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Global Solution to the Incompressible Oldroyd-B Model in Hybrid Besov Spaces

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Abstract. This paper is dedicated to the Cauchy problem of the incompressible Oldroyd-B model with general coupling constant $\omega \in (0,1)$. It is shown that this set of equations admits a unique global solution in a certain hybrid Besov spaces for small initial data in $\dot{H}^s \cap \dot{B}^{\frac{d}{2}}_{2,1}$ with $-\frac{d}{2} < s < \frac{d}{2} - 1$. In particular, if $d \geq 3$, and taking $s = 0$, then $\dot{H}^0 \cap \dot{B}^{\frac{d}{2}}_{2,1} = B^{\frac{d}{2}}_{2,1}$. Since $B^t_{2,\infty} \hookrightarrow B^{\frac{d}{2}}_{2,1}$ if $t > \frac{d}{2}$, this result extends the work by Chen and Miao [Nonlinear Anal.,68(2008), 1928–1939].

1. Introduction

We consider a typical model for viscoelastic fluids, the so called Oldroyd-B model [26] in this paper. This type of fluids is described by the following set of equations

$$\begin{cases} u_t + (u\cdot\nabla)u - \eta_s\Delta u + \nabla\Pi &= \mathrm{div}\tau, \\ \mathrm{div}u &= 0, \\ \lambda(\tau_t + (u\cdot\nabla)\tau + g_\alpha(\tau,\nabla u)) + \tau &= 2\eta_e D(u), \end{cases} \tag{1}$$

where u and τ are the velocity and symmetric tensor of constrains of the fluids, respectively. Π is the pressure which is the Lagrange multiplier for the divergence free condition. The quadratic form in $(\tau, \nabla u)$ is given by $g_\alpha(\tau,\nabla u) := \tau W(u) - W(u)\tau - \alpha\,(D(u)\tau + \tau D(u))$ for some $\alpha \in [-1,1]$, and $D(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top)$, $W(u) := \frac{1}{2}(\nabla u - (\nabla u)^\top)$ are the deformation tensor and the vorticity tensor, respectively. Moreover, the parameter $\eta_s := \eta\mu/\lambda$ denotes the solvent viscosity, and $\eta_e := \eta - \eta_s$ denotes the polymer viscosity, where η is the total viscosity of the fluid, $\lambda > 0$ is the relaxation time, and μ is the retardation time with $0 < \mu < \lambda$.

In the following, we would like to study system (1) in dimensionless variables, which takes the form

$$\begin{cases} \mathrm{Re}\,(u_t + (u\cdot\nabla)u) - (1-\omega)\Delta u + \nabla\Pi = \mathrm{div}\tau, \\ \mathrm{We}(\tau_t + (u\cdot\nabla)\tau + g_\alpha(\tau,\nabla u)) + \tau = 2\omega D(u), \\ \mathrm{div}u = 0, \end{cases} \tag{2}$$

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with parameters Reynolds number Re , Weissenberg number We and coupling constant $\omega := 1 - \frac{\mu}{\lambda} \in (0,1)$ of the fluid . For more details of the modeling, we refer to [9, 12, 28] and references therein.

Some of the previous works in this direction can be summarized as follows. To the best of our knowledge, the incompressible Oldroyd-B model was firstly studied by Guillopé and Saut [11], where they obtained a unique local strong solution to system (2) in suitable Sobolev spaces $H^s(\Omega)$ for the situation of a *bounded domain* $\Omega \subset \mathbb{R}^3$. Moreover, this solution is global provided both the data and the coupling constant ω between the two equations are sufficiently small. The extensions to these results to the L^p-setting can be found in [10]. Similar results on *exterior domains* were established by Hieber, Naito and Shibata [13]. The well-posedness results in *scaling invariant* Besov spaces on $\mathbb{R}^d, d \geq 2$ were first given by Chemin and Masmoudi [5].

All the results mentioned above were constructed under the assumption that the coupling constant ω is small enough. This means that the coupling effect between the velocity u and the symmetric tensor of constrains τ is weak and hence system (2) corresponds closely to the classical incompressible Navier-Stokes equations. From both the physical and mathematical point of view, it is more interesting to consider the strong coupling case, for which the coupling constant ω is not small. As a matter of fact, the studies in this direction have thrown up some interesting results. For the situation of *bounded domains*, the smallness restriction on the coupling constant ω in [11] was removed by Molinet and Talhouk [25]. As for the *exterior domains*, Fang, Hieber and the author [8] improved the main result in [13] to the situation of *non-small* coupling constant. In the *whole space* $\mathbb{R}^d$ case, Chen and Miao [6] obtained global solutions to system (1) with small initial data in $B^s_{2,\infty}, s > \frac{d}{2}$. For the critical L^p framework, the smallness restriction on ω in [5] was removed by Fang, Zhang and the author [30] very recently. Existence of global *weak solutions* for large data and strong coupling was proved by Lions and Masmoudi in [24] for the case $\alpha = 0$. The general case $\alpha \neq 0$ is still open up to now. For the Oldroyd-B fluids with *diffusive stress*, Constantin and Kliegl [7] proved the global regularity of solutions in two dimensional case.

Besides, we would like to point out that there are some other results on Oldroyd-B fluids in the literature. Indeed, Chemin and Masmoudi [5] gave some *blow-up criterions* both for 2D and 3D cases. Later on, the 2D case was improved by Lei, Masmoudi and Zhou in [20]. As for the 3D case, Kupferman, Mangoubi and Titi [15] established a Beale-Kato-Majda type blow-up criterion in terms of the $L^1_t(L^\infty_x)$ norm of τ in the *zero Reynolds number* regime. Further results, describing the *incompressible limit problems* for Oldroyd-B fluids, can be found in [12, 16] for well-prepared initial data, and in [9] for ill-prepared initial data. An approach based on *deformation tentor* was developed in [14, 17–19, 21–23, 27, 29].

The aim of this paper is to study the incompressible Oldroyd-B model (2) with *non-small* coupling constant ω. We establish the global solutions to system (2) with small data u_0 and τ_0 lying in $\dot{\mathcal{B}}^s = \dot{H}^s \cap \dot{B}^{\frac{d}{2}}_{2,1}, -\frac{d}{2} < s < \frac{d}{2} - 1$. Like all the previous results [6, 8, 24, 25] in L^2 framework with *non-small* coupling constant ω, the key point of the proof is to use the cancelation relation

$$(\mathrm{div}\tau|u) + (D(u)|\tau) = 0. \tag{3}$$

The global estimates can be divided into two parts. For the initial data in $\dot{B}^{\frac{d}{2}}_{2,1}$, owing to the Bernstein's inequality, we can obtain both the smoothing effect of the velocity u and the damping effect of the symmetric tensor of constrains τ in the high frequency case. While in the low frequency case, the estimate fails to be true since u and τ are treated as a whole, and $\|\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q \tau\|_{L^2}$ can not be dominated by $\|\nabla\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q \tau\|_{L^2}$ any more (see (26) for details). In order to deal with the low frequency part, we impose an extra condition on the initial dada. This leads to the estimates for initial data in $\dot{H}^s$. It is worth noting that the estimates of nonlinear terms necessitate bounding the term $\|u\|_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}$. To do so, we decompose $\|u\|_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}$ into $\|u\|^l_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}$ and $\|u\|^h_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}$. In particular, the low frequency part is bounded by $\|u\|^l_{L^2_t(\dot{H}^{s+1})}$; that is why we need $s < \frac{d}{2} - 1$. Combing the two parts estimates with initial data in $\dot{B}^{\frac{d}{2}}_{2,1}$ and $\dot{H}^s$, we obtain the global estimates for (u, τ). Finally, we would like to remark that although the results in [30] allow more general initial data (L^p type Besov spaces), we give a much easier proof without resorting to the Green matrix of the linearized system of (2) for the data lie in the L^2 type Besov spaces in this paper.

