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Global Well-Posedness for a Viscosity Problem of the Compressible Heisenberg Chain Equations

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Abstract. In this paper, we are concerned with the existence and uniqueness of global smooth solutions to a viscosity problem for the compressible Heisenberg chain equations in one dimension. Furthermore, we prove the global existence of weak solutions when the parameter A tends to zero by compactness method.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of global smooth solutions to the following periodic boundary value problem:

$$\vec{Z}_t = -\varepsilon \vec{Z} \times \left(\vec{Z} \times \left(G\left(\vec{Z}_x\right) \vec{Z}_x \right)_x \right) + \vec{Z} \times \left(G\left(\vec{Z}_x\right) \vec{Z}_x \right)_x, \ x \in \mathbb{R}, \ t \in \mathbb{R}_+, \tag{1}$$

$$\vec{Z}(x,0) = \vec{Z}_0(x), \ \vec{Z}(x+D,t) = \vec{Z}(x-D,t), \ |\vec{Z}_0(x)| \equiv 1, \ x \in \mathbb{R}, \tag{2}$$

where $G(\xi) = A + B|\xi|^2$ and $\varepsilon, A, B, D > 0$ are generic constants. If $\varepsilon = 0$, the above system is reduced to the compressible Heisenberg chain equations (see [5]). Ding, Guo and Su [2] proved the existence of measure-valued solutions to the compressible Heisenberg chain equations. If $B = 0$, the system is reduced to the Landau-Lifshitz equations, one can refer to [3, 4, 6, 9] and their references for related topics.

There is a fact we will use in this paper: (1) is equivalent to the following form in the classical sense:

$$\vec{Z}_t = \varepsilon G\left(\vec{Z}_x\right) |\vec{Z}_x|^2 \vec{Z} + \varepsilon \left(G\left(\vec{Z}_x\right) \vec{Z}_x \right)_x + \vec{Z} \times \left(G\left(\vec{Z}_x\right) \vec{Z}_x \right)_x. \tag{3}$$

Note that if $A = 0$, (3) is in fact the one-dimensional heat flow of p-harmonic map with values into sphere and $p = 4$ by neglecting the last term $\vec{Z} \times (G(\vec{Z}_x)\vec{Z}_x)_x$ on the right hand side of (3), one can refer to [1] for the global existence of weak solutions to p-harmonic maps in multi-dimensions. The readers can also refer to [7] for related topics.

We first establish the existence of local smooth solution to problem (1)-(2) by difference-differential method, and then give a *priori* estimates for such solutions to obtain the global existence of regular solutions

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for fixed $\varepsilon = 1$, positive A and B. Furthermore, observing that some basic energy estimates independent of A, we prove the global existence of weak solutions provided that $A = 0$. Let $\Omega = [-D, D]$. Now we state the main results as follows.

Theorem 1.1. *Let $\vec{Z}_0(x) \in H^k(\Omega)$, $k \geq 2$. Then for any given $T > 0$, problem (1)-(2) admits a unique global regular solution $\vec{Z}(x,t)$:*

$$\vec{Z}(x,t) \in \mathcal{F}(T) = \Big(\bigcap_{s=0}^{[\frac{k}{2}]} W^{s,\infty}(0,T; H^{k-2s}(\Omega)) \Big) \bigcap \Big(\bigcap_{s=0}^{[\frac{k+1}{2}]} H^s(0,T; H^{k+1-2s}(\Omega)) \Big).$$

Theorem 1.2. *Let $\vec{Z}_0(x) \in W^{1,4}(\Omega)$ and $A = 0$. Then for any given $T > 0$, problem (1)-(2) admits a global weak solution $\vec{Z}(x,t)$ such that*

$$\vec{Z}(x,t) \in L^\infty\left(0,T; W^{1,4}(\Omega)\right), \ \vec{Z}_t(x,t) \in L^2\left(0,T; L^2(\Omega)\right),$$

and (1) holds in the sense of distribution.

2. Local Smooth Solution

We need the following well known lemmas.

Lemma 2.1. *([8]) Let p be real number and j, m be integers such that $2 \leq p \leq \infty$, $0 \leq j < m$. Then*

$$\|\delta^j u_h\|_p \leq C \|u_h\|_2^{1-\alpha} \left(\|\delta^m u_h\|_2 + \frac{\|u_h\|_2}{(2D)^m} \right)^\alpha,$$

where $u_h = \{\, u_j = u(x_j) \mid j = 0, \pm1, \pm2, \ldots, \pm J \,\}$, $x_j = jh$, $h = 2D/J$, $\alpha = \frac{1}{m}(j + \frac{1}{2} - \frac{1}{p})$, C is a constant which is independent of u_h and h, and

$$\|\delta^k u_h\|_p = \left(\sum_{i=0}^{J-k} \left| \frac{\Delta_+^k u_i}{h^k} \right|^p h \right)^{\frac{1}{p}}, \quad \|\delta^k u_h\|_\infty = \max_{0 \leq j \leq J-k} \left| \frac{\Delta_+^k u_j}{h^k} \right|.$$

Lemma 2.2. *([8]) Let $u_h = \{\, u_j = u(x_j) \mid j = 0, \pm1, \pm2, \ldots, \pm J, \cdots \}$, $v_h = \{\, u_j = v(x_j) \mid j = 0, \pm1, \pm2, \ldots, \pm J, \cdots \}$, and $u_{j+J} = u_j$, $v_{j+J} = v_j$, we have*

$$\text{(i).} \quad \sum_{j=1}^{J} u_j \Delta_- v_j = -\sum_{j=0}^{J-1} v_j \Delta_+ u_j,$$

$$\text{(ii).} \quad \sum_{j=1}^{J} u_j \Delta_+ \Delta_- v_j = -\sum_{j=0}^{J-1} \Delta_+ u_j \Delta_+ v_j,$$

$$\text{(iii).} \ \Delta_+(u_j v_j) = u_{j+1} \Delta_+ v_j + v_j \Delta_+ u_j, \quad \Delta_-(u_j v_j) = u_{j-1} \Delta_- v_j + v_j \Delta_- u_j,$$

where Δ_+, Δ_- denote the forward and backward difference respectively.

