Filomat 28:7 (2014), 1457–1462
DOI 10.2298/FIL1407457B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A New Note on Absolute Riezs Summability.I

Hüseyin Bor[a]

[a] *P. O. Box 121, TR-06502 Bahçelievler, Ankara, Turkey*

Abstract. In [6], we proved a theorem dealing with absolute Riesz summability. In this paper, we prove that result under more weaker conditions. This theorem also includes some new results.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). We denote by u_n and t_n the nth $(C, 1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $\mid C, 1 \mid_k$, $k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n - u_{n-1} \mid^k = \sum_{n=1}^{\infty} \frac{\mid t_n \mid^k}{n} < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$R_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (R_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n, \theta_n \mid_k$, $k \geq 1$, if (see [10])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid R_n - R_{n-1} \mid^k < \infty. \tag{4}$$

2010 *Mathematics Subject Classification*. 26D15, 40D15, 40F05, 40G99
Keywords. Absolute summability factors; almost increasing sequences; Riesz mean; infinite series; Hölder inequality; Minkowski inequality
Received: 30 June 2013; Accepted: 19 March 2014
Communicated by Dragan S. Djordjevic
Email address: hbor33@gmail.com (Hüseyin Bor)

If we take $\theta_n = \frac{P_n}{p_n}$, then $| \bar{N}, p_n, \theta_n |_k$ summability reduces to $| \bar{N}, p_n |_k$ summability (see [2]). In the special case $p_n = 1$ for all values of n, $| \bar{N}, p_n |_k$ summability reduces to $| C, 1 |_k$ summability. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $| C, 1 |_k$ summability. Furthermore, if we take $\theta_n = n$, then $| \bar{N}, p_n, \theta_n |_k$ summability reduces to $| R, p_n |_k$ (see [4]) summability.

2. The Known Result

In [6], we proved the following main theorem dealing with $| \bar{N}, p_n |_k$ summability factors of infinite series.

Theorem A. Let (X_n) be an almost increasing sequence and let $(\frac{\theta_n p_n}{P_n})$ be a non-increasing sequence. Suppose also that there exists sequences (β_n) and (λ_n) such that

$$| \Delta\lambda_n | \leq \beta_n, \tag{5}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{6}$$

$$\sum_{n=1}^{\infty} n | \Delta\beta_n | X_n < \infty, \tag{7}$$

$$| \lambda_n | X_n = O(1). \tag{8}$$

If

$$\sum_{n=1}^{m} \theta_n^{k-1} \frac{| t_n |^k}{n^k} = O(X_m) \quad as \quad m \to \infty \tag{9}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{10}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{11}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $| \bar{N}, p_n, \theta_n |_k, k \geq 1$.
If we take $\theta_n = \frac{P_n}{p_n}$ and consider (10), then we get a result dealing with $| \bar{N}, p_n |_k$ summability factors of infinite series (see [5]). In this case, the condition " $(\frac{\theta_n p_n}{P_n})$ is a non-increasing sequence " is automatically satisfied.

3. The Main Result

The aim of this paper is to prove Theorem A under more weaker conditions. Now we shall prove the following theorem.

Theorem . Let (X_n) be an almost increasing sequence and let $\left(\frac{\theta_n p_n}{P_n}\right)$ be a non-increasing sequence. If the conditions (5)-(8), (10)-(11) and

$$\sum_{n=1}^{m} \theta_n^{k-1} \frac{| t_n |^k}{n^k X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{12}$$

are satisfied , then the series $\sum\limits_{n=1}^{\infty} a_n \dfrac{P_n \lambda_n}{n p_n}$ is summable $| \bar{N}, p_n, \theta_n |_k$, $k \geq 1$.