Notations. For $s \in \mathbb{R}$, set

$$\|u\|^{l}_{\dot{B}^{s}_{2,1}} := \sum_{q<0} 2^{qs}\|\dot{\Delta}_q u\|_{L^2}, \quad \text{and} \quad \|u\|^{h}_{\dot{B}^{s}_{2,1}} := \sum_{q\geq 0} 2^{qs}\|\dot{\Delta}_q u\|_{L^2}.$$

Further more, let us denote by $\dot{B}^s_{\mathrm{h}}$ the space which consists of distributions $u \in \mathcal{S}'$, such that $\|u\|^{h}_{\dot{B}^{s}_{2,1}} < \infty$. Throughout the paper, C denotes various "harmless" positive constants, and we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

We shall obtain the existence and uniqueness of a solution (u, τ) to (2) in the following space.

Definition 1.1. *For $T > 0$, and $s \in \mathbb{R}$, let us denote*

$$\mathcal{E}^s_T := \left(\tilde{C}_T(\dot{\mathcal{B}}^s) \cap L^2_T(\dot{H}^{s+1}) \cap L^1_T(\dot{B}^{\frac{d}{2}+1}_{\mathrm{h}})\right)^d \times \left(\tilde{C}_T(\dot{\mathcal{B}}^s) \cap L^2_T(\dot{H}^s) \cap L^1_T(\dot{B}^{\frac{d}{2}}_{\mathrm{h}})\right)^{d\times d}.$$

We use the notation $\mathcal{E}^s$ if $T = \infty$, changing $[0, T]$ into $[0, \infty)$ in the definition above. The definition of space $\dot{\mathcal{B}}^s$ can be found in Section 2.

Our main result reads as follows:

Theorem 1.1. *Let $d \geq 2$, $-\frac{d}{2} < s < \frac{d}{2} - 1$. Assume that $(u_0, \tau_0) \in \left(\dot{\mathcal{B}}^s\right)^d \times \left(\dot{\mathcal{B}}^s\right)^{d\times d}$ with $\mathrm{div} u_0 = 0$. There exist two positive constants c and M, depending on s, d, ω, Re and We, such that if*

$$\|u_0\|_{\dot{\mathcal{B}}^s} + \|\tau_0\|_{\dot{\mathcal{B}}^s} \leq c,$$

system (2) admits a unique global solution (u, τ) in $\mathcal{E}^s$ with

$$\|(u, \tau)\|_{\mathcal{E}^s} \leq M\left(\|u_0\|_{\dot{\mathcal{B}}^s} + \|\tau_0\|_{\dot{\mathcal{B}}^s}\right).$$

2. The Functional Tool Box

The results of the present paper rely on the use of a dyadic partition of unity with respect to the Fourier variables, the so-called the *Littlewood-Paley decomposition*. Let us briefly explain how it may be built in the case $x \in \mathbb{R}^d$ and the reader may see more details in [1, 3]. Let (χ, φ) be a couple of C^∞ functions satisfying

$$\mathrm{Supp}\chi \subset \{|\xi| \leq \frac{4}{3}\}, \quad \mathrm{Supp}\varphi \subset \{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\},$$

and

$$\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1,$$

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \text{for} \quad \xi \neq 0.$$

Set $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}(\varphi_q)$, and $\tilde{h} = \mathcal{F}^{-1}(\chi)$. The dyadic blocks and the low-frequency cutoff operators are defined for all $q \in \mathbb{Z}$ by

$$\dot{\Delta}_q u = \varphi(2^{-q}\mathrm{D})u = \int_{\mathbb{R}^d} h_q(y)u(x-y)dy,$$

$$\dot{S}_q u = \chi(2^{-q}\mathrm{D})u = \int_{\mathbb{R}^d} \tilde{h}_q(y)u(x-y)dy.$$

Then

$$u = \sum_{q\in\mathbb{Z}} \dot{\Delta}_q u, \tag{4}$$

holds for tempered distributions *modulo polynomials*. As working modulo polynomials is not appropriate for nonlinear problems, following Chapter 2 of [1], we shall restrict our attention to the set $\mathcal{S}'_h$ (see Definition 1.26 of [1]) of tempered distributions u such that

$$\lim_{q\to-\infty}\|\dot{S}_q u\|_{L^\infty}=0.$$

Note that (4) holds true whenever u is in $\mathcal{S}'_h$ and that one may write

$$\dot{S}_q u=\sum_{p\le q-1}\dot{\Delta}_p u.$$

Besides, we would like to mention that the Littlewood-Paley decomposition has a nice property of quasi-orthogonality:

$$\dot{\Delta}_p\dot{\Delta}_q u\equiv 0 \;\text{ if }\; |p-q|\ge 2 \;\text{ and }\; \dot{\Delta}_p(\dot{S}_{q-1}u\dot{\Delta}_q u)\equiv 0 \;\text{ if }\; |p-q|\ge 5. \tag{5}$$

One can now give the definition of homogeneous Besov spaces.

Definition 2.1. *For $s\in\mathbb{R}$, $(p,r)\in[1,\infty]^2$, and $u\in\mathcal{S}'_h(\mathbb{R}^d)$, we set*

$$\|u\|_{\dot{B}^s_{p,r}}=\left\|2^{sq}\|\dot{\Delta}_q u\|_{L^p}\right\|_{\ell^r}.$$

We then define the space $\dot{B}^s_{p,r}:=\{u\in\mathcal{S}'_h(\mathbb{R}^d),\ \|u\|_{\dot{B}^s_{p,r}}<\infty\}$.

Remark 2.1. *The inhomogeneous Besov spaces can be defined in a similar way. Indeed, for $u\in\mathcal{S}'(\mathbb{R}^d)$, we set*

$$\Delta_q u=0 \;\text{ if }\; q<-1,\quad \Delta_{-1}u=\chi(\mathrm{D})u,$$

$$\Delta_q u=\varphi(2^{-q}\mathrm{D})u \;\text{ if }\; q\ge 0,\quad \text{and}\quad S_q u=\sum_{p\le q-1}\Delta_p u.$$

Then for all $u\in\mathcal{S}'(\mathbb{R}^d)$, we have the inhomogeneous Littlewood-Paley decomposition $u=\sum_{q\in\mathbb{Z}}\Delta_q u$, and for $(p,r)\in[1,+\infty]^2, s\in\mathbb{R}$, we define the inhomogeneous Besov space $B^s_{p,r}$ as

$$B^s_{p,r}=\left\{u\in\mathcal{S}'(\mathbb{R}^d),\|u\|_{B^s_{p,r}}:=\left\|2^{sq}\|\Delta_q u\|_{L^p}\right\|_{\ell^r}<\infty\right\}$$

The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ will be used in this paper.