To get the existence of local smooth solution of (1)-(2), we apply the differential-difference method. Our aim is to construct the local solution (in time t) of (1)-(2) as limits of sequence $\{\vec{Z}_h\}$ when h tends to zero. We only consider the case $\varepsilon = 1$. Firstly, we establish the following difference-differential system:

$$\frac{d\vec{Z}_j}{dt} = -\vec{Z}_j \times \left(\vec{Z}_j \times \frac{\Delta_-\left(G\left(\frac{\Delta_+ \vec{Z}_j}{h} \right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h} \right) + \vec{Z}_j \times \frac{\Delta_-\left(G\left(\frac{\Delta_+ \vec{Z}_j}{h} \right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h}, \tag{4}$$

$$\vec{Z}_j|_{t=0} = \vec{Z}_{0j} = \vec{Z}_0(jh), \tag{5}$$

$$\vec{Z}_{j+J} = \vec{Z}_j, \tag{6}$$

where $j = 0, \pm 1, \pm 2, \ldots, \pm J, \cdots$, $h = \frac{2D}{J}$, $\vec{Z}_j = \vec{Z}(jh, t)$, $J > 0$.

It is clear that the initial value problem for ordinary differential equations (4)-(6) admits a local smooth solution. For such solution, we shall give some estimates uniformly in h and then get a local smooth solution to problem (1)-(2). In this section we always denote a smooth solution of (4)-(6) by $\vec{Z}_j$, $j = 0, \pm 1, \pm 2, \cdots$.

Lemma 2.3. *If $\vec{Z}_0(x) \in W^{1,4}(\Omega)$, then $\vec{Z}_j(t) \in S^2$ for all t and there are uniform constants $T_0 > 0, C > 0$ independent of h such that*

$$\sup_{0 \le t \le T_0} (\|\delta \vec{Z}_h(t)\|_2 + \|\delta \vec{Z}_h(t)\|_4) \le C. \tag{7}$$

Proof. Firstly, multiplying (4) by $\vec{Z}_j$, we obtain $\vec{Z}_j \cdot \vec{Z}_{jt} = 0$, $j \in \mathbb{Z}$. Then we have

$$|\vec{Z}_j(t)| = |\vec{Z}_{0j}| \equiv 1, \quad j \in \mathbb{Z}. \tag{8}$$

Secondly, it follows from (4) that

$$\frac{d\Delta_+ \vec{Z}_j}{dt} = -\Delta_+ \left(\vec{Z}_j \times \left(\vec{Z}_j \times \frac{\Delta_- \left(G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h} \right) \right) + \Delta_+ \left(\vec{Z}_j \times \frac{\Delta_- \left(G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h} \right). \tag{9}$$

It yields from multiplying (9) by $G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h}$ and then summing over j from 0 to $J - 1$ that

$$\sum_{j=0}^{J-1} G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \cdot \frac{d\Delta_+ \vec{Z}_j}{dt} = -\sum_{j=1}^{J} \left| \vec{Z}_j \times \frac{\Delta_- \left(G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h} \right|^2. \tag{10}$$

Therefore, we get

$$\frac{A}{2} \frac{d}{dt} \|\delta \vec{Z}_h\|_2^2 + \frac{B}{4} \frac{d}{dt} \|\delta \vec{Z}_h\|_4^4 + \sum_{j=1}^{J} \left| \vec{Z}_j \times \frac{\Delta_- \left(G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h} \right|^2 = 0. \tag{11}$$

Then we have

$$\|\delta \vec{Z}_h(t)\|_2 + \|\delta \vec{Z}_h(t)\|_4 + \int_0^t \sum_{j=1}^{J} \left| \vec{Z}_j \times \frac{\Delta_- \left(G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h} \right|^2 \mathrm{d}t \le C(\|\vec{Z}_0(x)\|_{W^{1,4}}). \tag{12}$$

□

Now we turn to get higher order estimates. Firstly, by noticing $\vec{Z}_j(t) \in S^2$, we have from (4) that

$$\frac{d\vec{Z}_j}{dt} = \frac{\Delta_- \left(G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h} - \left(\vec{Z}_j \cdot \frac{\Delta_- \left(G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h} \right) \vec{Z}_j + \vec{Z}_j \times \frac{\Delta_- \left(G\left(\frac{\Delta_+ \vec{Z}_j}{h}\right) \frac{\Delta_+ \vec{Z}_j}{h} \right)}{h}. \tag{13}$$

Then we get the following lemma.

Lemma 2.4. *If $\vec{Z}_0(x) \in H^2(\Omega)$, then there are uniform constants $T_0 > 0, C > 0$ independent of h such that*

$$\sup_{0\le t\le T_0}\left(\|\vec{Z}_{ht}(t)\|_2 + \|\delta^2\vec{Z}_h(t)\|_2 + \|\delta\vec{Z}_h(t)\|_\infty\right) + \int_0^{T_0}\left(\|\delta^3\vec{Z}_h(t)\|_2^2 + \|\delta\vec{Z}_{ht}(t)\|_2^2\right)\mathrm{d}t \le C. \tag{14}$$

Proof. Taking the forward difference Δ_+ of (13), multiplying the resulting equations by $\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}$, and then summing over j from 1 to J, we have

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1}\left|\frac{\Delta_+^2\vec{Z}_j}{h^2}\right|^2 h &= \sum_{j=1}^{J}\frac{\Delta_+\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\\
&\quad-\sum_{j=1}^{J}\Delta_+\left(\left(\vec{Z}_j\cdot\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\vec{Z}_j\right)\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\\
&\quad+\sum_{j=1}^{J}\Delta_+\left(\vec{Z}_j\times\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\\
&= I + II + III.
\end{aligned}\tag{15}$$

