Filomat 31:2 (2017), 363–370
DOI 10.2298/FIL1702363C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Drawing Graph Joins in the Plane with Restrictions on Crossings

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Abstract. A graph is called 1-planar if it can be drawn in the plane so that each of its edges is crossed by at most one other edge. In 2014, Zhang showed that the set of all 1-planar graphs can be decomposed into three classes C_0, C_1 and C_2 with respect to the types of crossings. He proved that every n-vertex 1-planar graph of class C_1 has a C_1-drawing with at most $\frac{3}{5}n - \frac{6}{5}$ crossings. Consequently, every n-vertex 1-planar graph of class C_1 has at most $\frac{18}{5}n - \frac{36}{5}$ edges.

In this paper we prove a stronger result. We show that every C_1-drawing of a 1-planar graph has at most $\frac{3}{5}n - \frac{6}{5}$ crossings. Next we present a construction of n-vertex 1-planar graphs of class C_1 with $\frac{18}{5}n - \frac{36}{5}$ edges. Finally, we present the decomposition of 1-planar join products.

1. Introduction

All graphs considered in this paper are finite, simple and undirected, unless otherwise stated. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph G, respectively. The *crossing number* of G, denoted by $cr(G)$, is the minimum possible number of crossings in a drawing of G in the plane.

A drawing of a graph is *1-planar* if each of its edges is crossed at most once. If a graph has a 1-planar drawing, then it is *1-planar*. Let G be a 1-planar graph drawn in the plane so that none of its edges is crossed more than once. The *associated plane graph* $G^\times$ of G is the plane graph obtained from G so that the crossings of G become new vertices of degree four; we call these vertices *false*. Vertices of $G^\times$ which are also vertices of G are called *true*. Similarly, the edges and faces of $G^\times$ are called false, if they are incident with a false vertex, and true otherwise. For a false vertex c let $N_{G^\times}(c)$ denote the set of neighbors of c in $G^\times$.

It is easy to see that if a graph has a 1-planar drawing in which two edges e_1, e_2 with a common endvertex cross, then the drawing of e_1 and e_2 can be changed so that these two edges no longer cross. Therefore, we may assume that adjacent edges never cross and that no edge is crossing itself. Consequently, every crossing involves two edges with four distinct endvertices, i.e. $|N_{G^\times}(c)| = 4$ for every false vertex c.

A 1-planar graph is of class C_0 if it has a 1-planar drawing D such that for any two false vertices c_1, c_2 of $D^\times$ it holds $|N_{D^\times}(c_1) \cap N_{D^\times}(c_2)| = 0$. This class of 1-planar graphs was investigated in [7, 9, 10] under the notion plane graphs with independent crossings. A 1-planar graph is of class C_i, $i \in \{1, 2\}$, if it is not of class C_k for any $k < i$ and it has a 1-planar drawing D such that for any two false vertices c_1, c_2 of $D^\times$ it holds $|N_{D^\times}(c_1) \cap N_{D^\times}(c_2)| \leq i$. The corresponding drawing is called C_i*-drawing*, $i = 0, 1, 2$. The class C_1 was investigated in [8] under the notion plane graphs with near-independent crossings.

2010 *Mathematics Subject Classification*. 05C10, 05C62
Keywords. Crossing number, join product, 1-planar graph
Received: 11 November 2014; Accepted: 29 April 2016
Communicated by Francesco Belardo
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The author of [8] proved that any *good C_1-drawing* (that is, a C_1-drawing with minimum possible number of crossings) of an n-vertex 1-planar graph of class C_1 has at most $\frac{3}{5}n - \frac{6}{5}$ crossings. In this paper we improve this result. We show that this bound holds for any C_1-drawing. From this result it follows that any n-vertex 1-planar graph of class C_1 has at most $\frac{18}{5}n - \frac{36}{5}$ edges. We show that this bound is tight.

The obtained results help us to determine the decomposition of 1-planar join products. The *join product* (or shortly, join) $G + H$ of two graphs G and H is obtained from vertex-disjoint copies of G and H by adding all edges between $V(G)$ and $V(H)$.

