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Duality of Herz-Morrey Spaces of Variable Exponent

Yoshihiro Mizuta[a]

[a] *Department of Mechanical Systems Engineering, Hiroshima Institute of Technology,
2-1-1 Miyake,Saeki-ku,Hiroshima 731-5193, Japan*

Abstract. In this paper, we discuss the duality between the inner and outer Herz-Morrey spaces of variable exponent.

1. Introduction

Variable exponent function spaces are useful for discussing nonlinear partial differential equations with non-standard growth condition, in connection with the study of elasticity, fluid mechanics; see [20].

Let G be a bounded open set in $\mathbf{R}^n$, whose diameter is denoted by d_G. Let $\omega(\cdot,\cdot) : G \times (0,\infty) \to (0,\infty)$ be a uniformly almost monotone function on $G \times (0,\infty)$ satisfying the uniformly doubling condition. Following Samko [21], for $x_0 \in G$ and variable exponents $p(\cdot)$ and $q(\cdot)$, we consider the inner (small) Herz-Morrey space $\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)$ and the outer (complementary) Herz-Morrey space $\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)$ of variable exponent; see also [2], [4], [5], [6], [8], [13], [16], [20], etc. . Following Di Fratta-Fiorenza [10] and Gogatishvili-Mustafayev [11], [12], we study the associate spaces among those Herz-Morrey spaces, as extensions of [17], [18]. This also gives another characterizations of Morrey spaces by Adams-Xiao [1] (see also [12]).

2. Variable Exponent Lebesgue Spaces

Let μ be a nonnegative Borel measure on an open set $G \subset \mathbf{R}^n$. Consider a measurable function $p(\cdot)$ on G satisfying

(P0) $1 \le p(x) \le \infty$ for all $x \in G$;

$p(\cdot)$ is referred to as a variable exponent. The variable Lebesgue space $L^{p(\cdot)}(G,\mu)$ is the family of all measurable functions f such that

$$\|f\|_{L^{p(\cdot)}(G,\mu)} = \inf\left\{\lambda > 0 : \int_{G_{p<\infty}} \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} d\mu(y) \le 1\right\} + \|f\|_{L^\infty(G_{p=\infty},\mu)} < \infty,$$

where $G_{p<\infty} = \{x \in G : p(x) < \infty\}$ and $G_{p=\infty} = \{x \in G : p(x) = \infty\}$. If μ is the Lebesgue measure on G, then we write $L^{p(\cdot)}(G)$, simply. For fundamental facts of the variable Lebesgue spaces, see [7] and [9].

Note here the following result:

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Email address: y.mizuta.5x@it-hiroshima.ac.jp (Yoshihiro Mizuta)

LEMMA 2.1. *$L^{p(\cdot)}(G)$ is a Banach function space in the sense of Benett-Sharpley [3].*

3. Herz-Morrey Spaces

We consider the family $\Omega(G)$ of all positive functions $\omega(\cdot,\cdot) : G \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(ω0) $\omega(x,0) = \lim_{r\to+0} \omega(x,r) = 0$ for all $x \in G$ or $\omega(x,0) = \infty$ for all $x \in G$;

(ω1) $\omega(x,\cdot)$ is uniformly almost monotone on $(0,\infty)$, that is, there exists a constant $A_1 > 0$ such that $\omega(x,\cdot)$ is uniformly almost increasing on $(0,\infty)$, that is,

$$\omega(x,r) \le A_1\omega(x,s) \quad \text{for all } x \in G \text{ and } 0 < r < s$$

or $\omega(x,\cdot)$ is uniformly almost decreasing on $(0,\infty)$, that is,

$$\omega(x,s) \le A_1\omega(x,r) \quad \text{for all } x \in G \text{ and } 0 < r < s;$$

(ω2) $\omega(x,\cdot)$ is uniformly doubling on $(0,\infty)$, that is, there exists a constant $A_2 > 1$ such that

$$A_2^{-1}\omega(x,r) \le \omega(x,2r) \le A_2\omega(x,r) \quad \text{for all } x \in G \text{ and } r > 0;$$

and

(ω3) there exists a constant $A_3 > 1$ such that

$$A_3^{-1} \le \omega(x,1) \le A_3 \quad \text{for all } x \in G.$$

Following Samko [21], for $x_0 \in G$, variable exponents $p(\cdot), q(\cdot)$ and a weight $\omega \in \Omega(G)$, we consider the inner (small) Herz-Morrey space $\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)$ consisting of all measurable functions f on G satisfying