It yields from direct calculations, Lemma 2.2 and (8) that

$$\begin{aligned}
I &= \sum_{j=1}^{J}G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\left|\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right|^2 h + 2B\sum_{j=1}^{J}\left|\frac{\Delta_+\vec{Z}_j}{h}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right|^2 h\\
&\quad+B\sum_{j=1}^{J}\left(\left(\frac{\Delta_+^2\vec{Z}_{j-1}}{h^2}\cdot\frac{\Delta_+\Delta_-\vec{Z}_j}{h}\right)+\left(\frac{\Delta_+^2\vec{Z}_j}{h}\cdot\frac{\Delta_+\Delta_-\vec{Z}_{j+1}}{h^2}\right)\right)\cdot\left(\frac{\Delta_+\vec{Z}_j}{h}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right)\\
&\quad+\sum_{j=1}^{J}\Delta_+\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\right)\left(\frac{\Delta_+\Delta_-\vec{Z}_{j+1}}{h^2}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right)\\
&\quad+\sum_{j=1}^{J}\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\right)\left(\frac{\Delta_+^2\vec{Z}_{j-1}}{h^2}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right),\\
II &= -\sum_{j=1}^{J}\Delta_+\left(\vec{Z}_j\cdot\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\left(\vec{Z}_{j+1}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right)\\
&\quad-\sum_{j=1}^{J}\left(\vec{Z}_j\cdot\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\left(\Delta_+\vec{Z}_j\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right)\\
&= \sum_{j=1}^{J}\Delta_+\left(\vec{Z}_j\cdot\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\left(\frac{\Delta_+\vec{Z}_{j+1}+\Delta_-\vec{Z}_{j+1}}{2h}\cdot\frac{\Delta_+\Delta_-\vec{Z}_{j+1}}{h^2}\right)\\
&\quad+\sum_{j=1}^{J}\Delta_+\left(\vec{Z}_j\cdot\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\left(\frac{3\Delta_+\vec{Z}_j+\Delta_-\vec{Z}_j}{2h}\cdot\frac{\Delta_+\Delta_-\vec{Z}_j}{h^2}\right)
\end{aligned}$$

$$-\sum_{j=1}^{J}\left(\vec{Z}_j\cdot\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\left(\Delta_+\vec{Z}_j\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right),$$

$$\begin{aligned}
III &= \sum_{j=1}^{J}\left(\Delta_+\vec{Z}_j\times\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\\
&\quad+\sum_{j=1}^{J}\left(\vec{Z}_{j+1}\times\frac{\Delta_+\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\\
&= \sum_{j=1}^{J}\left(\Delta_+\vec{Z}_j\times\frac{\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}\right)}{h}\right)\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\\
&\quad+2B\sum_{j=1}^{J}\left(\frac{\Delta_+\vec{Z}_j}{h}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^2}\right)\left(\vec{Z}_{j+1}\times\frac{\Delta_+\vec{Z}_j}{h}\right)\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\\
&\quad-\sum_{j=1}^{J}\left(\vec{Z}_{j+1}\times\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right)\cdot\Bigg[\Delta_+\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\right)\frac{\Delta_+\Delta_-\vec{Z}_{j+1}}{h^2}\\
&\quad+\Delta_-\left(G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\right)\frac{\Delta_+^2\vec{Z}_{j-1}}{h^2}+B\left(\frac{\Delta_+^2\vec{Z}_{j-1}}{h}\cdot\frac{\Delta_+\Delta_-\vec{Z}_j}{h^2}\right)\frac{\Delta_+\vec{Z}_j}{h}\\
&\quad+B\left(\frac{\Delta_+^2\vec{Z}_j}{h}\cdot\frac{\Delta_+\Delta_-\vec{Z}_{j+1}}{h^2}\right)\frac{\Delta_+\vec{Z}_j}{h}\Bigg].
\end{aligned}$$

Note that we can estimate the second term on the R.H.S. of III by the Cauchy inequality and (8)

$$\begin{aligned}
&2B\sum_{j=1}^{J}\left(\frac{\Delta_+\vec{Z}_j}{h}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^2}\right)\left(\vec{Z}_{j+1}\times\frac{\Delta_+\vec{Z}_j}{h}\right)\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\\
\leq\ &\frac{2}{3}B\sum_{j=1}^{J}\left|\frac{\Delta_+\vec{Z}_j}{h}\right|^2\left|\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right|^2 h+\frac{3}{2}B\sum_{j=1}^{J}\left|\frac{\Delta_+\vec{Z}_j}{h}\cdot\frac{\Delta_+^2\Delta_-\vec{Z}_j}{h^3}\right|^2 h. \qquad (16)
\end{aligned}$$

In conclusion, we have from the above calculation, Lemma 2.1, and the Cauchy inequality that

$$\begin{aligned}
&\frac{d}{dt}\|\delta^2\vec{Z}_h\|_2^2+A\|\delta^3\vec{Z}_h\|_2^2\\
\leq\ &\frac{A}{2}\|\delta^3\vec{Z}_h\|_2^2+C\|\delta\vec{Z}_h\|_\infty^2\|\delta^2\vec{Z}_h\|_\infty^2(\|\delta^2\vec{Z}_h\|_2^2+1)+C\left(\|\delta\vec{Z}_h\|_\infty^6+1\right)\left(\|\delta^2\vec{Z}_h\|_2^2+1\right)\\
\leq\ &\frac{3A}{4}\|\delta^3\vec{Z}_h\|_2^2+C\|\delta^2\vec{Z}_h\|_2^8+C, \qquad (17)
\end{aligned}$$

where we have used the following interpolations

$$\|\delta\vec{Z}_h\|_\infty\leq C\|\delta\vec{Z}_h\|_2^{\frac{1}{2}}\left(\|\delta^2\vec{Z}_h\|_2+\frac{\|\delta\vec{Z}_h\|_2}{2D}\right)^{\frac{1}{2}}, \qquad (18)$$

$$\|\delta^2\vec{Z}_h\|_\infty\leq C\|\delta^2\vec{Z}_h\|_2^{\frac{1}{2}}\left(\|\delta^3\vec{Z}_h\|_2+\frac{\|\delta^2\vec{Z}_h\|_2}{2D}\right)^{\frac{1}{2}}. \qquad (19)$$