The disjoint union of two graphs G_1 and G_2 will be denoted by $G_1 \cup G_2$ and the disjoint union of k copies of a graph G_1 will be denoted by kG_1.

2. Results

First we show (for the sake of completeness) that every 1-planar graph G has a 1-planar drawing D such that for any two false vertices c_1, c_2 of $D^\times$ it holds $|N_{D^\times}(c_1) \cap N_{D^\times}(c_2)| \leq 2$ (cf. Proposition 1.1 in [8]).

Assume that there are crossings c_1, c_2 in a 1-planar drawing D such that for the corresponding false vertices it holds $|N_{D^\times}(c_1) \cap N_{D^\times}(c_2)| \geq 3$. Let xy and zw be the edges which cross at c_1. Since $|N_{D^\times}(c_1) \cap N_{D^\times}(c_2)| \geq 3$, without loss of generality, we can assume that the crossing c_2 is the interior point of the edge xz. In this case we can redraw the edge xz such that it is crossing-free by following the edges that cross at c_1 from x and z until they meet in a close neighborhood of c_1. Therefore, if $D^\times$ contains false vertices c_1, c_2 such that $|N_{D^\times}(c_1) \cap N_{D^\times}(c_2)| \geq 3$, then we can eliminate one of them.

In the following we deal with the classification of 1-planar joins.

Lemma 2.1. *Let W be a 1-planar graph of class C_0. Then any C_0-drawing of W contains at most $\frac{|V(W)|}{4}$ crossings.*

Proof. It follows from the definition of C_0-drawing. □

Lemma 2.2. (*cf. Lemma 2.9 in [8]*) *Let W be a 1-planar graph of class C_1. If W has at most 8 vertices, then any C_1-drawing of W has at most two crossings.*

Proof. Let c_1, c_2, c_3 be crossings in a C_1-drawing D of W. Clearly, $|N_{D^\times}(c_1) \cup N_{D^\times}(c_2)| \geq 7$, since D is a C_1-drawing. Therefore, there is at most one true vertex in $D^\times$ which is incident neither c_1 nor c_2. The false vertex c_3 is incident with at most one vertex in $N_{D^\times}(c_1)$ and with at most one vertex in $N_{D^\times}(c_2)$. Consequently, c_3 has at most three (true) neighbors, a contradiction. □

Theorem 2.3. *[5] Let $K_{m,n}$ denote the complete bipartite graph on $m+n$ vertices. Then $cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ for $\min\{m, n\} \leq 6$.*

Lemma 2.4. *If $|V(G)| \geq |V(H)| \geq 4$ and $G + H$ is 1-planar, then $G + H$ is of class C_2.*

Proof. If $|V(G)| \geq |V(H)| \geq 4$, then $G + H$ contains $K_{4,4}$ as a subgraph. The graph $K_{4,4}$ is 1-planar, see [2]. From Theorem 2.3 we have $cr(K_{4,4}) = 4$, therefore any 1-planar drawing of $K_{4,4}$ contains at least four crossings. Hence, Lemma 2.1 and Lemma 2.2 imply that the graph $K_{4,4}$ is of class C_2 . The fact that $G + H$ contains a subgraph of class C_2 implies that $G + H$ also belongs to C_2. □

Lemma 2.5. *If $|V(G)| \geq 5$, $|V(H)| \geq 3$ and $G + H$ is 1-planar, then $G + H$ is of class C_2.*

Proof. In this case $G + H$ contains $K_{5,3}$ as a subgraph. The graph $K_{5,3}$ is 1-planar, see [2]. The crossing number of $K_{5,3}$ is four (see Theorem 2.3), hence (by Lemma 2.1 and Lemma 2.2) it is of class C_2. Consequently, the supergraph $G + H$ of $K_{5,3}$ is also of class C_2. □

From Lemma 2.4 and Lemma 2.5 we obtain, that there are only three possible cases for $G + H$ to belong to classes C_0 and C_1, namely:

- $|V(G)| = |V(H)| = 3$.
- $|V(G)| = 4$ and $|V(H)| = 3$.
- $|V(G)| \geq |V(H)|$ and $|V(H)| \leq 2$.