Definition 2.2. *Let $s\in\mathbb{R}$. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ (also denoted by $\dot{H}^s$) is the space of tempered distributions u over $\mathbb{R}^d$, the Fourier transform of which belongs to $L^1_{loc}(\mathbb{R}^d)$ and satisfies*

$$\|u\|^2_{\dot{H}^s}:=\int_{\mathbb{R}^d}|\xi|^{2s}|\hat{u}(\xi)|^2d\xi<\infty.$$

Besides, it is not difficult to verify that the homogeneous Besov space $\dot{B}^s_{2,2}$ coincides with $\dot{H}^s$ if $s<\frac{d}{2}$, see for example, page 63 of [1].

We also need the following hybrid Besov space in this paper.

Definition 2.3. *For $s\in\mathbb{R}$, and $u\in\mathcal{S}'_h(\mathbb{R}^d)$, we set*

$$\|u\|_{\dot{\mathcal{B}}^s}=\left(\sum_{q<0}2^{2qs}\|\dot{\Delta}_q u\|^2_{L^2}\right)^{\frac{1}{2}}+\sum_{q\ge 0}2^{q\frac{d}{2}}\|\dot{\Delta}_q u\|_{L^2}.$$

We then define the space $\dot{\mathcal{B}}^s:=\{u\in\mathcal{S}'_h(\mathbb{R}^d),\ \|u\|_{\dot{\mathcal{B}}^s}<\infty\}$.

Remark 2.2. *Clearly, for any $u \in \mathcal{S}'_h(\mathbb{R}^d)$ and $s \in \mathbb{R}$, there holds*

$$\|u\|_{\dot{\mathcal{B}}^s} \le \left(\sum_{q\in\mathbb{Z}} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\right)^{\frac{1}{2}} + \sum_{q\in\mathbb{Z}} 2^{q\frac{d}{2}}\|\dot{\Delta}_q u\|_{L^2} \lesssim \|u\|_{\dot{H}^s} + \|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}. \tag{6}$$

On the other hand, if $s \le \frac{d}{2}$, we have

$$\begin{aligned}
\|u\|_{\dot{H}^s} &\le C\left(\sum_{q\in\mathbb{Z}} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{q<0} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\right)^{\frac{1}{2}} + C\left(\sum_{q\ge 0} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{q<0} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\right)^{\frac{1}{2}} + C\left(\sum_{q\ge 0} 2^{qd}\|\dot{\Delta}_q u\|_{L^2}^2\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{q<0} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\right)^{\frac{1}{2}} + C\sum_{q\ge 0} 2^{q\frac{d}{2}}\|\dot{\Delta}_q u\|_{L^2}\\
&= C\|u\|_{\dot{\mathcal{B}}^s}.
\end{aligned} \tag{7}$$

If $s < \frac{d}{2}$, using Hölder's inequality yields

$$\begin{aligned}
\|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}} &= \sum_{q<0} 2^{q(\frac{d}{2}-s)}2^{qs}\|\dot{\Delta}_q u\|_{L^2} + \sum_{q\ge 0} 2^{q\frac{d}{2}}\|\dot{\Delta}_q u\|_{L^2}\\
&\le C\left(\sum_{q<0} 2^{2q(\frac{d}{2}-s)}\right)^{\frac{1}{2}}\left(\sum_{q<0} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\right)^{\frac{1}{2}} + \sum_{q\ge 0} 2^{q\frac{d}{2}}\|\dot{\Delta}_q u\|_{L^2}\\
&\le C\|u\|_{\dot{\mathcal{B}}^s}.
\end{aligned} \tag{8}$$

Combining (7) *with* (8), *we are led to*

$$\|u\|_{\dot{H}^s\cap\dot{B}^{\frac{d}{2}}_{2,1}} \le C\|u\|_{\dot{\mathcal{B}}^s} \quad \text{if} \quad s < \frac{d}{2}. \tag{9}$$

It follows from (6) *and* (9) *that*

$$\|u\|_{\dot{H}^s\cap\dot{B}^{\frac{d}{2}}_{2,1}} \approx \|u\|_{\dot{\mathcal{B}}^s} \quad \text{if} \quad s < \frac{d}{2}. \tag{10}$$

Therefore, we conclude that

$$\dot{\mathcal{B}}^s = \dot{H}^s \cap \dot{B}^{\frac{d}{2}}_{2,1} \quad \text{if} \quad s < \frac{d}{2}. \tag{11}$$

The following lemma describes the way derivatives act on spectrally localized functions.

Lemma 2.1 (Bernstein's inequalities). *Let $k \in \mathbb{N}$ and $0 < r < R$. There exists a constant C depending on r, R and d such that for all $(a, b) \in [1, \infty]^2$, we have for all $\lambda > 0$ and multi-index α*

- *If* $\mathrm{Supp}\hat{f} \subset B(0, \lambda R)$, *then* $\sup_{\alpha=k}\|\partial^\alpha f\|_{L^b} \le C^{k+1}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a}$.

- *If* $\operatorname{Supp}\hat{f} \subset \mathfrak{C}(0, \lambda r, \lambda R)$, *then* $C^{-k-1}\lambda^k\|f\|_{L^a} \le \sup_{|\alpha|=k}\|\partial^\alpha f\|_{L^a} \le C^{k+1}\lambda^k\|f\|_{L^a}$

Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Firstly introduced by J. M. Bony in [2], the paraproduct between f and g is defined by

$$\dot{T}_f g = \sum_{q\in\mathbb{Z}} \dot{S}_{q-1} f \dot{\Delta}_q g,$$

and the remainder is given by

$$\dot{R}(f,g) = \sum_{q\ge -1} \dot{\Delta}_q f \tilde{\dot{\Delta}}_q g$$

with

$$\tilde{\dot{\Delta}}_q g = (\dot{\Delta}_{q-1} + \dot{\Delta}_q + \dot{\Delta}_{q+1})g.$$

We have the following so-called Bony's decomposition:

$$fg = \dot{T}_f g + \dot{T}_g f + \dot{R}(f,g). \tag{12}$$

The paraproduct $\dot{T}$ and the remainder $\dot{R}$ operators satisfy the following continuous properties. The proof can be found in Theorem 2.47 and Theorem 2.52 of [1].