Following by Gronwall's inequality, we have there exist a $T_0 > 0$ independent of h, such that

$$\sup_{0\le t\le T_0} \|\delta^2 \vec{Z}_h(t)\|_2^2 + \int_0^{T_0} \|\delta^3 \vec{Z}_h(t)\|_2^2 \mathrm{d}t \le C.$$

Finally, by noticing that

$$\Delta_- \left(\frac{G\left(\frac{\Delta_+\vec{Z}_j}{h}\right)\frac{\Delta_+\vec{Z}_j}{h}}{h} \right) = G\left(\frac{\Delta_+\vec{Z}_{j-1}}{h}\right)\frac{\Delta_-\Delta_+\vec{Z}_j}{h^2} + B\left(\frac{\Delta_+\vec{Z}_{j-1}}{h}\cdot\frac{\Delta_-\Delta_+\vec{Z}_j}{h^2}\right)\frac{\Delta_+\vec{Z}_j}{h} + B\left(\frac{\Delta_+\vec{Z}_j}{h}\cdot\frac{\Delta_-\Delta_+\vec{Z}_j}{h^2}\right)\frac{\Delta_+\vec{Z}_j}{h}. \quad (20)$$

Then we get (14) by (13), (18), (19), Lemma 2.3 and the Cauchy inequality. □

By the similar method as in the proof of Lemma 2.3 and Lemma 2.4 and using the induction argument, one gets the following Lemma.

Lemma 2.5. *Let $\vec{Z}_0(x) \in H^k(\Omega)$ $(k \ge 2)$. There are constants $T_0 > 0, C > 0$ independent of h such that*

$$\sup_{0\le t\le T_0} \left\|\delta^{k-2s}\vec{Z}_{ht^s}(t)\right\|_2 + \int_0^{T_0} \left\|\delta^{k+1-2s}\vec{Z}_{ht^s}(t)\right\|_2^2 \mathrm{d}t \le C. \quad (21)$$

From Lemma 2.5, *a priori* estimates for solutions to the differential-difference equation (4)-(6), we conclude that there exists a generic constant $T_0 > 0$ such that problem (1)-(2) admits a smooth solution in $\Omega \times [0, T_0]$ by the standard procedure. This result is stated as follows.

Theorem 2.6. *Let $\vec{Z}_0(x) \in H^k(\Omega)$ $(k \ge 2)$. Then problem (1)-(2) admits at least one local smooth solution $\vec{Z}(x,t)$:*

$$\vec{Z}(x,t) \in \mathcal{F}(T_0) = \left(\bigcap_{s=0}^{[\frac{k}{2}]} W^{s,\infty}\left(0,T_0;H^{k-2s}(\Omega)\right)\right) \bigcap \left(\bigcap_{s=0}^{[\frac{k+1}{2}]} H^s\left(0,T_0;H^{k+1-2s}(\Omega)\right)\right).$$

3. $A > 0$: Global Smooth Solution

In section 2, we have obtained a local smooth solution for (1)-(2). In this section, we intend to prove the existence of global smooth solution to problem (1)-(2) by deriving the global (in time) estimates for given A and B. In the following, we always suppose $\vec{Z}(x,t)$ is a global smooth solution of problem (1)-(2). For $p \ge 1$, denote by $L^p = L^p(\Omega)$ the L^p space with the norm $\|\cdot\|_{L^p}$. For $k \ge 1$ and $p \ge 1$, denote by $W^{k,p} = W^{k,p}(\Omega)$ the Sobolev space whose norm is $\|\cdot\|_{W^{k,p}}$, $H^k = W^{k,2}(\Omega)$.

Lemma 3.1. *Let $\vec{Z}_0(x) \in W^{1,4}(\Omega)$, and suppose that $\vec{Z}(x,t)$ is a global smooth solution of problem (1)-(2). Then for any given $T > 0$, we have*

$$|\vec{Z}(x,t)| = 1, \quad \forall (x,t) \in \Omega \times [0,T]. \quad (22)$$

$$\sup_{0\le t\le T}\left(\frac{A}{2}\|\vec{Z}_x(\cdot,t)\|_{L^2}^2 + \frac{B}{4}\|\vec{Z}_x(\cdot,t)\|_{L^4}^4\right) + \int_0^T \|\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_x(\cdot,t)\|_{L^2}^2 \mathrm{d}t = \frac{A}{2}\|\vec{Z}_{0x}\|_{L^2}^2 + \frac{B}{4}\|\vec{Z}_{0x}\|_{L^4}^4, \quad (23)$$

$$\int_0^T \|\vec{Z}_t(\cdot,t)\|_{L^2}^2 \mathrm{d}t \le A\|\vec{Z}_{0x}\|_{L^2}^2 + B\|\vec{Z}_{0x}\|_{L^4}^4, \quad (24)$$

$$8B^2\int_0^T \||\vec{Z}_x|(\vec{Z}_x\cdot\vec{Z}_{xx})(\cdot,t)\|_{L^2}^2 \mathrm{d}t \le \frac{A}{2}\|\vec{Z}_{0x}\|_{L^2}^2 + \frac{B}{4}\|\vec{Z}_{0x}\|_{L^4}^4. \quad (25)$$

