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A Fixed Point Approach to the Fuzzy Stability of an AQCQ-Functional Equation

Choonkil Park[a], Dong Yun Shin[b], Reza Saadati[c], Jung Rye Lee[d]

[a] *Research Institute of Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea*
[b] *Department of Mathematics, University of Seoul, Seoul 130-743, Republic of Korea*
[c] *Department of Mathematics, Iran University of Science and Technology, Tehran, Iran*
[d] *Department of Mathematics, Daejin University, Kyeonggi 487-711, Republic of Korea*

Abstract. In [32, 33], the fuzzy stability problems for the Cauchy additive functional equation and the Jensen additive functional equation in fuzzy Banach spaces have been investigated.

Using the fixed point method, we prove the Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \tag{1}$$

in fuzzy Banach spaces.

1. Introduction and Preliminaries

Katsaras [27] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [19, 29, 45]. In particular, Bag and Samanta [3], following Cheng and Mordeson [10], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [28]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 32, 33] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the functional equation (0.1) in the fuzzy normed vector space setting.

Definition 1.1. *[3, 32–34] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(N_1) $N(x,t) = 0$ *for* $t \le 0$;

(N_2) $x = 0$ *if and only if* $N(x,t) = 1$ *for all* $t > 0$;

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Email addresses: baak@hanyang.ac.kr (Choonkil Park), dyshin@uos.ac.kr (Dong Yun Shin), rsaadati@iust.ac.ir (Reza Saadati), jrlee@daejin.ac.kr (Jung Rye Lee)

(N_3) *$N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;*
(N_4) *$N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;*
(N_5) *$N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;*
(N_6) *for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.*

The pair (X,N) is called a *fuzzy normed vector space*.
The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [32, 35].

Definition 1.2. *[3, 32–34] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.*

Definition 1.3. *[3, 32, 33] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4]).

The stability problem of functional equations originated from a question of Ulam [42] concerning the stability of group homomorphisms. Hyers [22] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [39] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [21] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [41] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [12] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 5, 6, 18, 23, 26, 40, 43, 44]).

In [25], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{2}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (2), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [30], Lee et al. considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{3}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (3), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.
We recall a fundamental result in fixed point theory.

Theorem 1.4. *[7, 14] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [24] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [8, 9, 35–38]).

The theory of probabilistic normed spaces(briefly, PN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The PN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics particularly in string theory and ε^∞ where studied by El. Naschie [15]. The Hyers-Ulam stability of different functional equations in random normed spaces, PN-spaces and fuzzy normed spaces has been recently studied in [31]-[34].

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1) in fuzzy Banach spaces for an odd case. In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1) in fuzzy Banach spaces for an even case.

Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

2. Hyers-Ulam Stability of the Functional Equation (1): an Odd Case

One can easily show that an odd mapping $f : X \to Y$ satisfies (1) if and only if the odd mapping mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x).$$

It was shown in [17, Lemma 2.2] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + 2f(2y) - 8f(y).$$

It was shown in [16, Lemma 2.1] that $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and that $f(x) = \frac{1}{12}g(x) - \frac{1}{12}h(x)$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x, y) : &= f(x + 2y) + f(x - 2y) - 4f(x + y) - 4f(x - y) + 6f(x) \\ &- f(2y) - f(-2y) + 4f(y) + 4f(-y) \end{aligned}$$

for all $x, y \in X$.

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in fuzzy Banach spaces: an odd case.

Theorem 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{8}\varphi(2x,2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$N(Df(x,y),t) \ge \frac{t}{t+\varphi(x,y)} \tag{4}$$

for all $x, y \in X$ and all $t > 0$. Then

$$C(x) := N\text{-}\lim_{n\to\infty} 8^n\left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(2x)-2f(x)-C(x),t) \ge \frac{(8-8L)t}{(8-8L)t+5L(\varphi(x,x)+\varphi(2x,x))} \tag{5}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y$ in (4), we get

$$N(f(3y)-4f(2y)+5f(y),t) \ge \frac{t}{t+\varphi(y,y)} \tag{6}$$

for all $y \in X$ and all $t > 0$.