Proposition 2.1. *For all $s \in \mathbb{R}$, $\sigma > 0$, and $1 \le p, p_1, p_2, r, r_1, r_2 \le \infty$, the paraproduct $\dot{T}$ is a bilinear, continuous operator from $L^\infty \times \dot{B}^s_{p,r}$ to $\dot{B}^s_{p,r}$ and from $\dot{B}^{-\sigma}_{\infty,r_1} \times \dot{B}^s_{p,r_2}$ to $\dot{B}^{s-\sigma}_{p,r}$ with $\frac{1}{r} = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$. The remainder $\dot{R}$ is bilinear continuous from $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ to $\dot{B}^{s_1+s_2}_{p,r}$ with $s_1 + s_2 > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \le 1$, and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \le 1$.*

In view of (12), Proposition 2.1 and Bernstein's inequalities, one easily deduces the following product estimates:

Corollary 2.1. *There hold:*

$$\|uv\|_{\dot{H}^s} \le C\|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|v\|_{\dot{H}^s}, \quad \text{if} \quad s \in (-\frac{d}{2}, \frac{d}{2}). \tag{13}$$

$$\|uv\|_{\dot{H}^s} \le C\|u\|_{\dot{H}^{s+1}}\|v\|_{\dot{B}^{\frac{d}{2}-1}_{2,\infty}}, \quad \text{if} \quad s \in (-\frac{d}{2}, \frac{d}{2} - 1). \tag{14}$$

and

$$\|uv\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \le C\|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|v\|_{\dot{B}^{\frac{d}{2}}_{2,1}}. \tag{15}$$

The study of non-stationary PDEs requires spaces of the type $L^\rho_T(X) = L^\rho(0,T;X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain the bounds in spaces which are not of the type $L^\rho(0,T;\dot{B}^s_{p,r})$. That naturally leads to the following definition introduced by Chemin and Lerner in [4].

Definition 2.4. *For $\rho \in [1,+\infty]$, $s \in \mathbb{R}$, and $T \in (0,+\infty)$, we set*

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} = \left\| 2^{qs}\|\dot{\Delta}_q u(t)\|_{L^\rho_T(L^p)} \right\|_{\ell^r}$$

and denote by $\tilde{L}^\rho_T(\dot{B}^s_{p,r})$ the subset of distributions $u \in \mathcal{D}'([0,T];\mathcal{S}'_h(\mathbb{R}^d))$ with finite $\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})}$ norm. When $T = +\infty$, the index T is omitted. We further denote $\tilde{C}_T(\dot{B}^s_{p,r}) = C([0,T];\dot{B}^s_{p,r}) \cap \tilde{L}^\infty_T(\dot{B}^s_{p,r})$.

Remark 2.3. *All the properties of continuity for the paraproduct, remainder, and product remain true for the Chemin-Lerner spaces. The exponent ρ just has to behave according to Hölder's inequality for the time variable.*

Remark 2.4. *The spaces $\tilde{L}^\rho_T(\dot{B}^s_{p,r})$ can be linked with the classical space $L^\rho_T(\dot{B}^s_{p,r})$ via the Minkowski inequality:*

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} \le \|u\|_{L^\rho_T(\dot{B}^s_{p,r})} \quad \text{if} \quad r \ge \rho, \qquad \|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} \ge \|u\|_{L^\rho_T(\dot{B}^s_{p,r})} \quad \text{if} \quad r \le \rho.$$

3. Global existence

In order to construct the global solutions to the incompressible Oldroyd-B model (2), we shall used the classical Friedrichs method to approximate the system (2) by a cut-off in the frequency space. Noting that this method has been applied to Oldroyd-B model in [5, 6] before, to avoid unnecessary repetition, we omit the details of approximation in this paper. In the following, the global estimates of (u,τ) will be given directly. To begin with, let us first of all localize the system (2) as follows,

$$\begin{cases}2\omega\mathrm{Re}\dot{\Delta}_q u_t - 2\omega(1-\omega)\dot{\Delta}_q\Delta u + 2\omega\nabla\dot{\Delta}_q\Pi = 2\omega\mathrm{div}\dot{\Delta}_q\tau - 2\omega\mathrm{Re}\dot{\Delta}_q(u\cdot\nabla u),\\ \mathrm{We}\left(\dot{\Delta}_q\tau_t + u\cdot\nabla\dot{\Delta}_q\tau\right) + \dot{\Delta}_q\tau = 2\omega D(\dot{\Delta}_q u) - \mathrm{We}\left([\dot{\Delta}_q, u]\cdot\nabla\tau + \dot{\Delta}_q g_\alpha(\tau,\nabla u)\right).\end{cases} \tag{16}$$

Taking the L^2 inner product of $(16)_1$ and $(16)_2$ with $\dot{\Delta}_q u$ and $\dot{\Delta}_q\tau$, respectively, using the relation $(\mathrm{div}\tau|u) + (D(u)|\tau) = 0$ and the divergence free condition of u, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(2\omega\mathrm{Re}\|\dot{\Delta}_q u\|_{L^2}^2 + \mathrm{We}\|\dot{\Delta}_q\tau\|_{L^2}^2\right) + 2\omega(1-\omega)\|\nabla\dot{\Delta}_q u\|_{L^2}^2 + \|\dot{\Delta}_q\tau\|_{L^2}^2\\ \le\ & 2\omega\mathrm{Re}\|\dot{\Delta}_q(u\cdot\nabla u)\|_{L^2}\|\dot{\Delta}_q u\|_{L^2} + \mathrm{We}\left(\|[\dot{\Delta}_q, u]\cdot\nabla\tau\|_{L^2} + \|\dot{\Delta}_q g_\alpha(\tau,\nabla u)\|_{L^2}\right)\|\dot{\Delta}_q\tau\|_{L^2}.\end{aligned} \tag{17}$$

It follows that

$$\begin{aligned}&\omega\mathrm{Re}\|u\|_{\tilde{L}_t^\infty(\dot{H}^s)}^2 + \frac{\mathrm{We}}{2}\|\tau\|_{\tilde{L}_t^\infty(\dot{H}^s)}^2 + 2\omega(1-\omega)\|\nabla u\|_{L_t^2(\dot{H}^s)}^2 + \|\tau\|_{L_t^2(\dot{H}^s)}^2\\ \le\ & \omega\mathrm{Re}\|u_0\|_{\dot{H}^s}^2 + \frac{\mathrm{We}}{2}\|\tau_0\|_{\dot{H}^s}^2 + 2\omega\mathrm{Re}\int_0^t\sum_{q\in\mathbb{Z}}2^{2qs}\|\dot{\Delta}_q(u\cdot\nabla u)\|_{L^2}\|\dot{\Delta}_q u\|_{L^2}dt'\\ &+\mathrm{We}\int_0^t\sum_{q\in\mathbb{Z}}2^{2qs}\left(\|[\dot{\Delta}_q, u]\cdot\nabla\tau\|_{L^2} + \|\dot{\Delta}_q g_\alpha(\tau,\nabla u)\|_{L^2}\right)\|\dot{\Delta}_q\tau\|_{L^2}dt'.\end{aligned} \tag{18}$$

Now we estimate the last three terms on the righthand side of (18) one by one. In fact, in view of Höder's inequality and the product estimate (13), we infer that for $-\frac{d}{2} < s < \frac{d}{2}$, there holds

$$\begin{aligned}&2\omega\mathrm{Re}\int_0^t\sum_{q\in\mathbb{Z}}2^{2qs}\|\dot{\Delta}_q(u\cdot\nabla u)\|_{L^2}\|\dot{\Delta}_q u\|_{L^2}dt'\\ \le\ & 2\omega\mathrm{Re}\int_0^t\|u\cdot\nabla u\|_{\dot{H}^s}\|u\|_{\dot{H}^s}dt'\\ \le\ & C\omega\mathrm{Re}\int_0^t\|\nabla u\|_{\dot{H}^s}\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|u\|_{\dot{H}^s}dt'\\ \le\ & C\omega\mathrm{Re}\|u\|_{L_t^\infty(\dot{H}^s)}\|\nabla u\|_{L_t^2(\dot{H}^s)}\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}.\end{aligned} \tag{19}$$