Proof. Multiplying (1) by $\vec{Z}(x,t)$, we have $\vec{Z}(x,t)\cdot\vec{Z}_t(x,t)=0$. This implies (22). Then differentiating (1) with respect to x, multiplying the resulting equation by $G(\vec{Z}_x)\vec{Z}_x$, and then integrating it over Ω , we have

$$\frac{A}{2}\frac{d}{dt}\|\vec{Z}_x\|_{L^2}^2+\frac{B}{4}\frac{d}{dt}\|\vec{Z}_x\|_{L^4}^4+\|\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_x\|_{L^2}^2=0. \tag{26}$$

This implies the basic energy equality

$$\frac{A}{2}\|\vec{Z}_x(\cdot,t\|_{L^2}^2+\frac{B}{4}\|\vec{Z}_x(\cdot,t)\|_{L^4}^4+\int_0^t\|\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_x\|_{L^2}^2\mathrm{d}t=\frac{A}{2}\|\vec{Z}_{0x}\|_{L^2}^2+\frac{B}{4}\|\vec{Z}_{0x}\|_{L^4}^4. \tag{27}$$

Note that

$$\|\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_x\|_{L^2}^2=\|\vec{Z}\times(\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_x)\|_{L^2}^2. \tag{28}$$

This, together with (1), yields that

$$\int_0^t\|\vec{Z}_t\|_{L^2}^2\mathrm{d}t\le A\|\vec{Z}_{0x}\|_{L^2}^2+B\|\vec{Z}_{0x}\|_{L^4}^4. \tag{29}$$

Observing that

$$\begin{aligned}
&\|\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_x\|_{L^2}^2\\
=\;&\|\vec{Z}\times(\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_x)\|_{L^2}^2\\
=\;&\|[\vec{Z}\cdot(G(\vec{Z}_x)\vec{Z}_x)_x]\vec{Z}-(G(\vec{Z}_x)\vec{Z}_x)_x\|_{L^2}^2\\
=\;&\|(G(\vec{Z}_x)\vec{Z}_x)_x\|_2^2-\|\vec{Z}\cdot(G(\vec{Z}_x)\vec{Z}_x)_x\|_{L^2}^2\\
=\;&\int_\Omega[G^2(\vec{Z}_x)|\vec{Z}_{xx}|^2+4B^2|(\vec{Z}_x\cdot\vec{Z}_{xx})\vec{Z}_x|^2+4BG(\vec{Z}_x)|\vec{Z}_x\cdot\vec{Z}_{xx}|^2]\mathrm{d}x-\int_\Omega G^2(\vec{Z}_x)(\vec{Z}\cdot\vec{Z}_{xx})^2\mathrm{d}x\\
\ge\;&\int_\Omega(4B^2|(\vec{Z}_x\cdot\vec{Z}_{xx})\vec{Z}_x|^2+4BG(\vec{Z}_x)|\vec{Z}_x\cdot\vec{Z}_{xx}|^2)\mathrm{d}x\\
\ge\;&\int_\Omega 8B^2|\vec{Z}_x|^2|\vec{Z}_x\cdot\vec{Z}_{xx}|^2\mathrm{d}x.
\end{aligned} \tag{30}$$

Therefore, (25) follows. □

Remark 3.2. *From the proof of (30), we have*

$$\begin{aligned}
&\|\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_x\|_{L^2}^2\\
\ge\;&\int_\Omega G^2(\vec{Z}_x)|\vec{Z}_{xx}|^2\mathrm{d}x-\int_\Omega G^2(\vec{Z}_x)(\vec{Z}\cdot\vec{Z}_{xx})^2\mathrm{d}x\\
\ge\;&\int_\Omega G^2(\vec{Z}_x)|\vec{Z}_{xx}|^2\mathrm{d}x-\int_\Omega G^2(\vec{Z}_x)(\vec{Z}\cdot\vec{Z}_{xx})^2\mathrm{d}x\\
\ge\;&\int_\Omega G^2(\vec{Z}_x)|\vec{Z}_{xx}|^2\mathrm{d}x-2A^2\int_\Omega|\vec{Z}_x|^4\mathrm{d}x-2B^2\int_\Omega|\vec{Z}_x|^8\mathrm{d}x\\
\ge\;&\int_\Omega(A^2+B^2|\vec{Z}_x|^2)|\vec{Z}_{xx}|^2\mathrm{d}x-2A^2\int_\Omega|\vec{Z}_x|^4\mathrm{d}x-2B^2\||\vec{Z}_x|^4\|_{L^\infty}\int_\Omega|\vec{Z}_x|^4\mathrm{d}x.
\end{aligned} \tag{31}$$

It yields from the one dimensional Sobolev embedding $W^{1,1}(\Omega)\hookrightarrow L^\infty(\Omega)$ that

$$\begin{aligned}
\||\vec{Z}_x|^4\|_{L^\infty}&\le c_s\||\vec{Z}_x|^4\|_{L^1}+c_s\|(|\vec{Z}_x|^4)_x\|_{L^1}\\
&\le c_s\|\vec{Z}_x\|_{L^4}^4+2c_s\||\vec{Z}_x|^2(\vec{Z}_x\cdot\vec{Z}_{xx})\|_{L^1}\\
&\le c_s\|\vec{Z}_x\|_{L^4}^4+2c_s\|\vec{Z}_x\|_{L^2}\||\vec{Z}_x|(\vec{Z}_x\cdot\vec{Z}_{xx})\|_{L^2},
\end{aligned} \tag{32}$$

where c_s is a Sobolev constant. If we assume that $A < 1$, then by using Lemma 3.1, (31) and (32), we can conclude that

$$A^2 \int_0^T \|\vec{Z}_{xx}(\cdot,t)\|_{L^2}^2 \mathrm{d}t \le C(B, c_s, \|\vec{Z}_{0x}\|_{W^{1,4}}, T), \tag{33}$$

where the generic constant $C(B, c_s, \|\vec{Z}_{0x}\|_{W^{1,4}})$ is independent of A. This remark will be useful in the next section for the sake of establishing the global existence of weak solutions when $A \to 0$.