Replacing x by $2y$ in (4), we get

$$N(f(4y)-4f(3y)+6f(2y)-4f(y),t) \ge \frac{t}{t+\varphi(2y,y)} \tag{7}$$

for all $y \in X$ and all $t > 0$.

By (6) and (7),

$$\begin{aligned} &N(f(4y)-10f(2y)+16f(y),4t+t)\\ &\quad\ge \min\{N(4(f(3y)-4f(2y)+5f(y)),4t),\\ &\qquad N(f(4y)-4f(3y)+6f(2y)-4f(y),t)\}\\ &\quad\ge \frac{t}{t+\varphi(y,y)+\varphi(2y,y)} \end{aligned} \tag{8}$$

for all $y \in X$ and all $t > 0$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$N\left(g(x)-8g\left(\frac{x}{2}\right),5t\right) \ge \frac{t}{t+\varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2}\right)} \tag{9}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \ge \frac{t}{t+\varphi(x,x)+\varphi(2x,x)},\ \ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete. (See the proof of [31, Lemma 2.1].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{8}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \frac{L}{8}(\varphi(x, x) + \varphi(2x, x))} \\ &= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (9) that

$$N\left(g(x) - 8g\left(\frac{x}{2}\right), \frac{5L}{8}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{5L}{8}$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{10}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(g(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(g, C) \leq \frac{1}{1-L}d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{5L}{8 - 8L}.$$

This implies that the inequality (5) holds.

By (4),

$$N\left(8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 8^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right) \geq \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(DC(x, y), t) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $C : X \to Y$ is cubic, as desired. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$N(Df(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{11}$$

for all $x, y \in X$ and all $t > 0$. Then

$$C(x) := N\text{-}\lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(2x) - 2f(x) - C(x), t) \geq \frac{(2^p - 8)t}{(2^p - 8)t + 5(3 + 2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (4)*. Then*

$$C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1}x\right) - 2f(2^n x)\right)$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(2x) - 2f(x) - C(x), t) \geq \frac{(8 - 8L)t}{(8 - 8L)t + 5\varphi(x, x) + 5\varphi(2x, x)} \tag{12}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8} g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{1}{8} g(2x) - \frac{1}{8} h(2x), L\varepsilon t\right) \\ &= N(g(2x) - h(2x), 8L\varepsilon t) \\ &\geq \frac{8Lt}{8Lt + \varphi(2x, 2x) + \varphi(4x, 2x)} \geq \frac{8Lt}{8Lt + 8L(\varphi(x, x) + \varphi(2x, x))} \\ &= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq L d(g, h)$$

for all $g, h \in S$.

It follows from (9) that

$$N\left(g(x) - \frac{1}{8} g(2x), \frac{5}{8} t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{5}{8}$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C(2x) = 8C(x) \tag{13}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (13) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(g(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{1}{8^n} g(2^n x) = C(x)$$

for all $x \in X$;

(3) $d(g, C) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{5}{8 - 8L}.$$

This implies that the inequality (12) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (11)*. Then*

$$C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1}x\right) - 2f(2^n x)\right)$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N\left(f(2x) - 2f(x) - C(x), t\right) \geq \frac{(8-2^p)t}{(8-2^p)t + 5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (4)*. Then*

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n\left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(2x) - 8f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + 5L(\varphi(x,x) + \varphi(2x,x))} \tag{14}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Letting $y := \frac{x}{2}$ and $h(x) := f(2x) - 8f(x)$ for all $x \in X$ in (8), we get

$$N\left(h(x) - 2h\left(\frac{x}{2}\right), 5t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \tag{15}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}(\varphi(x,x) + \varphi(2x,x))} \\ &= \frac{t}{t + \varphi(x,x) + \varphi(2x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (15) that

$$N\left(h(x) - 2h\left(\frac{x}{2}\right), \frac{5L}{2}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq \frac{5L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{16}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (16) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(h(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(h, A) \leq \frac{1}{1-L} d(h, Jh)$, which implies the inequality

$$d(h, A) \leq \frac{5L}{2 - 2L}.$$

This implies that the inequality (14) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (11). *Then*