2.1. The first case: $|V(G)| = |V(H)| = 3$

If the graphs G and H have together (at most) six vertices, then $G + H$ is always 1-planar, since it is a subgraph of the complete graph on six vertices K_6 which is 1-planar, see [2].

Lemma 2.6. *If W is a 1-planar graph on at most six vertices, then W is either of class $\mathcal{C}_0$ or $\mathcal{C}_2$.*

Proof. If W has a 1-planar drawing with at most one crossing, then it is of class $\mathcal{C}_0$. If any 1-planar drawing D of W has at least two crossings, say c_1, c_2, then it is of class $\mathcal{C}_2$, since $|N_{D^\times}(c_1) \cap N_{D^\times}(c_2)| > 1$. □

Let C_n and P_n denote the cycle and the path on n vertices, respectively.

Lemma 2.7. *The graphs $C_3 + (P_2 \cup P_1)$ and $P_3 + P_3$ are of class $\mathcal{C}_0$.*

Proof. $\mathcal{C}_0$-drawings of the graphs $C_3 + (P_2 \cup P_1)$ and $P_3 + P_3$ are shown in Figure 1. □

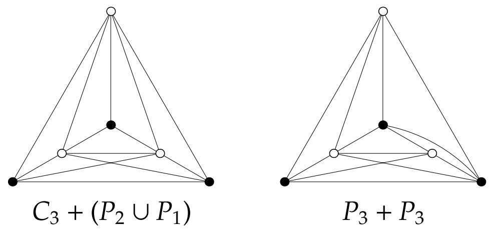


Figure 1: $\mathcal{C}_0$-drawings of the graphs $C_3 + (P_2 \cup P_1)$ and $P_3 + P_3$.

The crossing numbers of join products of cycles and paths were studied in [6].

Theorem 2.8. *[6] $cr(C_n + P_m) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1$ for $m \geq 2, n \geq 3$ with $\min\{m, n\} \leq 6$.*

Lemma 2.9. *The graph $C_3 + P_3$ is of class $\mathcal{C}_2$.*

Proof. The join $C_3 + P_3$ is 1-planar, since it is a subgraph of K_6, which is 1-planar. From Theorem 2.8 it follows $cr(C_3 + P_3) = 2$. Hence, Lemma 2.1 and Lemma 2.6 imply that $C_3 + P_3$ is of class $\mathcal{C}_2$. □

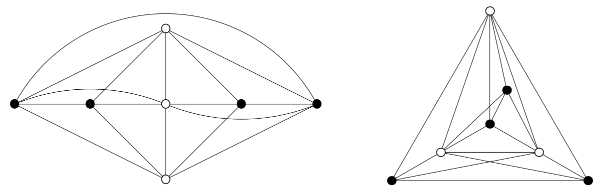
2.2. The second case: $|V(G)| = 4$ *and* $|V(H)| = 3$

Lemma 2.10. *If $|V(G)| = 4$ and $|V(H)| = 3$, then the graph $G + H$ cannot be of class $\mathcal{C}_0$.*

Proof. The graph $G + H$ contains $K_{4,3}$ as a subgraph whose crossing number is two (by Theorem 2.3). This means that any drawing of $G + H$ contains at least two crossings. Therefore, $G + H$ cannot be of class $\mathcal{C}_0$ (see Lemma 2.1). □

Lemma 2.11. *The graphs $P_4 + P_3$ and $2P_2 + C_3$ are of class $\mathcal{C}_1$.*

Proof. From Lemma 2.10 it follows that these graphs cannot be of class $\mathcal{C}_0$. $\mathcal{C}_1$-drawings of the graphs $P_4 + P_3$ and $2P_2 + C_3$ are shown in Figure 2. □



$P_4 + P_3$ $2P_2 + C_3$

Figure 2: $\mathcal{C}_1$-drawings of the graphs $P_4 + P_3$ and $2P_2 + C_3$.