In the following lemmas, denote by C the uniform constants depending on A and B, but independent of $\vec{Z}(x,t)$.

Lemma 3.3. *Let $\vec{Z}_0(x) \in H^2(\Omega)$, and suppose that $\vec{Z}(x,t)$ is a global smooth solution of problem (1)-(2). Then for any given $T > 0$, we have*

$$\sup_{0\le t\le T}(\|\vec{Z}_t(\cdot,t)\|_{L^2} + \|\vec{Z}_{xx}(\cdot,t)\|_{L^2} + \|\vec{Z}_x(\cdot,t)\|_{L^\infty} + \|(G(\vec{Z}_x)\vec{Z}_x)_x(\cdot,t)\|_{L^2}) + \int_0^T (\|\vec{Z}_{xt}(\cdot,t)\|_{L^2}^2 + \|\vec{Z}_{xxx}(\cdot,t)\|_{L^2}^2)\mathrm{d}t \le C. \tag{34}$$

Proof. Differential (3) with respect to t, then multiplying the resulting equation by $\vec{Z}_t$ and integrating it over Ω, one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\vec{Z}_t|^2 \mathrm{d}x = \int_\Omega (G(\vec{Z}_x)\vec{Z}_x)_{xt}\cdot\vec{Z}_t \mathrm{d}x + \int_\Omega G(\vec{Z}_x)|\vec{Z}_x|^2|\vec{Z}_t|^2\mathrm{d}x + \int_\Omega (\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_{xt})\cdot\vec{Z}_t\mathrm{d}x. \tag{35}$$

We get from the first term on the right-hand side of (35) that

$$\begin{aligned}
&\int_\Omega (G(\vec{Z}_x)\vec{Z}_x)_{xt}\cdot\vec{Z}_t\mathrm{d}x \\
=\;& -\int_\Omega (G(\vec{Z}_x)\vec{Z}_x)_t\cdot\vec{Z}_{xt}\mathrm{d}x \\
=\;& -\int_\Omega [2B(\vec{Z}_x\cdot\vec{Z}_{xt})\vec{Z}_x + (A+B|\vec{Z}_x|^2)\vec{Z}_{xt}]\cdot\vec{Z}_{xt}\mathrm{d}x \\
=\;& -2B\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xt}|^2\mathrm{d}x - A\int_\Omega |\vec{Z}_{xt}|^2\mathrm{d}x - B\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xt}|^2\mathrm{d}x.
\end{aligned} \tag{36}$$

For the second term on the right-hand side of (35), we have

$$\int_\Omega G(\vec{Z}_x)|\vec{Z}_x|^2|\vec{Z}_t|^2\mathrm{d}x \le (A\||\vec{Z}_x|^2\|_{L^\infty} + B\||\vec{Z}_x|^2\|_{L^\infty}^2)\|\vec{Z}_t\|_{L^2}^2 \le C(1+\||\vec{Z}_x|(\vec{Z}_x\cdot\vec{Z}_{xx})\|_{L^2}^2)\|\vec{Z}_t\|_{L^2}^2, \tag{37}$$

where we have used Lemma 3.1, (32) and $\||\vec{Z}_x|^2\|_{L^\infty} \le C\||\vec{Z}_x|^4\|_{L^\infty}^{\frac{1}{2}}$.

We finally deal with the third term as follows,

$$\begin{aligned}
&\int_\Omega [\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_{xt}]\cdot\vec{Z}_t\mathrm{d}x \\
=\;& -\int_\Omega [\vec{Z}\times(G(\vec{Z}_x)\vec{Z}_x)_t]\cdot\vec{Z}_{xt}\mathrm{d}x - \int_\Omega [\vec{Z}_x\times(G(\vec{Z}_x)\vec{Z}_x)_t]\cdot\vec{Z}_t\mathrm{d}x \\
=\;& -2B\int_\Omega (\vec{Z}_x\cdot\vec{Z}_{xt})(\vec{Z}\times\vec{Z}_x)\cdot\vec{Z}_{xt}\mathrm{d}x - \int_\Omega G(\vec{Z}_x)(\vec{Z}_x\times\vec{Z}_{xt})\cdot\vec{Z}_t\mathrm{d}x \\
\le\;& \frac{2B}{3}\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xt}|^2\mathrm{d}x + \frac{3B}{2}\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xt}|^2\mathrm{d}x + \frac{B}{6}\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xt}|^2 + C\int_\Omega G^2(\vec{Z}_x)|\vec{Z}_t|^2\mathrm{d}x \\
=\;& \frac{5B}{6}\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xt}|^2\mathrm{d}x + \frac{3B}{2}\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xt}|^2\mathrm{d}x + C(1+\||\vec{Z}_x|^4\|_{L^\infty})\int_\Omega |\vec{Z}_t|^2\mathrm{d}x
\end{aligned}$$

$$\leq \frac{5B}{6}\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xt}|^2\mathrm{d}x + \frac{3B}{2}\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xt}|^2\mathrm{d}x + C\left(1+\left\||\vec{Z}_x|(\vec{Z}_x\cdot\vec{Z}_{xx})\right\|_{L^2}^2\right)\int_\Omega |\vec{Z}_t|^2\mathrm{d}x. \tag{38}$$

In conclusion, we have

$$\frac{d}{dt}\|\vec{Z}_t\|_{L^2}^2 + A\int_\Omega |\vec{Z}_{xt}|^2\mathrm{d}x + B\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xt}|^2\mathrm{d}x \leq C(\||\vec{Z}_x|(\vec{Z}_x\cdot\vec{Z}_{xx})\|_{L^2}^2+1)\|\vec{Z}_t\|_{L^2}^2. \tag{39}$$