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)} = \|\omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\|_{L^{q(\cdot)}((0,d_G),dt/t)} < \infty;$$

more precisely,

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)} = \inf\left\{\lambda > 0 : \int_0^{d_G} \left(\omega(x_0,t)\|f/\lambda\|_{L^{p(\cdot)}(B(x_0,t))}\right)^{q(t)} \frac{dt}{t}\right\}.$$

Set

$$\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G) = \bigcap_{x_0\in G} \underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)$$

and

$$\underset{\sim}{\mathcal{H}}^{p(\cdot),q(\cdot),\eta}(G) = \sum_{x_0\in G} \underline{\mathcal{H}}^{p(\cdot),q(\cdot),\eta}_{\{x_0\}}(G),$$

whose quasi-norms are defined by

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G)} = \sup_{x_0\in G} \|f\|_{\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)}$$

and

$$\|f\|_{\underset{\sim}{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G)} = \inf_{|f|=\sum_j |f_j|, \{x_j\}\subset G} \sum_j \|f_j\|_{\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_j\}}(G)},$$

respectively.

We further consider the outer (complementary) Herz-Morrey space $\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)$ consisting of all measurable functions f on G satisfying

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)} = \|\omega(x_0,t)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,t))}\|_{L^{q(\cdot)}((0,d_G),dt/t)} < \infty.$$

Set

$$\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G) = \bigcap_{x_0\in G} \overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)$$

and

$$\widetilde{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G) = \sum_{x_0\in G} \overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G),$$

whose quasi-norms are defined by

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G)} = \sup_{x_0\in G} \|f\|_{\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)}$$

and

$$\|f\|_{\widetilde{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G)} = \inf_{|f|=\sum_j |f_j|,\{x_j\}\subset G} \sum_j \|f_j\|_{\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_j\}}(G)},$$

respectively.

Then note that

$$\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G) \supset L^{p(\cdot)}(G) \qquad \text{when } \|\omega(x_0,\cdot)\|_{L^{q(\cdot)}((0,d_G),dt/t)} < \infty$$

and

$$\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G) = L^p(G) \qquad \text{when } \|\omega(x,\cdot)\|_{L^{q(\cdot)}((0,d_G),dt/t)} < \infty \text{ for all } x \in G.$$

Further,

$$\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G) = \{0\} \qquad \text{when } \|\omega(x_0,\cdot)\|_{L^{q(\cdot)}((0,d_G),dt/t)} = \infty.$$

Is is worth to see the following facts.

Lemma 3.1. *Both* $\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)$ *and* $\overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)$ *are Banach function spaces in the sense of Benett-Sharpley [3].*

Lemma 3.2. *For* $x_0 \in G$,

$$\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G) \subset \underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G) \qquad \text{and} \qquad \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G) \subset \overline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G).$$

Throughout this note, let C denote various positive constants independent of the variables in question.

We now show that the Herz-Morrey spaces coincide with those with a constant exponent q if $q(\cdot)$ is Hölder continuous at the origin.

Lemma 3.3. *Suppose $q(\cdot)$ is log-Hölder continuous at 0, that is, there is a constant $c_q > 0$ such that*

(Q) $|q(t) - q(0)| \le \dfrac{c_q}{\log(1+t^{-1})}$ *for* $0 < t < d_G$.

Then

$$\underline{\mathcal{H}}_{\{x_0\}}^{p(\cdot),q(\cdot),\omega}(G) = \underline{\mathcal{H}}_{\{x_0\}}^{p(\cdot),q(0),\omega}(G) \qquad \text{for all } x_0 \in G$$

and

$$\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G) = \underline{\mathcal{H}}^{p(\cdot),q(0),\omega}(G).$$

The same is true for $\overline{\mathcal{H}}_{\{x_0\}}^{p(\cdot),q(\cdot),\omega}(G)$.