Lemma 2.12. *Let* $|V(G)| = 4$ *and* $|V(H)| = 3$*. If* $G + H$ *is 1-planar and* G *contains a vertex of degree three, then* $G + H$ *is of class* $\mathcal{C}_2$*.*

Proof. In this case the graph G contains $K_{3,1}$ as a subgraph. Hence, $G + H$ contains $K_{3,3,1}$ as a subgraph. The crossing number of $K_{3,3,1}$ is 3, see [1]. Therefore, from Lemma 2.2 it follows that $K_{3,3,1}$ does not have a $\mathcal{C}_1$-drawing. Consequently, if its supergraph $G + H$ is 1-planar, then it must be of class $\mathcal{C}_2$. □

Lemma 2.13. *The graph* $C_4 + 3P_1$ *is of class* $\mathcal{C}_2$*.*

Proof. The join $C_4 + 3P_1$ is 1-planar, see [3]. From Lemma 2.10 it follows that $C_4 + 3P_1$ cannot be of class $\mathcal{C}_0$. Assume that it is of class $\mathcal{C}_1$. Color the edges of C_4 with red and the other edges of $C_4 + 3P_1$ with black (the edges which join vertices of C_4 and $3P_1$). Any drawing of $C_4 + 3P_1$ has at least two crossings which are incident with only black edges, since the black edges induce $K_{4,3}$. Therefore, any $\mathcal{C}_1$-drawing of $C_4 + 3P_1$ has exactly two crossings (see Lemma 2.2). This means that in any $\mathcal{C}_1$-drawing of $C_4 + 3P_1$ no red edge is crossed. The red cycle divides the plane into two parts. If all vertices of $3P_1$ belong to the same part, then we remove one of them, after that we insert the removed vertex to the other part and we join it with the vertices of C_4. Clearly, we again obtain a $\mathcal{C}_1$-drawing of $C_4 + 3P_1$. So we can assume that the inner part of C_4 contains exactly two vertices of $3P_1$. Consequently, all crossings are inside the red C_4, since the black edges which are outside the red C_4 are incident with a common vertex and no red edge is crossed. Therefore, if we remove the vertex which lies outside the red C_4 we obtain a $\mathcal{C}_1$-drawing of a graph on six vertices (with two crossings), a contradiction (see Lemma 2.6). □

Lemma 2.14. *The graph* $(C_3 \cup P_1) + 3P_1$ *is of class* $\mathcal{C}_2$*.*

Proof. The join $(C_3 \cup P_1) + 3P_1$ is 1-planar, see [3]. From Lemma 2.10 it follows that $(C_3 \cup P_1) + 3P_1$ cannot be of class $\mathcal{C}_0$. Assume that it is of class $\mathcal{C}_1$. Then any $\mathcal{C}_1$-drawing of $(C_3 \cup P_1) + 3P_1$ has exactly two crossings. Let D be a $\mathcal{C}_1$-drawing of $(C_3 \cup P_1) + 3P_1$. The associated plane graph $D^\times$ has 9 vertices and 19 edges. Any plane triangulation on 9 vertices has 21 edges. This implies that $D^\times$ has either a face of size 5 or two faces of size 4. If $D^\times$ has a face f of size 5, then there are at least 3 true vertices on the boundary of f (since false vertices cannot be adjacent). We claim that we can add two diagonals e_1, e_2 to f which join only true vertices. This is not possible if and only if at least one of these edges is already present in $(C_3 \cup P_1) + 3P_1$. Assume that e_1 is in $(C_3 \cup P_1) + 3P_1$. If it is crossed by an other edge, then by relocating e_1 to the inner part of f we can decrease the number of crossings to one, which is not possible. If e_1 is not crossed, then its endvertices form a 2-vertex-cut in $D^\times$. In [4] it was proved, that the associated plane graph of a 3-connected 1-plane graph with minimum number of crossings is also 3-connected. Since $(C_3 \cup P_1) + 3P_1$ is 3-connected (it contains a 3-connected induced subgraph $K_{4,3}$), it cannot contain a 2-vertex-cut.

If $D^\times$ contains two faces of size 4, then we can proceed similarly as above.