Then following by Gronwall's inequality and (25), we get

$$\sup_{0\leq t\leq T}\|\vec{Z}_t\|_{L^2} + A\int_0^T\int_\Omega |\vec{Z}_{xt}|^2\mathrm{d}x\mathrm{d}t + B\int_0^T\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xt}|^2\mathrm{d}x\mathrm{d}t \leq C. \tag{40}$$

Multiply (3) by $\vec{Z}_{xx}$, and then integrating it over Ω, we have

$$\begin{aligned}
& 2B\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xx}|^2\mathrm{d}x + \int_\Omega (A+B|\vec{Z}_x|^2)|\vec{Z}_{xx}|^2\mathrm{d}x \\
= & \int_\Omega \vec{Z}_t\cdot\vec{Z}_{xx}\mathrm{d}x + \int_\Omega G(\vec{Z}_x)|\vec{Z}_x|^4\mathrm{d}x - 2B\int_\Omega (\vec{Z}_x\cdot\vec{Z}_{xx})(\vec{Z}\times\vec{Z}_x)\cdot\vec{Z}_{xx}\mathrm{d}x \\
\leq & \frac{A}{2}\int_\Omega |\vec{Z}_{xx}|^2\mathrm{d}x + C\int_\Omega |\vec{Z}_t|^2\mathrm{d}x + \||\vec{Z}_x|^4\|_{L^\infty}\int_\Omega (A+B|\vec{Z}_x|^2)\mathrm{d}x \\
& +\frac{3B}{2}\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xx}|^2\mathrm{d}x + \frac{2B}{3}\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xx}|^2\mathrm{d}x \\
\leq & \frac{A}{2}\int_\Omega |\vec{Z}_{xx}|^2\mathrm{d}x + C + C\||\vec{Z}_x|^2(\vec{Z}_x\cdot\vec{Z}_{xx})\|_{L^1} + \frac{3B}{2}\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xx}|^2\mathrm{d}x + \frac{2B}{3}\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xx}|^2\mathrm{d}x \\
\leq & \frac{A}{2}\int_\Omega |\vec{Z}_{xx}|^2\mathrm{d}x + C + C\int_\Omega |\vec{Z}_x|^4\mathrm{d}x + \frac{5B}{3}\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xx}|^2\mathrm{d}x + \frac{2B}{3}\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xx}|^2\mathrm{d}x \\
\leq & \frac{A}{2}\int_\Omega |\vec{Z}_{xx}|^2\mathrm{d}x + C + \frac{5B}{3}\int_\Omega |\vec{Z}_x\cdot\vec{Z}_{xx}|^2\mathrm{d}x + \frac{2B}{3}\int_\Omega |\vec{Z}_x|^2|\vec{Z}_{xx}|^2\mathrm{d}x.
\end{aligned} \tag{41}$$

Then we conclude that $\sup\limits_{0\leq t\leq T}\|\vec{Z}_{xx}(\cdot,t)\|_2 \leq C$, and then the estimate about $\|\vec{Z}_x\|_\infty$ follows by one dimensional Sobolev embedding.

Finally, multiplying (3) by $(G(\vec{Z}_x)\vec{Z}_x)_x$, and then integrating over Ω, we have

$$\begin{aligned}
\int_\Omega \vec{Z}_t\cdot(G(\vec{Z}_x)\vec{Z}_x)_x\mathrm{d}x &= \int_\Omega |(G(\vec{Z}_x)\vec{Z}_x)_x|^2\mathrm{d}x - \int_\Omega G(\vec{Z}_x)^2|\vec{Z}_x|^4\mathrm{d}x \\
&= \int_\Omega |(G(\vec{Z}_x)\vec{Z}_x)_x|^2\mathrm{d}x - \int_\Omega A^2\left(|\vec{Z}_x|^4 + B^2|\vec{Z}_x|^8 + 2AB|\vec{Z}_x|^6\right)\mathrm{d}x.
\end{aligned} \tag{42}$$

Then (34) follows from the interpolation inequalities. □

Using the mathematical induction, we have the following Lemma.

Lemma 3.4. *Let $\vec{Z}_0(x)\in H^k(\Omega)$, $k\geq 2$ and suppose that $\vec{Z}(x,t)$ is a global smooth solution of problem (1)-(2) . Then for any given $T>0$, there is $C>0$ such that*

$$\sup_{0\leq t\leq T}\|\partial_t^s\partial_x^{k-2s}\vec{Z}(\cdot,t)\|_{L^2} + \int_0^T \|\partial_t^s\partial_x^{k+1-2s}\vec{Z}(\cdot,t)\|_{L^2}^2\mathrm{d}t \leq C, \quad 0\leq s\leq [k/2]. \tag{43}$$

Combining the local existence obtained in section 2 and the global in time estimates in Lemma 3.4, we can get the existence of global smooth solution to problem (1)-(2) in the following sense.

Theorem 3.5. *Let $\vec{Z}_0(x) \in H^k(\Omega)$, $k \geq 2$. Then for any given $T > 0$, problem (1)-(2) admits at least one global smooth solution $\vec{Z}(x,t)$:*

$$\vec{Z}(x,t) \in \mathcal{F}(T) = \left(\bigcap_{s=0}^{[\frac{k}{2}]} W^{s,\infty}\left(0,T;H^{k-2s}(\Omega)\right)\right)\bigcap\left(\bigcap_{s=0}^{[\frac{k+1}{2}]} H^{s}\left(0,T;H^{k+1-2s}(\Omega)\right)\right).$$

The uniqueness of the global smooth solution can be proved by standard discussion.