Noting that if $s < \frac{d}{2} - 1$, we can bound $\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}$ as follows:

$$\begin{aligned}\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} &\le \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^l + \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^h\\ &\le C\left(\|u\|_{L_t^2(\dot{H}^{s+1})}^l + \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}+\frac{1}{2}})}^h\right)\\ &\le C\left(\|u\|_{L_t^2(\dot{H}^{s+1})} + \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h\|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h\right)^{\frac{1}{2}}\right)\end{aligned} \tag{20}$$

Inserting (20) into (19), we arrive at

$$
\begin{aligned}
& 2\omega\mathrm{Re}\int_0^t \sum_{q\in\mathbb{Z}} 2^{2qs}\|\dot\Delta_q(u\cdot\nabla u)\|_{L^2}\|\dot\Delta_q u\|_{L^2}dt' \\
\leq\ & C\omega\mathrm{Re}\|u\|_{L^\infty_t(\dot H^s)}\|\nabla u\|_{L^2_t(\dot H^s)}\left(\|u\|_{L^2_t(\dot H^{s+1})}+\left(\|u\|^h_{L^\infty_t(\dot B^{\frac d2}_{2,1})}\|u\|^h_{L^1_t(\dot B^{\frac d2+1}_{2,1})}\right)^{\frac12}\right), \qquad (21)
\end{aligned}
$$

with $-\frac d2 < s < \frac d2 - 1$. Similarly, we have

$$
\begin{aligned}
& \mathrm{We}\int_0^t \sum_{q\in\mathbb{Z}} 2^{2qs}\|\dot\Delta_q g_\alpha(\tau,\nabla u)\|_{L^2}\|\dot\Delta_q\tau\|_{L^2}dt' \\
\leq\ & \mathrm{We}\int_0^t \|g_\alpha(\tau,\nabla u)\|_{\dot H^s}\|\tau\|_{\dot H^s}dt' \\
\leq\ & C\mathrm{We}\int_0^t \|\tau\|_{\dot B^{\frac d2}_{2,1}}\|\nabla u\|_{\dot H^s}\|\tau\|_{\dot H^s}dt' \\
\leq\ & C\mathrm{We}\|\tau\|_{\tilde L^\infty_t(\dot B^{\frac d2}_{2,1})}\|\nabla u\|_{L^2_t(\dot H^s)}\|\tau\|_{L^2_t(\dot H^s)}, \quad \text{for} \quad -\frac d2 < s < \frac d2. \qquad (22)
\end{aligned}
$$

Finally, using Hölder's inequality and the commutator estimate, c. f. [1], for $-\frac d2 - 1 < s < \frac d2$, we are led to

$$
\begin{aligned}
& \mathrm{We}\int_0^t \sum_{q\in\mathbb{Z}} 2^{2qs}\|[\dot\Delta_q, u]\cdot\nabla\tau\|_{L^2}\|\dot\Delta_q\tau\|_{L^2}dt' \\
\leq\ & \mathrm{We}\int_0^t \left(\sum_{q\in\mathbb{Z}} 2^{2qs}\|[\dot\Delta_q, u]\cdot\nabla\tau\|^2_{L^2}\right)^{\frac12}\|\tau\|_{\dot H^s}dt' \\
\leq\ & C\mathrm{We}\int_0^t \|\nabla u\|_{\dot B^{\frac d2}_{2,1}}\|\tau\|^2_{\dot H^s}dt' \\
=\ & C\mathrm{We}\int_0^t \|\nabla u\|^l_{\dot B^{\frac d2}_{2,1}}\|\tau\|^2_{\dot H^s}+\|\nabla u\|^h_{\dot B^{\frac d2}_{2,1}}\|\tau\|^2_{\dot H^s}dt' \\
\leq\ & C\mathrm{We}\|\nabla u\|^l_{L^2_t(\dot B^{\frac d2}_{2,1})}\|\tau\|_{L^\infty_t(\dot H^s)}\|\tau\|_{L^2_t(\dot H^s)}+C\mathrm{We}\|\nabla u\|^h_{L^1_t(\dot B^{\frac d2}_{2,1})}\|\tau\|^2_{L^\infty_t(\dot H^s)} \\
\leq\ & C\mathrm{We}\|\nabla u\|_{L^2_t(\dot H^s)}\|\tau\|_{L^\infty_t(\dot H^s)}\|\tau\|_{L^2_t(\dot H^s)}+C\mathrm{We}\|\nabla u\|^h_{L^1_t(\dot B^{\frac d2}_{2,1})}\|\tau\|^2_{L^\infty_t(\dot H^s)}. \qquad (23)
\end{aligned}
$$

Substituting (21)–(23) into (18) yields

$$
\begin{aligned}
& \omega\mathrm{Re}\|u\|^2_{\tilde L^\infty_t(\dot H^s)}+\frac{\mathrm{We}}{2}\|\tau\|^2_{\tilde L^\infty_t(\dot H^s)}+2\omega(1-\omega)\|\nabla u\|^2_{L^2_t(\dot H^s)}+\|\tau\|^2_{L^2_t(\dot H^s)} \\
\leq\ & \omega\mathrm{Re}\|u_0\|^2_{\dot H^s}+\frac{\mathrm{We}}{2}\|\tau_0\|^2_{\dot H^s}+C\mathrm{We}\|\tau\|_{\tilde L^\infty_t(\dot B^{\frac d2}_{2,1})}\|\nabla u\|_{L^2_t(\dot H^s)}\|\tau\|_{L^2_t(\dot H^s)} \\
& +C\mathrm{We}\|\nabla u\|_{L^2_t(\dot H^s)}\|\tau\|_{L^\infty_t(\dot H^s)}\|\tau\|_{L^2_t(\dot H^s)}+C\mathrm{We}\|\nabla u\|^h_{L^1_t(\dot B^{\frac d2}_{2,1})}\|\tau\|^2_{L^\infty_t(\dot H^s)} \\
& +C\omega\mathrm{Re}\|u\|_{L^\infty_t(\dot H^s)}\|\nabla u\|_{L^2_t(\dot H^s)}\left(\|u\|_{L^2_t(\dot H^{s+1})}+\left(\|u\|^h_{L^\infty_t(\dot B^{\frac d2}_{2,1})}\|u\|^h_{L^1_t(\dot B^{\frac d2+1}_{2,1})}\right)^{\frac12}\right). \qquad (24)
\end{aligned}
$$