Consequently, we can add two edges to $D^\times$ which join only true vertices. If at least one of these two edges, say e_1, joins two vertices of $C_3 \cup P_1$, then we obtain a $\mathcal{C}_1$-drawing of $G + 3P_1$, where G is a graph $C_3 \cup P_1$ with the edge e_1. Since G contains a vertex of degree 3, Lemma 2.12 implies that $G + 3P_1$ does not belong to the class $\mathcal{C}_1$, a contradiction. Therefore, the two edges e_1, e_2 must join vertices of $3P_1$. In this case we obtain a $\mathcal{C}_1$-drawing of $(C_3 \cup P_1) + P_3$, what is impossible, since its subgraph $C_3 + P_3$ is of class $\mathcal{C}_2$, see Lemma 2.9. □

Lemma 2.15. *The graph* $(P_3 \cup P_1) + C_3$ *is of class* $\mathcal{C}_2$.

Proof. It follows from Lemma 2.9. □

2.3. The last case: $|V(G)| \geq |V(H)|$ *and* $|V(H)| \leq 2$

Note that the graphs $nP_1 + 2P_1 = K_{n,2}$ and $nP_1 + P_1 = K_{n,1}$ are planar, hence they belong to $\mathcal{C}_0$. Therefore, if the graph H has at most two vertices, then there exists graph G with arbitrarily many vertices such that the join $G + H$ is 1-planar. Hence, it is not possible to describe, in this case, belonging of $G + H$ without additional constraints on G.

2.3.1. The maximum degree of G

Let $\Delta(G)$ denote the maximum degree of a graph G.

Lemma 2.16. *If* $G + 2P_1$ *or* $G + P_2$ *is of class* $\mathcal{C}_0$*, then* $\Delta(G) \leq 3$*. Moreover, this bound is tight.*

Proof. If G has a vertex of degree at least four, then it contains $K_{4,1}$ as a subgraph. Therefore, $K_{4,3}$ is a subgraph of $G + H$. The crossing number of $K_{4,3}$ is two, therefore $K_{4,3}$ and its supergraph $G + H$ cannot be of class $\mathcal{C}_0$, see Lemma 2.1.

Now we show that the bound is sharp. Let $C_k = v_1v_2 \dots v_kv_1$ be a cycle on $k \geq 6$ vertices. The plane drawing of this cycle divides the plane into two parts. Insert the edges v_1v_3 and v_4v_6 into different parts. We obtain a graph G_k which has k vertices, $k + 2$ edges and maximum degree three, moreover, if we put the vertices of $2P_1$ into different faces of size $k - 1$, then we can easily obtain a $\mathcal{C}_0$-drawing of $G_k + 2P_1$.

Let G_k^- be the graph obtained from G_k by removing the edge v_3v_4. Clearly, $\Delta(G_k^-) = 3$ and the graph $G_k^- + P_2$ has a $\mathcal{C}_0$-drawing. □

Lemma 2.17. *If* $G + 2P_1$ *or* $G + P_2$ *is of class* $\mathcal{C}_1$*, then* $\Delta(G) \leq 4$*. Moreover, this bound is tight.*

Proof. If G has a vertex of degree at least five, then it contains $K_{5,1}$ as a subgraph. Therefore, $G + H$ contains $K_{5,3}$ as a subgraph, moreover, $K_{5,3}$ is of class $\mathcal{C}_2$ (see the proof of Lemma 2.5). Consequently, the supergraph $G + H$ of $K_{5,3}$ cannot be of class $\mathcal{C}_1$.

Figure 3 describes a graph G of maximum degree four and a $\mathcal{C}_1$-drawing of $G + 2P_1$, therefore the upper bound 4 for $\Delta(G)$ is sharp. □

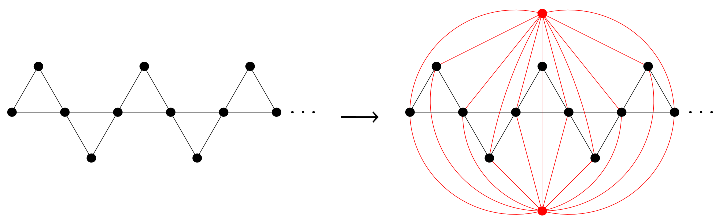


Figure 3: The graph G and a $\mathcal{C}_1$-drawing of $G + 2P_1$.

