a_jt}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)^{\mu_j}e^{xt}\\=&\sum_{n=0}^{\infty}BH_n\left(x|a;b;\lambda;\mu\right)\frac{t^n}{n!},\end{aligned}$$

and are called the Barnes' multiple Bernoulli and generalized Barnes' multiple Frobenius-Euler mixed-type polynomials. These are studied in [8].

5. We apply Theorem 1 to the Hermite polynomial $s_n\left(x\right)=H_n^{(\nu)}\left(x\right)\sim\left(e^{\frac{\nu t^2}{2}},t\right)$. Then we have

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}H_n^{(\nu)}\left(x+a_1i_1+\cdots+a_li_l\right)\\=&\frac{1}{\left(n+l\right)_l}\sum_{i=0}^{l}\left(-1\right)^{l-i}\left(BH^{(\nu)}\left(a_1,\ldots,a_l\right)\right)_{n+l,i}\left(x|\left(m_1+1\right)a_1,\ldots,\left(m_l+1\right)a_l\right)\\=&\frac{1}{\left(n+l\right)_l}\sum_{i=0}^{l}\left(-1\right)^{l-i}\sum_{\substack{J\subset[1,l]\\|J|=i}}BH_{n+l}^{(\nu)}\left(x+\left(\left(m+1\right)a\right)_J|a_1,\ldots,a_l\right).\end{aligned}$$

Here $BH^{(\nu)}\left(a_1,\ldots,a_l\right)_n\left(x\right)=BH_n^{(\nu)}\left(x|a_1,\ldots,a_l\right)$ are given by the generating function to be

$$\prod_{j=1}^{l}\left(\frac{t}{e^{a_jt}-1}\right)e^{-\nu t^2/2}e^{xt}=\sum_{n=0}^{\infty}BH_n^{(\nu)}\left(x|a_1,\ldots,a_l\right)\frac{t^n}{n!},$$

and are called the Barnes' multiple Bernoulli and Hermite mixed-type polynomials. These mixed-type polynomials are studied in [10].

As a very special case, we have

$$\sum_{i=0}^{m}H_n^{(\nu)}\left(i\right)=\frac{1}{n+1}\left(BH_{n+1}^{(\nu)}\left(m+1\right)-BH_{n+1}^{(\nu)}\right),$$

where $BH_n^{(\nu)}\left(x\right)$ are the polynomials given by

$$\frac{t}{e^t-1}e^{xt-\nu t^2/2}=\sum_{n=0}^{\infty}BH_n^{(\nu)}\left(x\right)\frac{t^n}{n!}.$$

An explicit expression for the Hermite polynomials is given in [16] by

$$H_n^{(\nu)}\left(x\right)=\sum_{j=0}^{\left[\frac{n}{2}\right]}\left(-\frac{\nu}{2}\right)^j\frac{\left(n\right)_{2j}}{j!}x^{n-2j}.$$

Thus, we see that

$$\begin{aligned}&\sum_{j=0}^{\left[\frac{n}{2}\right]}\left(-\frac{\nu}{2}\right)^{j}\frac{(n)_{2j}}{j!}\frac{1}{n-2j+1}\left(B_{n-2j+1}(m+1)-B_{n-2j+1}\right)\\=&\frac{1}{n+1}\left(BH_{n+1}^{(\nu)}(m+1)-BH_{n+1}^{(\nu)}\right).\end{aligned}$$

4. Examples on Theorem 2

1. From Corollary 2 (e) with $s_n(x) = x^n \sim (1, t)$ and $l = 1$, we have the classical alternating power sum identitiy:

$$\sum_{i=0}^{m}(-1)^i i^n = \frac{1}{2}\left((-1)^m E_n(m+1)+E_n\right).$$

But it is instructive to go over our proof in this simplest possible case.

$$\begin{aligned}&\sum_{i=0}^{m}(-1)^i i^n\\=&\left\langle\frac{(-1)^m e^{(m+1)t}+1}{e^t+1}\middle|x^n\right\rangle\\=&\left\langle\frac{(-1)^m e^{(m+1)t}+1}{2}\middle|\frac{2}{e^t+1}x^n\right\rangle\\=&\left\langle\frac{(-1)^m e^{(m+1)t}+1}{2}\middle|E_n(x)\right\rangle\\=&\frac{1}{2}\left((-1)^m E_n(m+1)+E_n\right).\end{aligned}$$