Remark 1. It should be noted that condition (12) is reduced to the condition (9) when k=1 . When $k > 1$, the condition (12) is weaker than the condition (9), but the converse is not true. As in [12] we can show that if (9) is satisfied, then we get that

$$\sum_{n=1}^{m} \theta_n^{k-1} \frac{| t_n |^k}{n^k X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \theta_n^{k-1} \frac{| t_n |^k}{n^k} = O(X_m).$$

If (12) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} \theta_n^{k-1} \frac{| t_n |^k}{n^k} = \sum_{n=1}^{m} \theta_n^{k-1} X_n^{k-1} \frac{| t_n |^k}{n^k X_n^{k-1}} = O(X_m^{k-1}) \sum_{n=1}^{m} \theta_n^{k-1} \frac{| t_n |^k}{n^k X_n^{k-1}} = O(X_m^k) \neq O(X_m).$$

Remark 2. It should be noted that under the conditions on the sequence (λ_n) we have that; (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).
We require the following lemmas for the proof of the theorem.

Lemma 1 ([9]). If (X_n) is an almost increasing sequence, then under the conditions of the Theorem we have that

$$nX_n\beta_n = O(1), \tag{13}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{14}$$

Lemma 2 ([3]). If the conditions (10) and (11) are satisfied, then $\Delta(P_n/p_n n^2) = O(1/n^2)$.

5. Proof of the Theorem

Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{v p_v}. \tag{15}$$

Then for $n \geq 1$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v v \lambda_v}{v^2 p_v}.$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\left(\frac{P_{v-1} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} \\ &+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v (v+1) \frac{t_v}{v^2 p_v} \\ &- \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} (v+1) t_v \Delta(P_v/v^2 p_v) \\ &+ \lambda_n t_n (n+1)/n^2 \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1,2,3,4. \tag{16}$$

Now, applying Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \mid t_v \mid\mid \lambda_v \mid \frac{1}{v} \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \\
&\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid p_v \mid t_v \mid^k \frac{1}{P_v} \frac{1}{v^k} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} \mid \lambda_v \mid \frac{\mid t_v \mid^k}{v^k {X_v}^{k-1}} \quad \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \theta_v^{k-1} \frac{\mid t_v \mid^k}{v^k {X_v}^{k-1}} \mid \lambda_v \mid \\
&= O(1) \sum_{v=1}^{m-1} \Delta \mid \lambda_v \mid \sum_{r=1}^{v} \theta_r^{k-1} \frac{\mid t_r \mid^k}{r^k {X_r}^{k-1}} + O(1) \mid \lambda_m \mid \sum_{v=1}^{m} \theta_v^{k-1} \frac{\mid t_v \mid^k}{v^k {X_v}^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) \mid \lambda_m \mid X_m = O(1)
\end{aligned}$$

as $m \to \infty$, by virtue of the hypotheses of the theorem and Lemma 1. By using (10), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} \mid \Delta\lambda_v \mid p_v \mid t_v \mid \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k \mid \Delta\lambda_v \mid^k \mid t_v \mid^k p_v \right\} \\
&\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k} \mid \Delta\lambda_v \mid^{k} \mid t_v \mid^{k} p_v \sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k} \frac{p_n}{P_n P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} \mid \Delta\lambda_v \mid^{k} \mid t_v \mid^{k} \theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m} \mid \Delta\lambda_v \mid^{k-1} \mid \Delta\lambda_v \mid\mid t_v \mid^{k} \theta_v^{k-1}\\
&= O(1)\sum_{v=1}^{m} {\beta_v}^{k-1}\beta_v \mid t_v \mid^{k} \theta_v^{k-1}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{1}{vX_v}\right)^{k-1} \beta_v \mid t_v \mid^{k} \theta_v^{k-1}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\theta_r^{k-1}\frac{\mid t_r \mid^{k}}{r^k {X_r}^{k-1}} + O(1)m\beta_m\sum_{v=1}^{m}\theta_v^{k-1}\frac{\mid t_v \mid^{k}}{v^k {X_v}^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