2.3.2. The number of edges of G

Theorem 2.18. *[9] Let W be a 1-planar graph of class* $\mathcal{C}_0$*. Then* $|E(W)| \leq \frac{13}{4}|V(W)| - 6$*. Moreover, this bound is tight.*

The following assertion improves Theorem 2.2 in [8]. The author of [8] considered only such drawings which have the minimum number of crossings.

Lemma 2.19. *Let W be an n-vertex 1-planar graph of class* $\mathcal{C}_1$*. Then every* $\mathcal{C}_1$*-drawing of W has at most* $\frac{3}{5}n - \frac{6}{5}$ *crossings.*

Proof. Let D be a $\mathcal{C}_1$-drawing of W. Let c denote the number of crossings in D. The associated plane graph $D^\times$ has $n + c$ vertices. Note that no two false vertices are adjacent in $D^\times$. Hence, we can extend $D^\times$ to a plane semitriangulation T (i.e. a plane multigraph triangulating the plane) by adding some edges into non-triangular faces of $D^\times$ which join only true vertices.

The obtained semitriangulation T has $2n+2c-4$ faces (Let $F(T)$ denote the face set of T. Clearly, $3|F(T)| = 2|E(T)|$, since T is a semitriangulation. Combining this equality with Euler's formula $|V(T)|-|E(T)|+|F(T)| = 2$, we obtain $|F(T)| = 2|V(T)| - 4$) and $4c$ of them are false.

Observe that every true edge in T is incident with at most one false face. Therefore, the number of false faces cannot be greater than the number of true edges. On the other hand, every true edge is incident with a true face. Hence, the number of true edges is at most triple the number of true faces. Consequently, $4c \leq 3t$, where t denotes the number of true faces.

Therefore, $2n + 2c - 4 = 4c + t \geq 4c + \frac{4}{3}c$. Consequently, $2n - 4 \geq \frac{10}{3}c$, which implies $c \leq \frac{3}{5}n - \frac{6}{5}$. □

Lemma 2.19 implies that every 1-planar graph G of class $\mathcal{C}_1$ has at most $\frac{18}{5}|V(G)| - \frac{36}{5}$edges, see [8]. Now we show that this bound is tight.

We can construct graphs with the desired property using the graphs depicted in Figure 4.

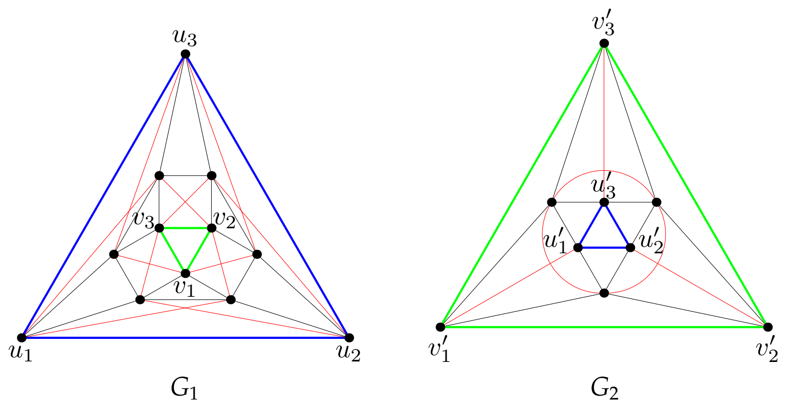


Figure 4: The graphs G_1 and G_2.

Let S be a graph obtained from G_1 by inserting the graph G_2 into the central triangle $v_1v_2v_3$ of G_1 (by identifying the triangles $v_1v_2v_3$ and $v'_1v'_2v'_3$). Let T be a graph obtained from S by inserting the graph G_1 into the central triangle $u'_1u'_2u'_3$ of S (by identifying the triangles $u'_1u'_2u'_3$ and $u_1u_2u_3$). This graph has 27 vertices and 90 edges, moreover, $\frac{18}{5} \cdot 27 - \frac{36}{5} = 90$. Note that, if we iterate this procedure (in the second step we begin with T) we can produce an infinite family of examples with the desired property.