More generally, from Corollary 2 (c), we can derive

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}(-1)^{i_1+\cdots+i_l}(i_1+i_2+\cdots+i_l)^n\\=&\sum_{i=0}^{m_1+\cdots+m_l}p_i(m_1,\ldots,m_l)(-1)^i i^n\\=&2^{-l}\sum_{i=0}^{l}\tilde{E}_{n,i}^{(l)}(m_1+1,\ldots,m_l+1)\\=&2^{-l}\sum_{i=0}^{l}\sum_{\substack{J\subset[1,l]\\|J|=i}}(-1)^{m_J}E_n^{(l)}\left((m+1)_J\right).\end{aligned}$$

2. Applying Theorem 2 to the poly-Bernoulli polynomial of index k

$$s_n(x) = \mathbb{B}_n^{(k)}(x) \sim \left(\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}, t\right),$$

we have

$$\begin{aligned}
&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}(-1)^{i_1+\cdots+i_l}\,\mathbb{B}_n^{(k)}\left(x+a_1i_1+\cdots+a_li_l\right)\\
=&2^{-l}\sum_{i=0}^{l}\left(\tilde{E}\left(a_1,\ldots,a_l\right)\mathbb{B}^{(k)}\right)_{n,i}\left(x|\left(m_1+1\right)a_1,\ldots,\left(m_l+1\right)a_l\right)\\
=&2^{-l}\sum_{i=0}^{l}\sum_{\substack{J\subset[1,l]\\|J|=i}}(-1)^{m_J}\,T_n^{(l,k)}\left(x+\left((m+1)\,a\right)_J|a_1,\ldots,a_l\right),
\end{aligned}$$

where $T_n^{(l,k)}\left(x|a_1,\ldots,a_l\right)=T_n^{(l,k)}\left(x|a_1,\ldots,a_l;-1,\ldots,-1\right)$ are the special $\lambda_1=\cdots=\lambda_l=-1$ case of the Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials $T_n^{(l,k)}\left(a_1,\ldots,a_l;\lambda_1,\ldots,\lambda_l\right)$ studied in [11] whose generating function is given by

$$\begin{aligned}
&\prod_{j=1}^{l}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k\left(1-e^{-t}\right)}{1-e^{-t}}e^{xt}\\
=&\sum_{n=0}^{\infty}T_n^{(l,k)}\left(x|a_1,\ldots,a_l;\lambda_1,\ldots,\lambda_l\right)\frac{t^n}{n!}.
\end{aligned}$$

In the special case of $m_1=\cdots=m_l=m$ and $a_1=\cdots=a_l=1$, we have

$$\begin{aligned}
&\sum_{i_1,\ldots,i_l=0}^{m}(-1)^{i_1+\cdots+i_l}\,\mathbb{B}_n^{(k)}\left(x+i_1+\cdots+i_l\right)\\
=&2^{-l}\sum_{i=0}^{l}(-1)^{mi}\binom{l}{i}T_n^{(l,k)}\left(x+(m+1)\,i\right),
\end{aligned}$$

where $T_n^{(l,k)}\left(x\right)=T_n^{(l,k)}\left(x|-1\right)$ are the special $\lambda=-1$ case of the higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials $T_n^{(l,k)}\left(x|\lambda\right)$ studied in [6] whose generating function is defined by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^l\frac{\mathrm{Li}_k\left(1-e^{-t}\right)}{1-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}T_n^{(l,k)}\left(x|\lambda\right)\frac{t^n}{n!}.$$

For $l=1$ and $x=0$, we get

$$\sum_{i=0}^{m}(-1)^i\,\mathbb{B}_n^{(k)}\left(i\right)=\frac{1}{2}\left\{(-1)^m\,T_n^{(1,k)}\left(m+1\right)+T_n^{(1,k)}\right\}.$$

Again, using

$$\mathbb{B}_n^{(k)}\left(x\right)=\sum_{l=0}^{n}\binom{n}{l}\left\{\sum_{j=0}^{n-l}\frac{(-1)^{n-j-l}\,j!}{(j+1)^k}S_2\left(n-l,j\right)\right\}x^l,$$

we obtain

$$\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l}\left\{\sum_{j=0}^{n-l}\frac{(-1)^{n-j-l}\,j!}{(j+1)^k}S_2\left(n-l,j\right)\right\}\frac{1}{2}\left((-1)^m\,E_l\left(m+1\right)+E_l\right)\\
=&\frac{1}{2}\left\{(-1)^m\,T_n^{(1,k)}\left(m+1\right)+T_n^{(1,k)}\right\}.
\end{aligned}$$