in view of the hypotheses of the theorem and Lemma 1. By using Lemma 2, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1} \mid T_{n,3} \mid^{k} &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^{k}\frac{1}{P_{n-1}^{k}}\left\{\sum_{v=1}^{n-1} P_v \mid \lambda_{v+1} \mid\mid t_v \mid \frac{1}{v}\frac{v+1}{v}\right\}^{k}\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^{k}\frac{1}{P_{n-1}^{k}}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v \mid \lambda_{v+1} \mid \frac{1}{v} \mid t_v \mid\right\}^{k}\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^{k}\frac{1}{P_{n-1}^{k}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{k} p_v\frac{1}{v^k} \mid \lambda_{v+1} \mid^{k} \mid t_v \mid^{k}\right\}\\
&\times \left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k} p_v\frac{1}{v^k} \mid \lambda_{v+1} \mid^{k-1} \mid \lambda_{v+1} \mid\mid t_v \mid^{k}\\
&\times \left\{\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_n P_{n-1}}\right\}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}\left(\frac{1}{X_v}\right)^{k-1} \mid \lambda_{v+1} \mid\mid t_v \mid^{k} \theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} \mid \lambda_{v+1} \mid \theta_v^{k-1} \frac{\mid t_v \mid^k}{v^k X_v{}^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1} \Delta \mid \lambda_{v+1} \mid \sum_{r=1}^{v} \theta_r^{k-1} \frac{\mid t_r \mid^k}{r^k X_r{}^{k-1}} + O(1) \mid \lambda_{m+1} \mid \sum_{v=1}^{m} \theta_v^{k-1} \frac{\mid t_v \mid^k}{v^k X_v{}^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta\lambda_{v+1} \mid X_{v+1} + O(1) \mid \lambda_{m+1} \mid X_{m+1} \\
&= O(1)\sum_{v=2}^{m} \mid \Delta\lambda_{v} \mid X_{v} + O(1) \mid \lambda_{m+1} \mid X_{m+1} \\
&= O(1)\sum_{v=1}^{m} \beta_v X_v + O(1) \mid \lambda_{m+1} \mid X_{m+1} \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 1. Finally, as in $T_{n,3}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,4} \mid^k &= O(1)\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{n+1}{n}\right)^k \frac{1}{n^k} \mid \lambda_n \mid^k \mid t_n \mid^k \\
&= O(1)\sum_{n=1}^{m} \theta_n^{k-1} \frac{1}{n^k} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid\mid t_n \mid^k \\
&= O(1)\sum_{n=1}^{m} \theta_n^{k-1} \frac{1}{n^k} \left(\frac{1}{X_n}\right)^{k-1} \mid \lambda_n \mid\mid t_n \mid^k \\
&= O(1)\sum_{n=1}^{m} \mid \lambda_n \mid \theta_n^{k-1} \frac{\mid t_n \mid^k}{n^k X_n{}^{k-1}} = O(1) \quad as \quad m \to \infty.
\end{aligned}
$$

This completes the proof of the theorem. If we take $p_n = 1$ for all values of n, then we get a new result concerning the $\mid C, 1, \theta_n \mid_k$ summability. Also, if we take $\theta_n = n$, then we have another new result dealing with $\mid R, p_n \mid_k$ summability. Finally, if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get a result concerning the $\mid C, 1 \mid_k$ summability.

Acknowledgement. The author expresses his thanks to the referee for his/her useful comments and suggestions.

References

[1] N. K. Bari and S. B. Stečkin, Best approximation and differential properties of two conjugate functions, Trudy. Moskov. Mat. Obšč. 5 (1956) 483-522 (Russian).
[2] H. Bor, On two summability methods, Math. Proc. Camb. Philos. Soc. 97 (1985) 147-149.
[3] H. Bor, Absolute summability factors for infinite series, Indian J. Pure Appl. Math. 19 (1988) 664-671.
[4] H. Bor, On the relative strength of two absolute summability methods, Proc. Amer. Math. Soc. 113 (1991) 1009-1012.
[5] H. Bor, A note on absolute Riesz summability factors, Math. Inequal. Appl. 10 (2007) 619-625.
[6] H. Bor, A new note on absolute Riesz summability, Math. Comput. Modelling. 55 (2012) 1639-1643.
[7] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc. 7 (1957) 113-141.
[8] G. H. Hardy, Divergent Series, Oxford Univ. Press. Oxford (1949).
[9] S. M. Mazhar, A note on absolute summability factors, Bull. Inst. Math. Acad. Sinica. 25 (1997) 233-242.
[10] W. T. Sulaiman, On some summability factors of infinite series, Proc. Amer. Math. Soc. 115 (1992) 313-317.
[11] W. T. Sulaiman, A note on $\mid A \mid_k$ summability factors of infinite series, Appl. Math. Comput. 216 (2010) 2645-2648.