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(2x) - 8f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 5(3 + 2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Theorem 2.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (4). *Then*

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right)$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(2x) - 8f(x) - A(x), t\right) \ge \frac{(2-2L)t}{(2-2L)t + 5\varphi(x, x) + 5\varphi(2x, x)} \tag{17}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{1}{2}g(2x) - \frac{1}{2}h(2x), L\varepsilon t\right) \\ &= N\left(g(2x) - h(2x), 2L\varepsilon t\right) \\ &\ge \frac{2Lt}{2Lt + \varphi(2x, 2x) + \varphi(4x, 2x)} \ge \frac{2Lt}{2Lt + 2L(\varphi(x, x) + \varphi(2x, x))} \\ &= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (15) that

$$N\left(h(x) - \frac{1}{2}h(2x), \frac{5}{2}t\right) \ge \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \le \frac{5}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(2x) = 2A(x) \tag{18}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (18) such that there exists a $\mu \in (0,\infty)$ satisfying

$$N(h(x)-A(x),\mu t) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{1}{2^n} h(2^n x) = A(x)$$

for all $x \in X$;

(3) $d(h,A) \leq \frac{1}{1-L} d(h,Jh)$, which implies the inequality

$$d(h,A) \leq \frac{5}{2-2L}.$$

This implies that the inequality (17) holds.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Corollary 2.8. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (11). Then*

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right)$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(2x)-8f(x)-A(x),t\right) \geq \frac{(2-2^p)t}{(2-2^p)t+5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.7 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. $\square$

3. Hyers-Ulam Stability of the Functional Equation (1): an Even Case

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in fuzzy Banach spaces: an even case.

Theorem 3.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{16}\varphi\left(2x,2y\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (4). Then

$$Q(x) := N\text{-}\lim_{n\to\infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(2x)-4f(x)-Q(x),t\right) \geq \frac{(16-16L)t}{(16-16L)t+5L(\varphi(x,x)+\varphi(2x,x))} \tag{19}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y$ in (4), we get

$$N\left(f(3y) - 6f(2y) + 15f(y), t\right) \geq \frac{t}{t + \varphi(y, y)} \tag{20}$$

for all $y \in X$ and all $t > 0$.

Replacing x by $2y$ in (4), we get

$$N\left(f(4y) - 4f(3y) + 4f(2y) + 4f(y), t\right) \geq \frac{t}{t + \varphi(2y, y)} \tag{21}$$

for all $y \in X$ and all $t > 0$.

By (20) and (21),

$$\begin{aligned} N&\left(f(4x) - 20f(2x) + 64f(x), 4t + t\right) \\ &\geq \min\left\{N\left(4(f(3x) - 6f(2x) + 15f(x)), 4t\right),\right. \\ &\qquad \left.N\left(f(4x) - 4f(3x) + 4f(2x) + 4f(x)), t\right)\right\} \\ &\geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned} \tag{22}$$

for all $x \in X$ and all $t > 0$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$N\left(g(x) - 16g\left(\frac{x}{2}\right), 5t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \tag{23}$$

for all $x \in X$ and all $t > 0$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(16g\left(\frac{x}{2}\right) - 16h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{16}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \frac{L}{16}(\varphi(x, x) + \varphi(2x, x))} \\ &= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (23) that

$$N\left(g(x) - 16g\left(\frac{x}{2}\right), \frac{5L}{16}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{5L}{16}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{24}$$

for all $x \in X$. Since $g : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (24) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(g(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 16^n g\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(g, Q) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, Q) \leq \frac{5L}{16 - 16L}.$$

This implies that the inequality (19) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let $\theta \geq 0$ and let p be a real number with $p > 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (11). Then*

$$Q(x) := N\text{-}\lim_{n\to\infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(2x) - 4f(x) - Q(x), t\right) \geq \frac{(2^p - 16)t}{(2^p - 16)t + 5(3 + 2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{4-p}$ and we get the desired result. □