Proof. Let f be a nonnegative measurable function on G satisfying

$$\|f\|_{\underline{\mathcal{H}}_{\{x_0\}}^{p(\cdot),q(\cdot),\omega}(G)} \le 1.$$

Then

$$\int_0^{d_G} \left(\omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\right)^{q(t)} \, dt/t \le 1.$$

We have

$$\omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))} \le C$$

for $0 < t < d_G$. Take $a > 0$. Then condition (Q) gives

$$\left(\omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\right)^{q(0)} \le C\left(\omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\right)^{q(t)} + t^{aq(0)},$$

which gives

$$\int_0^{d_G} \left(\omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\right)^{q(0)} \, dt/t \le C.$$

Thus it follows that

$$\|f\|_{\underline{\mathcal{H}}_{\{x_0\}}^{p(\cdot),q(0),\omega}(G)} \le C,$$

which implies that

$$\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}(G) \subset \underline{\mathcal{H}}^{p(\cdot),q(0),\omega}(G).$$

The converse can be treated similarly. □

We consider the grand Lebesgue space $L^{p)}(G)$ consisting of measurable functions f on G such that

$$\sup_{0<\varepsilon<p-1} \varepsilon \int_G |f(y)|^{p-\varepsilon} dy < \infty.$$

Grand Lebesgue spaces were introduced in [14] for the study of Jacobian, which is useful for the theory of partial differential equations (see [15]).

Proposition 3.4. *Set* $\omega(x, r) = r^{\nu}$ *with* $\nu > 0$. *If* $q = np/(n + \nu) > 1$, *then*

$$\overline{\mathcal{H}}^{p,\infty,\omega}_{\{x_0\}}(G) \subset L^{q)}(G).$$

Proof. Let f be a nonnegative measurable function on G such that

$$t^{\nu}\|f\|_{L^p(G\setminus B(x_0,t))} \le 1 \qquad \text{for } 0 < t < d_G.$$

Then for $a > n(1 - (q - \varepsilon)/p)$ and $\varepsilon > 0$ we have

$$\begin{aligned}
\int_G f(y)^{q-\varepsilon}\,dy &= \int_G f(y)^{q-\varepsilon}\left(a|x_0 - y|^{-a}\int_0^{|x_0-y|} t^{a-1}\,dt\right) dy \\
&= a\int_0^{d_G} t^{a-1}\left(\int_{G\setminus B(x_0,t)} |x_0 - y|^{-a} f(y)^{q-\varepsilon}\,dy\right) dt \\
&\le a\int_0^{d_G} t^{a-1}\left(\int_{G\setminus B(x_0,t)} f(y)^p\,dy\right)^{(q-\varepsilon)/p} \\
&\quad \times\left(\int_{G\setminus B(x_0,t)} |x_0 - y|^{-a/(1-(q-\varepsilon)/p)}\,dy\right)^{1-(q-\varepsilon)/p} dt \\
&\le C\int_0^{d_G} t^{a-1}t^{-\nu(q-\varepsilon)}t^{-a+n(1-(q-\varepsilon)/p)}dt \\
&\le C\int_0^{d_G} t^{(\nu+n/p)\varepsilon-1}\,dt \\
&\le C\varepsilon^{-1},
\end{aligned}$$

so that

$$\varepsilon\int_G f(y)^{q-\varepsilon}\,dy \le C,$$

as required. □

Remark 3.5. The same conclusion as Proposition 3.4 is true for the variable Herz-Morrey space $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ if $p(\cdot)$ satisfies

(P2) $p(\cdot)$ is log-Hölder continuous in G; that is, there is $c_p > 0$ such that

$$|p(x) - p(y)| \le \frac{c_p}{\log(1 + |x - y|^{-1})} \qquad \text{for } x, y \in G.$$

This can also be extended to the logarithmic weight case as in [17, Theorem 10.1] and [18, Theorem 8.1].

4. Associate Spaces

Let X be a function space consisting of measurable functions on G, with norm $\|\cdot\|_X$. Then X' denotes the associate space of X consisting of all measurable functions f on G such that

$$\|f\|_{X'} \equiv \sup_{\{g:\|g\|_X\le 1\}} \int_G f(x)g(x)\,dx < \infty;$$

see Benett-Sharpley [3]; the usual dual space of X is denoted by X^*.

Note here that

$$\left(L^{p(\cdot)}(G)\right)' = L^{p'(\cdot)}(G).$$

and

$$\left(L^{p(\cdot)}(G)\right)^* = L^{p'(\cdot)}(G).$$

when $1 < p^- \le p^+ \le \infty$.

5. Duality of Herz-Morrey Spaces

Now we are ready to show the duality of Herz-Morrey spaces as extensions of [17] and [18].