To close (24), we have to estimate the high frequency part of u and τ in the space $\dot B^{\frac d2}_{2,1}$. To this end, we first notice that

$$
2\omega\mathrm{Re}\|\dot\Delta_q u\|^2_{L^2}+\mathrm{We}\|\dot\Delta_q\tau\|^2_{L^2}\approx\left(\sqrt{2\omega\mathrm{Re}}\|\dot\Delta_q u\|_{L^2}+\sqrt{\mathrm{We}}\|\dot\Delta_q\tau\|_{L^2}\right)^2, \qquad (25)
$$

and for $q \geq 0$,

$$\begin{aligned}
2\omega(1-\omega)\|\nabla\dot{\Delta}_q u\|_{L^2}^2 + \|\dot{\Delta}_q\tau\|_{L^2}^2 &\approx 2\omega(1-\omega)2^{2q}\|\dot{\Delta}_q u\|_{L^2}^2 + \|\dot{\Delta}_q\tau\|_{L^2}^2 \\
&\approx \left(\sqrt{2\omega(1-\omega)}2^q\|\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q\tau\|_{L^2}\right)^2 \\
&\geq \min\left\{\sqrt{\frac{1-\omega}{\mathrm{Re}}}, \frac{1}{\sqrt{\mathrm{We}}}\right\}\left(\sqrt{2\omega(1-\omega)}2^q\|\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q\tau\|_{L^2}\right) \\
&\quad \times\left(\sqrt{2\omega\mathrm{Re}}\|\dot{\Delta}_q u\|_{L^2} + \sqrt{\mathrm{We}}\|\dot{\Delta}_q\tau\|_{L^2}\right). \qquad (26)
\end{aligned}$$

It follows from (17), (25) and (26) that, if $q \geq 0$, there holds

$$\begin{aligned}
&\frac{d}{dt}\sqrt{2\omega\mathrm{Re}\|\dot{\Delta}_q u\|_{L^2}^2 + \mathrm{We}\|\dot{\Delta}_q\tau\|_{L^2}^2} \\
&\qquad + \min\left\{\sqrt{\frac{1-\omega}{\mathrm{Re}}}, \frac{1}{\sqrt{\mathrm{We}}}\right\}\left(\sqrt{2\omega(1-\omega)}2^q\|\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q\tau\|_{L^2}\right) \\
\lesssim\ & \sqrt{2\omega\mathrm{Re}}\|\dot{\Delta}_q(u\cdot\nabla u)\|_{L^2} + \sqrt{\mathrm{We}}\left(\|[\dot{\Delta}_q, u]\cdot\nabla\tau\|_{L^2} + \|\dot{\Delta}_q g_\alpha(\tau,\nabla u)\|_{L^2}\right). \qquad (27)
\end{aligned}$$

Integrating the above equation with respect to t, multiplying the resulting inequality by $2^{q\frac{d}{2}}$, and then summing w. r. t. q over all the nonnegative integers, we find that

$$\begin{aligned}
&\sqrt{2\omega\mathrm{Re}}\|u\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \sqrt{\mathrm{We}}\|\tau\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} \\
&\qquad + \min\left\{\sqrt{\frac{1-\omega}{\mathrm{Re}}}, \frac{1}{\sqrt{\mathrm{We}}}\right\}\left(\sqrt{2\omega(1-\omega)}\|u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|\tau\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right) \\
\lesssim\ & \sqrt{2\omega\mathrm{Re}}\|u_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + \sqrt{\mathrm{We}}\|\tau_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + \sqrt{2\omega\mathrm{Re}}\int_0^t \|u\cdot\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\,dt' \\
&+ \sqrt{\mathrm{We}}\int_0^t \sum_{q\in\mathbb{Z}} 2^{q\frac{d}{2}}\|[\dot{\Delta}_q, u]\cdot\nabla\tau\|_{L^2} + \|g_\alpha(\tau,\nabla u)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\,dt'. \qquad (28)
\end{aligned}$$

Product estimate (15) in Besov space and (20) imply that, for $-\frac{d}{2} < s < \frac{d}{2} - 1$,

$$\begin{aligned}
&\sqrt{2\omega\mathrm{Re}}\int_0^t \|u\cdot\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\,dt' \\
\leq\ & C\sqrt{2\omega\mathrm{Re}}\int_0^t \|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\,dt' \\
=\ & C\sqrt{2\omega\mathrm{Re}}\int_0^t \|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\left(\|\nabla u\|^l_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|\nabla u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\right)dt' \\
\leq\ & C\sqrt{2\omega\mathrm{Re}}\left(\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\nabla u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|u\|_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\nabla u\|^l_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right) \\
\leq\ & C\sqrt{2\omega\mathrm{Re}}\left(\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\nabla u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \left(\|u\|_{L^2_t(\dot{H}^{s+1})} + \left(\|u\|^h_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)^{\frac{1}{2}}\right)\|\nabla u\|_{L^2_t(\dot{H}^s)}\right). \qquad (29)
\end{aligned}$$

Using commutator estimate and product estimate (15) in Besov space again, we get for $-\frac{d}{2} < s < \frac{d}{2}$,

$$\begin{aligned}
&\sqrt{\mathrm{We}}\int_0^t \sum_{q\in\mathbb{Z}} 2^{q\frac{d}{2}}\|[\dot{\Delta}_q, u]\cdot\nabla\tau\|_{L^2} + \|g_\alpha(\tau,\nabla u)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\,dt' \\
\leq\;& C\sqrt{\mathrm{We}}\int_0^t \|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\tau\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\,dt' \\
\leq\;& C\sqrt{\mathrm{We}}\left(\|\tau\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\nabla u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|\tau\|_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\nabla u\|^l_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right) \\
\leq\;& C\sqrt{\mathrm{We}}\left(\|\tau\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\nabla u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \left(\|\tau\|_{L^2_t(\dot{H}^s)} + \left(\|\tau\|^h_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\tau\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right)^{\frac{1}{2}}\right)\|\nabla u\|_{L^2_t(\dot{H}^s)}\right).
\end{aligned} \tag{30}$$

Substituting (29) and (30) into (28) yields

$$\begin{aligned}
&\sqrt{2\omega\mathrm{Re}}\|u\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \sqrt{\mathrm{We}}\|\tau\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} \\
&\qquad + \min\left\{\sqrt{\frac{1-\omega}{\mathrm{Re}}}, \frac{1}{\sqrt{\mathrm{We}}}\right\}\left(\sqrt{2\omega(1-\omega)}\|u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|\tau\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right) \\
\leq\;& \sqrt{2\omega\mathrm{Re}}\|u_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + \sqrt{\mathrm{We}}\|\tau_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} \\
&+ C\sqrt{2\omega\mathrm{Re}}\left(\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\nabla u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \left(\|u\|_{L^2_t(\dot{H}^{s+1})} + \left(\|u\|^h_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)^{\frac{1}{2}}\right)\|\nabla u\|_{L^2_t(\dot{H}^s)}\right) \\
&+ C\sqrt{\mathrm{We}}\left(\|\tau\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\nabla u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \left(\|\tau\|_{L^2_t(\dot{H}^s)} + \left(\|\tau\|^h_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\tau\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right)^{\frac{1}{2}}\right)\|\nabla u\|_{L^2_t(\dot{H}^s)}\right).
\end{aligned} \tag{31}$$