4. $A = 0$: Global Weak Solution

Lemma 4.1. *([1]) Let $p \geq 2$. Then there holds for all $a, b \in \mathbb{R}^k$*

$$\left(|a|^{p-2}a - |b|^{p-2}b\right)\cdot(a-b) \geq 2^{p-2}|a-b|^p. \tag{44}$$

Proof of Theorem 1.2 For any $\vec{Z}_0(x) \in W^{1,4}(\Omega)$, we can construct a approximate sequence $\{\vec{Z}_0^{(k)}(x)\}_{k=1}^{\infty}$ such that $\vec{Z}_0^{(k)}(x) \in H^k(\Omega)$ ($k \geq 2$) and $\vec{Z}_0^{(k)}(x) \to \vec{Z}_0(x)$ in $W^{1,4}(\Omega)$. Suppose that $\{\vec{Z}^{(k)}(x,t)\}_{k=1}^{\infty}$ is the sequence of regular solutions corresponding to the initial data $\{\vec{Z}_0^{(k)}(x)\}_{k=1}^{\infty}$. Then by Lemma 3.1 in Section three, we have from the estimates uniform in the parameter A that

$$\{\vec{Z}^{(k)}(x,t)\}_{k=1}^{\infty} \text{ is a bounded set in } L^{\infty}(0,T;W^{1,4}(\Omega)), \tag{45}$$

$$\{\partial_t\vec{Z}^{(k)}(x,t)\}_{k=1}^{\infty} \text{ is a bounded set in } L^{2}(0,T;L^{2}(\Omega)). \tag{46}$$

Then one can pass to a subsequence, without changing notation, to get that as $k \to \infty$,

$$\vec{Z}^{(k)} \rightharpoonup \vec{Z} \text{ weakly}* \text{ in } L^{\infty}(0,T;W^{1,4}(\Omega)), \tag{47}$$

$$\vec{Z}_t^{(k)} \rightharpoonup \vec{Z}_t \text{ weakly in } L^{2}(0,T;L^{2}(\Omega)). \tag{48}$$

By the compactness argument, one have

$$\vec{Z}^{(k)} \to \vec{Z} \text{ strongly in } C(Q_T),\ Q_T = \Omega\times[0,T]. \tag{49}$$

Now we claim that

$$\vec{Z}_x^{(k)} \to \vec{Z}_x \text{ strongly in } L^4(Q_T). \tag{50}$$

In fact, by Lemma 4.1, we have

$$\begin{aligned}\int_{Q_T}|\vec{Z}_x^{(k)}-\vec{Z}_x|^4\mathrm{d}x\mathrm{d}t &\leq \frac{1}{4}\int_{Q_T}\left(|\vec{Z}_x^{(k)}|^2\vec{Z}_x^{(k)}-|\vec{Z}_x|^2\vec{Z}_x\right)\cdot\left(\vec{Z}_x^{(k)}-\vec{Z}_x\right)\mathrm{d}x\mathrm{d}t\\ &= \frac{1}{4}\int_{Q_T}|\vec{Z}_x^{(k)}|^2\vec{Z}_x^{(k)}\cdot\left(\vec{Z}_x^{(k)}-\vec{Z}_x\right)-\frac{1}{4}\int_{Q_T}|\vec{Z}_x|^2\vec{Z}_x\cdot\left(\vec{Z}_x^{(k)}-\vec{Z}_x\right)\mathrm{d}x\mathrm{d}t\\ &= I_1+I_2.\end{aligned} \tag{51}$$

Note that $|\vec{Z}_x|^2\vec{Z}_x \in L^{\infty}(0,T;L^{\frac{4}{3}})$ and $L^{\frac{4}{3}} = (L^4)^*$ (the dual space of L^4) and by the weak convergence of $\{\vec{Z}^{(k)}(x,t)\}_{k=1}^{\infty}$, we have

$$I_2 \to 0,\ k\to\infty. \tag{52}$$

On the other hand, recalling (3), Lemma 3.1 and Remark 3.2, we get

$$I_1 = -\frac{1}{4}\langle\left(|\vec{Z}_x^{(k)}|^2\vec{Z}_x^{(k)}\right)_x, \vec{Z}^{(k)}-\vec{Z}\rangle$$

$$
\begin{aligned}
&= -\frac{1}{4}\langle\Big(|\vec{Z}_x^{(k)}|^2\vec{Z}_x^{(k)}+\frac{A}{B}\vec{Z}_x^{(k)}\Big)_x,\vec{Z}^{(k)}-\vec{Z}\rangle+\frac{1}{4}\langle\frac{A}{B}\vec{Z}_{xx}^{(k)},\vec{Z}^{(k)}-\vec{Z}\rangle\\
&= -\frac{1}{4B}\langle\vec{Z}_t^{(k)}-G(\vec{Z}_x^{(k)})|\vec{Z}_x^{(k)}|^2\vec{Z}^{(k)}-\vec{Z}^{(k)}\times(G(\vec{Z}_x^{(k)})\vec{Z}_x^{(k)})_x,\vec{Z}^{(k)}-\vec{Z}\rangle+\frac{1}{4}\langle\frac{A}{B}\vec{Z}_{xx}^{(k)},\vec{Z}^{(k)}-\vec{Z}\rangle\\
&\leq C_B\|\vec{Z}^{(k)}-\vec{Z}\|_{C(Q_T)}\Big(\|\vec{Z}_t^{(k)}\|_{L^2(Q_T)}+\|G(\vec{Z}_x^{(k)})|\vec{Z}_x^{(k)}|^2\|_{L^1(Q_T)}+\|\vec{Z}^{(k)}\times(G(\vec{Z}_x^{(k)})\vec{Z}_x^{(k)})_x\|_{L^2(Q_T)}+A^2\|\vec{Z}_{xx}^{(k)}\|_{L^2(Q_T)}\Big)\\
&\leq C_B\|\vec{Z}^{(k)}-\vec{Z}\|_{C(Q_T)}\to 0,\ k\to\infty, \qquad (53)
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in $L^2(Q_T)$ space. Thus we finish the proof of the claim.

Finally, by the convergence (47)-(50), we can easily conclude that (1) holds in the sense of distribution. □

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