Theorem 3.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 16L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (4). Then

$$Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{16^n}\left(f\left(2^{n+1}x\right) - 4f(2^n x)\right)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(2x) - 4f(x) - Q(x), t\right) \geq \frac{(16-16L)t}{(16-16L)t + 5\varphi(x,x) + 5\varphi(2x,x)} \tag{25}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16} g\left(2x\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{1}{16} g\left(2x\right) - \frac{1}{16} h\left(2x\right), L\varepsilon t\right) \\ &= N\left(g\left(2x\right) - h\left(2x\right), 16L\varepsilon t\right) \\ &\geq \frac{16Lt}{16Lt + \varphi\left(2x, 2x\right) + \varphi(4x, 2x)} \geq \frac{16Lt}{16Lt + 16L(\varphi(x,x) + \varphi(2x,x))} \\ &= \frac{t}{t + \varphi(x,x) + \varphi(2x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (23) that

$$N\left(g(x) - \frac{1}{16} g(2x), \frac{5}{16} t\right) \geq \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{5}{16}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(2x\right) = 16Q(x) \tag{26}$$

for all $x \in X$. Since $g : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (26) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(g(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty}\frac{1}{16^n}g(2^n x) = Q(x)$$

for all $x \in X$;

(3) $d(g, Q) \leq \frac{1}{1-L}d(g, Jg)$, which implies the inequality

$$d(g, Q) \leq \frac{5}{16-16L}.$$

This implies that the inequality (25) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (11). Then*

$$Q(x) := N\text{-}\lim_{n\to\infty}\frac{1}{16^n}\left(f\left(2^{n+1}x\right) - 4f(2^n x)\right)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(2x) - 4f(x) - Q(x), t\right) \geq \frac{(16-2^p)t}{(16-2^p)t + 5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-4}$ and we get the desired result. □

Theorem 3.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (4). Then

$$T(x) := N\text{-}\lim_{n\to\infty} 4^n\left(f\left(\frac{x}{2^{n-1}}\right) - 16f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$N\left(f(2x) - 16f(x) - T(x), t\right) \geq \frac{(4-4L)t}{(4-4L)t + 5L(\varphi(x,x) + \varphi(2x,x))} \tag{27}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Letting $h(x) := f(2x) - 16f(x)$ for all $x \in X$ in (22), we get

$$N\left(h(x) - 4h\left(\frac{x}{2}\right), 5t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \tag{28}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 4h\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, x\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}(\varphi(x, x) + \varphi(2x, x))} \\ &= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (28) that

$$N\left(h(x) - 4h\left(\frac{x}{2}\right), \frac{5L}{4}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq \frac{5L}{4}$.

By Theorem 1.4, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, i.e.,

$$T\left(\frac{x}{2}\right) = \frac{1}{4}T(x) \tag{29}$$

for all $x \in X$. Since $h : X \to Y$ is even, $T : X \to Y$ is an even mapping. The mapping T is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that T is a unique mapping satisfying (29) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(h(x) - T(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, T) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n h\left(\frac{x}{2^n}\right) = T(x)$$

for all $x \in X$;

(3) $d(h, T) \leq \frac{1}{1-L}d(h, Jh)$, which implies the inequality

$$d(h, T) \leq \frac{5L}{4 - 4L}.$$

This implies that the inequality (27) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.6. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (11). Then*

$$T(x) := N\text{-}\lim_{n\to\infty} 4^n \left(f\left(\frac{x}{2^{n-1}}\right) - 16f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$N(f(2x) - 16f(x) - T(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 5(3 + 2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.5 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Theorem 3.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (4). Then

$$T(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} \left(f\left(2^{n+1}x\right) - 16f(2^n x)\right)$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$N(f(2x) - 16f(x) - T(x), t) \geq \frac{(4 - 4L)t}{(4 - 4L)t + 5\varphi(x, x) + 5\varphi(2x, x)} \tag{30}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.
Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{4}h(2x)$$

for all $x \in X$.
Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{1}{4}g(2x) - \frac{1}{4}h(2x), L\varepsilon t\right) \\ &= N(g(2x) - h(2x), 4L\varepsilon t) \\ &\geq \frac{4Lt}{4Lt + \varphi(2x, 2x) + \varphi(4x, 2x)} \geq \frac{4Lt}{4Lt + 4L(\varphi(x, x) + \varphi(2x, x))} \\ &= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (28) that

$$N\left(h(x) - \frac{1}{4}h(2x), \frac{5}{4}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq \frac{5}{4}$.