Let us define

$$\begin{aligned}
E_1(t) &:= \sqrt{\omega\mathrm{Re}}\|u\|_{\tilde{L}^\infty_t(\dot{H}^s)} + \sqrt{\mathrm{We}}\|\tau\|_{\tilde{L}^\infty_t(\dot{H}^s)} + \sqrt{\omega(1-\omega)}\|\nabla u\|_{L^2_t(\dot{H}^s)} + \|\tau\|_{L^2_t(\dot{H}^s)}, \\
E_2(t) &:= \sqrt{\omega\mathrm{Re}}\|u\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \sqrt{\mathrm{We}}\|\tau\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \sqrt{\omega(1-\omega)}\|\nabla u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|\tau\|^h_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})},
\end{aligned}$$

$$E_1(0) := \sqrt{\omega\mathrm{Re}}\|u_0\|_{\dot{H}^s} + \sqrt{\mathrm{We}}\|\tau_0\|_{\dot{H}^s}, \qquad E_2(0) := \sqrt{\omega\mathrm{Re}}\|u_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + \sqrt{\mathrm{We}}\|\tau_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}},$$

and

$$E(t) := E_1(t) + E_2(t), \qquad E(0) := E_1(0) + E_2(0).$$

Moreover, we denote

$$\begin{aligned}
\kappa_1 &:= \max\left\{\left(\frac{\mathrm{We}}{\omega(1-\omega)}\right)^{\frac{1}{4}}, \frac{1}{(\omega(1-\omega))^{\frac{1}{4}}}, \frac{(\omega\mathrm{Re})^{\frac{1}{4}}}{\sqrt{\omega(1-\omega)}}, \frac{(\omega\mathrm{Re})^{\frac{1}{8}}}{(\omega(1-\omega))^{\frac{3}{8}}}\right\}, \\
\kappa_2 &:= \max\left\{1, \sqrt{\frac{\mathrm{Re}}{1-\omega}}, \sqrt{\mathrm{We}}\right\}, \\
\kappa_3 &:= \max\left\{\frac{1}{\sqrt{\omega(1-\omega)}}, \frac{\sqrt{\mathrm{Re}}}{\sqrt{\omega}(1-\omega)}, \frac{(\mathrm{Re})^{\frac{1}{4}}}{\omega^{\frac{1}{2}}(1-\omega)^{\frac{3}{4}}}, \frac{(\mathrm{We})^{\frac{1}{4}}}{\sqrt{\omega(1-\omega)}}\right\}.
\end{aligned}$$

Then (24) and (31) read as follows:

$$E_1(t) \le E_1(0) + C\kappa_1 E(t)^{\frac{3}{2}}, \quad \text{for} \quad -\frac{d}{2} < s < \frac{d}{2} - 1, \tag{32}$$

and

$$E_2(t) \le \kappa_2 E_2(0) + C\kappa_2\kappa_3 E(t)^2, \quad \text{for} \quad -\frac{d}{2} < s < \frac{d}{2} - 1, \tag{33}$$

Consequently,

$$E(t) \le \kappa_2 E(0) + C\left(\kappa_1 E(t)^{\frac{3}{2}} + \kappa_2\kappa_3 E(t)^2\right), \quad \text{for} \quad -\frac{d}{2} < s < \frac{d}{2} - 1. \tag{34}$$

By using standard continuity method, we infer from (34) that

$$E(t) \le 2\kappa_2 E(0), \tag{35}$$

provided $E(0)$ is small enough. Then the existence part of Theorem 1.1 follows immediately. □

4. Uniqueness

Let (u_1, τ_1) and (u_2, τ_2) be the solution to the system (2) with the same initial data obtained in Section 3. Denote $(w, \sigma) := (u_1 - u_2, \tau_1 - \tau_2)$, and $p = \Pi_1 - \Pi_2$. Then it is easy to verify that (w, σ) satisfies the following system:

$$\begin{cases} \mathrm{Re}\ \partial_t w - (1-\omega)\Delta w + \nabla p = \mathrm{div}\sigma - \mathrm{Re}\ (w\cdot\nabla u_1 + u_2\cdot\nabla w), \\ \mathrm{We}\ (\partial_t\sigma + u_1\cdot\nabla\sigma) + \sigma = 2\omega D(w) - \mathrm{We}\ w\cdot\nabla\tau_2 - \mathrm{We}\ g_\alpha(\sigma,\nabla u_1) - \mathrm{We}\ g_\alpha(\tau_2,\nabla w). \end{cases} \tag{36}$$

Applying the localized operator $\dot\Delta_q$ to system (36) yields

$$\begin{cases} \mathrm{Re}\ \partial_t\dot\Delta_q w - (1-\omega)\Delta\dot\Delta_q w + \nabla\dot\Delta_q p = \mathrm{div}\dot\Delta_q\sigma - \mathrm{Re}\ \dot\Delta_q(w\cdot\nabla u_1 + u_2\cdot\nabla w), \\ \mathrm{We}\ (\partial_t\dot\Delta_q\sigma + u_1\cdot\nabla\dot\Delta_q\sigma) + \dot\Delta_q\sigma = 2\omega D(\dot\Delta_q w) - \mathrm{We}\ [\dot\Delta_q, u_1]\cdot\nabla\sigma - \mathrm{We}\ \dot\Delta_q(w\cdot\nabla\tau_2) \\ \qquad\qquad -\mathrm{We}\ \dot\Delta_q g_\alpha(\sigma,\nabla u_1) - \mathrm{We}\ \dot\Delta_q g_\alpha(\tau_2,\nabla w). \end{cases} \tag{37}$$

Using the cancelation relation $(\mathrm{div}\dot\Delta_q\sigma|\dot\Delta_q w) + (D(\dot\Delta_q w)|\dot\Delta_q\sigma) = 0$, similar to (17), we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(2\omega\mathrm{Re}\|\dot\Delta_q w\|_{L^2}^2 + \mathrm{We}\|\dot\Delta_q\sigma\|_{L^2}^2\right) + 2\omega(1-\omega)\|\nabla\dot\Delta_q w\|_{L^2}^2 + \|\dot\Delta_q\sigma\|_{L^2}^2 \\ \le\ & 2\omega\mathrm{Re}\|\dot\Delta_q(w\cdot\nabla u_1 + u_2\cdot\nabla w)\|_{L^2}\|\dot\Delta_q w\|_{L^2} + \mathrm{We}\left(\|[\dot\Delta_q, u_1]\cdot\nabla\sigma\|_{L^2} + \|\dot\Delta_q(w\cdot\nabla\tau_2)\|_{L^2}\right)\|\dot\Delta_q\sigma\|_{L^2} \\ & + \mathrm{We}\ \left(\|\dot\Delta_q g_\alpha(\sigma,\nabla u_1)\|_{L^2} + \|\dot\Delta_q g_\alpha(\tau_2,\nabla w)\|_{L^2}\right)\|\dot\Delta_q\sigma\|_{L^2}. \end{aligned} \tag{38}$$