Theorem 5.1. *Let $x_0 \in G$. Suppose there exist constants $a, b, Q > 0$ such that*

(ω4.1) *$t^a\omega(x_0, t)$ is quasi-increasing, that is,*

$$\sup_{0<s<t} s^a\omega(x_0, s) \le Qt^a\omega(x_0, t); \text{ and}$$

(ω4.2) *$t^b\omega(x_0, t)$ is quasi-decreasing, that is,*

$$\sup_{t<s<d_G} s^b\omega(x_0, s) \le Qt^b\omega(x_0, t)$$

for all $0 < t < d_G$. Set $\eta(x, r) = \omega(x, r)^{-1}$. Then for a constant exponent $1 \le q \le \infty$

$$\left(\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)' = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)$$

and

$$\left(\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G).$$

With the aid of Lemma 3.3, Theorem 5.1 gives the following result.

Corollary 5.2. *Let $x_0 \in G$ and ω, η be as in Theorem 5.1. If $1 < p^- \le p^+ < \infty$, $1 < q^- \le q^+ < \infty$ and $q(\cdot)$ is log-Hölder at 0, then*

$$\left(\underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G)\right)^* = \overline{\mathcal{H}}^{p'(\cdot),q'(\cdot),\eta}_{\{x_0\}}(G)$$

and

$$\left(\overline{\mathcal{H}}^{p'(\cdot),q'(\cdot),\eta}_{\{x_0\}}(G)\right)^* = \underline{\mathcal{H}}^{p(\cdot),q(\cdot),\omega}_{\{x_0\}}(G).$$

Example 5.3. *The typical examples of ω and η are*

$$\omega(x_0, t) = \eta(x_0, t)^{-1} = t^{-\varepsilon}$$

for $\varepsilon > 0$.

In necessary modifications of the proof of Theorem 5.1, we can treat logarithmic weights in the following manner.

Theorem 5.4. *Let* $x_0 \in G$. *Set* $\omega(x_0,t) = \left(\log \frac{2d_G}{t}\right)^{\varepsilon}$ *and* $\eta(x_0,t) = \left(\log \frac{2d_G}{t}\right)^{-1-\varepsilon}$ *for* $\varepsilon > -1/q$. *Then for a constant exponent* $1 \le q \le \infty$

$$\left(\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)' = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)$$

and

$$\left(\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G).$$

Corollary 5.5. *Let* ω *and* η *be as in Theorem 5.1. Then*

$$\left(\widetilde{\mathcal{H}}^{p'(\cdot),q',\eta}(G)\right)' = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}(G).$$

Remark 5.6. Let ω and η be as in Theorem 5.1. Then

$$\left(\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)\right)' = \left(L^{p(\cdot)}(G)\right)' = L^{p'(\cdot)}(G) = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}(G).$$

For a proof of Theorem 5.1, we have to prepare the following lemmas; see more precisely the proofs in [17] and [18].

Lemma 5.7. *Let* $1 < q \le \infty$ *and* $x_0 \in G$. *Suppose there exist constants* $b > 0, Q > 1$ *such that*

(ω4.1) $\displaystyle\int_0^{d_G} \eta(x_0,t)^{q'} \frac{dt}{t} < \infty$; *and*

(ω4.2) $\displaystyle\int_t^{2d_G} s^{-b}\omega(x_0,s)^{-q'} \frac{ds}{s} \le Qt^{-b}\eta(x_0,t)^{q'}$ *for all* $0 < t < d_G$.

Then there exists a constant $C > 0$ *such that*

$$\int_G |f(x)g(x)|dx \le C\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}\|g\|_{\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)}$$

for all measurable functions f *and* g *on* G.

Lemma 5.8. *Let* $x_0 \in G$ *and* $1 < q \le \infty$. *Suppose there exist* $a > 0$ *and* $Q > 0$ *such that*

(ω4.3) $\displaystyle\int_0^{t} s^{-a}\eta(x_0,s)^{q'} \frac{ds}{s} \le Qt^{-a}\omega(x_0,t)^{-q'}$ *for all* $0 < t < d_G$.

Set $X = \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$. *Then there exists a constant* $C > 0$ *such that*

$$\|g\|_{\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)} \le C \sup_f \int_G |f(x)g(x)|dx = C\|g\|_{X'}$$

for all measurable functions g *on* G, *where the supremum is taken over all measurable functions* f *on* G *such that* $\|f\|_X \le 1$.

It is worth to note that conditions (ω4.1) – (ω4.3) holds if and only if ω satisfies all the conditions in Theorem 5.1.

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