By Theorem 1.4, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, i.e.,

$$T(2x) = 4T(x) \tag{31}$$

for all $x \in X$. Since $h : X \to Y$ is even, $T : X \to Y$ is an even mapping. The mapping T is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that T is a unique mapping satisfying (31) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(h(x) - T(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, T) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{1}{4^n} h(2^n x) = T(x)$$

for all $x \in X$;

(3) $d(h, T) \leq \frac{1}{1-L} d(h, Jh)$, which implies the inequality

$$d(h, T) \leq \frac{5}{4 - 4L}.$$

This implies that the inequality (30) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.8. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (11). Then $T(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n}\left(f\left(2^{n+1}x\right) - 16f(2^n x)\right)$ exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that*

$$N(f(2x) - 16f(x) - T(x), t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 5(3 + 2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.7 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

References

[1] M. Akkouchi, *Stability of certain functional equations via a fixed point of Ćirić*, Filomat **25** (2011), no. 2, 121–127.
[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[3] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.
[4] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems **151** (2005), 513–547.
[5] V. Berinde, *On the solution of Steinhaus functional equation using weakly Picard operators*, Filomat **25** (2011), no. 1, 69–79.
[6] A. Bodaghi, *Stability of a mixed type additiveand quartic functional equation*, Filomat **28** (2014), 1629–1640.
[7] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[8] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[9] L. Cădariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory and Applications **2008**, Art. ID 749392 (2008).
[10] S.C. Cheng and J.M. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.
[11] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
[12] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.
[13] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.
[14] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[15] M.S. El Nasche, *On the unification heterotic string theory and ε^{∞} theory*, Chaos, Solitons and Fractals **11** (2000), 237–252.
[16] M. Eshaghi Gordji, S. Abbaszadeh and C. Park, *On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces*, J. Inequal. Appl. **2009**, Article ID 153084 (2009).
[17] M. Eshaghi Gordji, S. Kaboli-Gharetapeh, C. Park and S. Zolfaghri, *Stability of an additive-cubic-quartic functional equation*, Adv. Difference Equ. **2009**, Article ID 395693 (2009).
[18] M. Eshaghi Gordji and M. B. Savadkouhi, *Stability of a mixed type additive, quadratic, cubic functional equation in random normed spaces*, Filomat **25** (2011), no. 3, 43–54.
[19] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Systems **48** (1992), 239–248.
[20] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.
[21] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[22] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
[23] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkh*ä*user, Basel, 1998.
[24] G. Isac and Th.M. Rassias, *Stability of ψ-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.
[25] K. Jun and H. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274** (2002), 867–878.
[26] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.
[27] A.K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems **12** (1984), 143–154.
[28] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.
[29] S.V. Krishna and K.K.M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), 207–217.
[30] S. Lee, S. Im and I. Hwang, *Quartic functional equations*, J. Math. Anal. Appl. **307** (2005), 387–394.
[31] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[32] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems **159** (2008), 730–738.
[33] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems **159** (2008), 720–729.
[34] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy approximately cubic mappings*, Inform. Sci. **178** (2008), 3791–3798.
[35] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
[36] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory and Applications **2007**, Art. ID 50175 (2007).
[37] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory and Applications **2008**, Art. ID 493751 (2008).
[38] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[39] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[40] R. Saadati, Gh. Sadeghi and Th.M. Rassias, *Approximate generalized additive mappings in proper multi-CQ*-algebras*, Filomat **28** (2014), 677–694.
[41] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[42] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[43] J. Wang and Z. Lin, *Ulam's type stability of Hadamard type fractional integral equations*, Filomat **28** (2014), 1323–1331.
[44] Z. Wang, Th.M. Rassias and R. Saadati, *Intuitionistic fuzzy stability of Jensen-type quadratic functional equations*, Filomat **28** (2014), 663–676.
[45] J.Z. Xiao and X.H. Zhu, *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets and Systems **133** (2003), 389–399.