Lemma 2.20. *If $G + 2P_1$ is of class $\mathcal{C}_0$, then $|E(G)| \leq |V(G)| + 2$. Moreover, this bound is tight.*

Proof. Let D be a $\mathcal{C}_0$-drawing of $G + 2P_1$. Remove the two vertices of $2P_1$ from D thereby obtaining a $\mathcal{C}_0$-drawing of G. First we show that this $\mathcal{C}_0$-drawing of G contains no crossings. Assume that, in this drawing of G, the edges xy and zw cross each other at a crossing c. Now consider a subgraph $\{xy, zw\} + 2P_1$ of $G + 2P_1$ in the drawing D. Lemma 2.1 implies that this drawing of $\{xy, zw\} + 2P_1$ can contain at most one crossing. Now we draw the edges xz, zy, yw, wx to $\{xy, zw\} + 2P_1$ such that they are crossing-free by following the edges that cross at c from the endvertices until they meet in a close neighborhood of c. In this way we obtain a $\mathcal{C}_0$-drawing of K_6 minus one edge. Any planar graph on 6 vertices has at most 12 edges. The graph K_6 minus one edge has 6 vertices and 14 edges. Therefore, any drawing of K_6 minus one edge has at least two crossings, consequently, it cannot admit a $\mathcal{C}_0$-drawing (see Lemma 2.1), a contradiction.

Since the drawing D without $2P_1$ is crossing-free, every crossed edge in D has an endvertex in $2P_1$. Hence, D contains at most two crossings (since it is a $\mathcal{C}_0$-drawing). If we remove one crossed edge for each crossing in D, then we obtain a drawing without crossings. This implies $|E(G)| + 2|V(G)| = |E(G + 2P_1)| \leq 3|V(G + 2P_1)| - 6 + 2 = 3|V(G)| + 2$, which proves the claim.

To see that the bound is sharp it is sufficient to consider the graph G_k defined in the proof of Lemma 2.16. □

Lemma 2.21. *If $G + P_2$ is of class C_0, then $|E(G)| \leq |V(G)| + 1$. Moreover, this bound is tight.*

Proof. We can proceed similarly as in the proof of Lemma 2.20. □

Lemma 2.22. *If $G + P_1$ is of class C_0, then $|E(G)| \leq \frac{9}{4}|V(G)| - \frac{11}{4}$. Moreover, this bound is tight.*

Proof. From Theorem 2.18 we obtain $|E(G)| + |V(G)| = |E(G + P_1)| \leq \frac{13}{4}|V(G + P_1)| - 6 = \frac{13}{4}|V(G)| - \frac{11}{4}$ which proves the claim.

Now we prove that the bound is sharp. Put $n = 2k$ with $k \geq 2$ being even. Take two paths $a_1a_2 \dots a_{k-1}a_k$, $b_1b_2 \dots b_{k-1}$ and, for each $i \in \{1, \dots, k-1\}$, add new edges $a_ib_i, a_{i+1}b_i$ and the edge $a_{k-2}a_k$; in addition, for each even $j \in \{2, \dots, k-2\}$, add new edges b_ja_{j-1}. The resulting graph G_{n-1} has $n-1$ vertices and $\frac{9}{4}(n-1) - \frac{11}{4}$ edges and a 1-planar drawing in which the edges $a_ib_{i+1}, a_{i+1}b_i$ cross, for each odd $i \in \{1, \dots, k-3\}$ and the other edges are crossing-free (see Figure 5). If we put a new vertex v into the outer face of $G^{\times}_{n-1}$ and join it with all vertices of $G^{\times}_{n-1}$ such that the edge va_{k-1} cross the edge $b_{k-1}a_k$ and the other edges incident with v are crossing-free, then we obtain a C_0-drawing of $G_{n-1} + P_1$. □

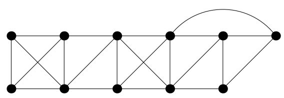


Figure 5: The graph G_{11}.