Integrating (38) w. r. t. time t, multiplying the resulting inequality by 2^{2qs}, and then taking sum w. r. t. q over $\mathbb{Z}$, using Hölder's inequality, we are led to

$$\begin{aligned} &\omega\mathrm{Re}\|w(t)\|_{\dot H^s}^2 + \frac{\mathrm{We}}{2}\|\sigma(t)\|_{\dot H^s}^2 + 2\omega(1-\omega)\|\nabla w\|_{L_t^2(\dot H^s)}^2 + \|\sigma\|_{L_t^2(\dot H^s)}^2 \\ \le\ & 2\omega\mathrm{Re}\int_0^t \|w\cdot\nabla u_1 + u_2\cdot\nabla w\|_{\dot H^s}\|w\|_{\dot H^s}dt' + \mathrm{We}\int_0^t\left(\sum_{q\in\mathbb{Z}} 2^{2qs}\|[\dot\Delta_q, u_1]\cdot\nabla\sigma\|_{L^2}^2\right)^{\frac{1}{2}}\|\sigma\|_{\dot H^s}dt' \\ & + \mathrm{We}\int_0^t \left(\|w\cdot\nabla\tau_2\|_{\dot H^s} + \|g_\alpha(\sigma,\nabla u_1)\|_{\dot H^s} + \|g_\alpha(\tau_2,\nabla w)\|_{\dot H^s}\right)\|\sigma\|_{\dot H^s}dt'. \end{aligned} \tag{39}$$

By virtue of the product estimates (13), (14) and commutator estimate in Besov spaces, we have

$$\|w\cdot\nabla u_1+u_2\cdot\nabla w\|_{\dot H^s}\lesssim\|\nabla u_1\|_{\dot B^{\frac d2}_{2,1}}\|w\|_{\dot H^s}+\|u_2\|_{\dot B^{\frac d2}_{2,1}}\|\nabla w\|_{\dot H^s},\quad\text{for}\quad -\frac d2<s<\frac d2,$$

$$\Big(\sum_{q\in\mathbb Z}2^{2qs}\|[\dot\Delta_q,u_1]\cdot\nabla\sigma\|_{L^2}^2\Big)^{\frac12}\lesssim\|\nabla u_1\|_{\dot B^{\frac d2}_{2,1}}\|\sigma\|_{\dot H^s},\quad\text{for}\quad -\frac d2-1<s<\frac d2,$$

$$\|w\cdot\nabla\tau_2\|_{\dot H^s}\lesssim\|w\|_{\dot H^{s+1}}\|\nabla\tau_2\|_{\dot B^{\frac d2-1}_{2,\infty}}\lesssim\|\nabla w\|_{\dot H^s}\|\tau_2\|_{\dot B^{\frac d2}_{2,1}},\quad\text{for}\quad -\frac d2<s<\frac d2-1,$$

$$\|g_\alpha(\sigma,\nabla u_1)\|_{\dot H^s}+\|g_\alpha(\tau_2,\nabla w)\|_{\dot H^s}\lesssim\|\nabla u_1\|_{\dot B^{\frac d2}_{2,1}}\|\sigma\|_{\dot H^s}+\|\tau_2\|_{\dot B^{\frac d2}_{2,1}}\|\nabla w\|_{\dot H^s},\quad\text{for}\quad -\frac d2<s<\frac d2,$$

Substituting these estimates into (39) yields that, for $-\frac d2<s<\frac d2-1$, there holds

$$\begin{aligned}&\omega\mathrm{Re}\|w(t)\|_{\dot H^s}^2+\mathrm{We}\|\sigma(t)\|_{\dot H^s}^2+2\omega(1-\omega)\|\nabla w\|_{L^2_t(\dot H^s)}^2+\|\sigma\|_{L^2_t(\dot H^s)}^2\\ \le\;& C\int_0^t\|\nabla u_1\|_{\dot B^{\frac d2}_{2,1}}\Big(\omega\mathrm{Re}\,\|w\|_{\dot H^s}^2+\mathrm{We}\,\|\sigma\|_{\dot H^s}^2\Big)dt'\\ &+C\int_0^t\Big(\omega\mathrm{Re}\,\|u_2\|_{\dot B^{\frac d2}_{2,1}}\|w\|_{\dot H^s}+\mathrm{We}\,\|\tau_2\|_{\dot B^{\frac d2}_{2,1}}\|\sigma\|_{\dot H^s}\Big)\|\nabla w\|_{\dot H^s}dt'.\end{aligned}\tag{40}$$

Noting that by Cauchy's inequality, there exists a positive constant C depending on Re , We and ω, such that

$$\begin{aligned}&\int_0^t\Big(\omega\mathrm{Re}\,\|u_2\|_{\dot B^{\frac d2}_{2,1}}\|w\|_{\dot H^s}+\mathrm{We}\,\|\tau_2\|_{\dot B^{\frac d2}_{2,1}}\|\sigma\|_{\dot H^s}\Big)\|\nabla w\|_{\dot H^s}dt'\\ \le\;&\omega(1-\omega)\|\nabla w\|_{L^2_t(\dot H^s)}^2+C\int_0^t\Big(\omega\mathrm{Re}\,\|u_2\|^2_{\dot B^{\frac d2}_{2,1}}+\mathrm{We}\,\|\tau_2\|^2_{\dot B^{\frac d2}_{2,1}}\Big)\Big(\omega\mathrm{Re}\,\|w\|_{\dot H^s}^2+\mathrm{We}\,\|\sigma\|_{\dot H^s}^2\Big)dt',\end{aligned}$$

combining this inequality with (40), we obtain

$$\begin{aligned}&\omega\mathrm{Re}\|w(t)\|_{\dot H^s}^2+\mathrm{We}\|\sigma(t)\|_{\dot H^s}^2+\omega(1-\omega)\|\nabla w\|_{L^2_t(\dot H^s)}^2+\|\sigma\|_{L^2_t(\dot H^s)}^2\\ \le\;& C\int_0^t\Big(\|\nabla u_1\|_{\dot B^{\frac d2}_{2,1}}+\omega\mathrm{Re}\,\|u_2\|^2_{\dot B^{\frac d2}_{2,1}}+\mathrm{We}\,\|\tau_2\|^2_{\dot B^{\frac d2}_{2,1}}\Big)\Big(\omega\mathrm{Re}\,\|w\|_{\dot H^s}^2+\mathrm{We}\,\|\sigma\|_{\dot H^s}^2\Big)dt'.\end{aligned}$$

Thanks to the embedding in low frequency, we infer from (35) that

$$\begin{aligned}&\int_0^t\Big(\|\nabla u_1\|_{\dot B^{\frac d2}_{2,1}}+\omega\mathrm{Re}\,\|u_2\|^2_{\dot B^{\frac d2}_{2,1}}+\mathrm{We}\,\|\tau_2\|^2_{\dot B^{\frac d2}_{2,1}}\Big)dt'\\ \lesssim\;&\|\nabla u_1\|^h_{L^1_t(\dot B^{\frac d2}_{2,1})}+t^{\frac12}\|\nabla u_1\|_{L^2_t(\dot H^s)}+\omega\mathrm{Re}\,t\Big(\|u_2\|^h_{\tilde L^\infty_t(\dot B^{\frac d2}_{2,1})}+\|u_2\|_{\tilde L^\infty_t(\dot H^s)}\Big)^2\\ &+\mathrm{We}\,t\Big(\|\tau_2\|^h_{\tilde L^\infty_t(\dot B^{\frac d2}_{2,1})}+\|\tau_2\|_{\tilde L^\infty_t(\dot H^s)}\Big)^2\\ \le\;&\infty,\end{aligned}$$

for any $t>0$. Then the uniqueness follows from Glonwall's inequality immediately. This completes the proof of Theorem 1.1.□

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