3. We now apply Theorem 2 to the Barnes' multiple Euler polynomial $s_n(x) = E_n(x|b_1,\dots,b_k) = E(b_1,\dots,b_k)_n(x)$. Then, we obtain

$$\begin{aligned}
&\sum_{i_1=0}^{m_1}\cdots\sum_{i_l=0}^{m_l}(-1)^{i_1+\cdots+i_l}E_n(x+a_1i_1+\cdots+a_li_l|b_1,\dots,b_k)\\
=&2^{-l}\sum_{i=0}^{l}\tilde{E}(a_1,\dots,a_l,b_1,\dots,b_k)_{n,i}(x|(m_1+1)a_1,\dots,(m_l+1)a_l)\\
=&2^{-l}\sum_{i=0}^{l}\sum_{\substack{J\subset[1,l]\\|J|=i}}(-1)^{m_J}E_n\left(x+((m+1)a)_J|a_1,\dots,a_l,b_1,\dots,b_l\right).
\end{aligned}$$

References

[1] S. Araci and M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 3, 399–406.

[2] A. Baszó, Á. Pintér and H. M. Srivastava, A refinement of Faulhaber's theorem concerning sums of powers of natural numbers, *Appl. Math. Lett.* **25** (2012), 486–489.

[3] A. Bayad, *Modular properties of elliptic Bernoulli and Euler functions*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 3, 389–401.

[4] D. V. Dolgy, D. S. Kim, T. Kim, and T. Komatsu, *Barnes' multiple Bernoulli and poly-Bernoulli mixed-type polynomials*, J. Comput. Anal. Appl. 18 (2015), no. 5, 933–951.

[5] S. Gaboury, R. Tremblay, and B.-J. Fugère, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials*, Proc. Jangjeon Math. Soc. **17** (2014), no. 1, 115–123.

[6] D. S. Kim and T. Kim, *Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials*, Adv. Difference Equ. (2013), 2013:251.

[7] D. S. Kim and T. Kim,, *Higher-order Bernoulli and poly-Bernoulli mixed-type polynomial*, Georgian Math. J. 22(2015), no.2, 265–272.

[8] D. S. Kim, T. Kim, T. Komatsu, and H. I. Kwon, *Barnes' multiple Bernoulli and generalized Barnes' multiple Frobenius-Euler mixed-type polynomials*, Adv. Difference Equ. (2014), 2014:238.

[9] D. S. Kim, T. Kim, and S.-H. Lee, *A note on poly-Bernoulli polynomilas arising from umbral calculus*, Adv. Stud. Theor. Phys. **7** (2013), no. 15, 731–744.

[10] D. S. Kim, T. Kim, S.-H. Rim, and D. V. Dolgy, *Barnes' multiple Bernoulli and Hermite mixed-type polynomials*, Proc. Jangjeon Math. Soc. **18** (2015), no. 1, 7–19.

[11] D. S. Kim, T. Kim, J.-J. Seo, and T. Komatsu, *Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials*, Adv. Difference Equ. (2014), 2014:92.

[12] T. Kim, *On Euler-Barnes multiple zeta functions*, Russ. J. Math. Phys. **10** (2003), no. 3, 261–267.

[13] Q.-M. Luo, *q-extensions of some results involving the Luo-Srivastava Generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, Filomat **28** (2014), no. 2, 329–351.

[14] Á. Pintér and H. M. Srivastava, Addition theorems for the Appell polynomials and the associated classes of polynomial expansions, *Aequationes Math.* **85** (2013), 483–495.

[15] Feng, Qi, *Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind*, Filomat **28** (2014), no. 2, 319–327.

[16] S. Roman, *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.

[17] E. Şen, *Theorems on Apostol-Euler polynomials of higher order arising from Euler basis*, Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 2, 337–345.

[18] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inform. Sci.* **5** (2011), 390–444.

[19] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[20] Z. Zhang and H. Yang, *Some closed formulas for generalized Bernoulli-Euler numbers and polynomials*, Proc. Jangjeon Math. Soc. **11** (2008), no. 2, 191–198.