Lemma 2.23. *If $G + 2P_1$ is of class C_1, then $|E(G)| \leq \frac{8}{5}|V(G)|$.*

Proof. Since every 1-planar graph G of class C_1 has at most $\frac{18}{5}|V(G)| - \frac{36}{5}$ edges, we obtain $|E(G)| + 2|V(G)| = |E(G + 2P_1)| \leq \frac{18}{5}|V(G + 2P_1)| - \frac{36}{5} = \frac{18}{5}|V(G)|$ which proves the claim. □

Lemma 2.24. *There is a graph G with $|E(G)| = \frac{3}{2}|V(G)|$ such that $G + 2P_1$ is of class C_1.*

Proof. Let $C = v_1v_2 \dots v_{4\ell}v_1$ be a cycle on $4\ell \geq 8$ vertices. The plane drawing of this cycle divides the plane into two parts. Add the edges $v_{4k-2}v_{4k}$, $k = 1, \dots, \ell$, to the inner part and the edges $v_{4\ell}v_2, v_{4k}v_{4k+2}$, $k = 1, \dots, \ell - 1$, to the outer part. In such a way we obtain a graph G with 4ℓ vertices and 6ℓ edges. Moreover, $G + 2P_1$ has a C_1-drawing. □

Lemma 2.25. *If $G + P_1$ is of class C_1, then $|E(G)| \leq \frac{13}{5}|V(G)| - \frac{18}{5}$.*

Proof. Similarly as in the proof of Lemma 2.23 we obtain $|E(G)| + |V(G)| = |E(G + P_1)| \leq \frac{18}{5}|V(G + P_1)| - \frac{36}{5} = \frac{18}{5}|V(G)| - \frac{18}{5}$ which proves the claim. □

Lemma 2.26. *There is a graph G with $|E(G)| = \frac{12}{5}|V(G)| - \frac{19}{5}$ such that $G + P_1$ is of class C_1.*

Proof. Let G_1 be a graph depicted in Figure 6. Let G_k, $k \geq 2$, be a graph obtained from G_{k-1} and G_1 by identifying the edges v_1v_2 of G_{k-1} and u_1u_2 of G_1. The graph G_k, $k \geq 2$, has $3k + 1$ vertices of degree three, k vertices of degree six, $k - 1$ vertices of degree nine and 2 vertices of degree four. Therefore, it has $12k + 1$ edges. On the other hand, this graph has $5k + 2$ vertices. Consequently, $|E(G_k)| = \frac{12}{5}|V(G_k)| - \frac{19}{5}$.

The graph $G_k + P_1$ has a C_1-drawing, since G_k is of class C_1 and all true vertices of $G^{\times}_k$ are incident with the outer face. □

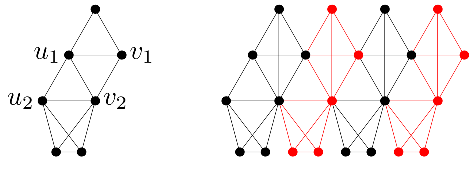


Figure 6: The graphs G_1 and G_4.

3. Conclusion

In this paper we showed that every 1-planar graph is of class C_i for some $i \in \{0, 1, 2\}$. After that we proved that the join $G + H$ is of class C_0 if and only if the pair $[G, H]$ is subgraph-majorized (that is, both G and H are subgraphs of graphs of the major pair) by one of pairs $[C_3, P_2 \cup P_1], [P_3, P_3]$ and is of class C_1 if and only if the pair $[G, H]$ is subgraph-majorized by one of pairs $[2P_2 \cup C_3], [P_4, P_3]$ in the case when both factors of the graph join have at least three vertices.

In [3] it was proved that the join $G + H$ is 1-planar if and only if the pair $[G, H]$ is subgraph-majorized by one of pairs $[C_3 \cup C_3, C_3]$, $[C_4, C_4]$, $[C_4, C_3]$, $[K_{2,1,1}, P_3]$ in the case when both factors of the graph join have at least three vertices. Therefore we have full characterization of 1-planar joins in the case when both factors have at least three vertices.

Finally, we proved several necessary conditions for the bigger factor in the case when the smaller one has at most two vertices; in addition, we improved two results of Zhang [8].

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