



Stochastic Periodic Solutions of Stochastic Periodic Differential Equations

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Abstract. In this paper, the periodic stochastic differential equations are studied. By applying the theory of Lyapunov's second method, contraction mapping principle and establishing new lemmas, the existence and uniqueness of stochastic periodic solutions to stochastic periodic differential equations are obtained. Moreover, several examples are introduced to illustrate our theoretical results.

1. Introduction

Periodic movement is very universal in nature, for example, celestial bodies motion, the wave vibration, climate changes in four seasons, etc. And these periodic movements may be modeled by differential systems. So investigating the periodic solution is of great significance. An important aspect is to ensure whether there exist periodic solutions for the system. The existence of periodic solutions of ordinary and functional differential equations has been discussed extensively. Readers can see the books [1–3], papers [4–10] and the references therein. There are many methods to study the existence of periodic solutions, for example, coincidence degree, fixed point theorem, bifurcation theory and the method of Lyapunov function, etc.

However, in the practical case, the systems are often subject to stochastic perturbation. So stochastic differential equations (SDEs) have attracted great interests due to their applications in many ways such as in insurance, finance, population dynamics, control and so on (see, e.g.[11]). Many researchers have studied qualitative properties such as existence, uniqueness, boundedness and stability for various stochastic differential systems, for instance [12–21]. However, as far as we know, there are few papers on periodic stochastic differential equation. R. Z. Hasminskii [22] gave the definition of periodic Markov process and discussed the sufficient and necessary conditions for the existence of periodic Markov process. In [23],

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Zhao and Zheng presented the definition of random periodic solutions of random dynamical systems. They proved the existence of such periodic solutions for a C^1 perfect cocycle on a cylinder using a random invariant set, the Lyapunov exponents and the pullback of the cocycle. In [24, 25], Jiang et al. showed that $\mathbb{E}[1/x(t)]$ has a unique positive T -periodic solution $\mathbb{E}[1/x^p(t)]$ for the stochastic periodic Logistic model, where $x(t)$ is the population size at time t . In [26], Wang and Hu proved that stochastic periodic Logistic model is asymptotic stable in distribution and gave the explicit stochastic periodic solution to this model.

In this paper, adopting the definition of periodic stochastic process in [26], we deal with the existence and uniqueness of stochastic periodic solutions to stochastic periodic differential equations in the abstract form

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB_t \quad (1)$$

on $t \geq 0$ with initial value $X(0) = x$, where the coefficients $f(t, x) : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g(t, x) : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are both Borel measurable, \mathcal{F}_t -adapted and satisfy

$$f(t + T, x) = f(t, x), \quad g(t + T, x) = g(t, x)$$

for all $x \in \mathbb{R}^n$ and some $T > 0$, B_t is a one-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a σ -field filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The main technique is based upon contraction mapping principle and Lyapunov's second method. To the best of our knowledge to date, there are no papers which study stochastic periodic solutions to stochastic periodic differential equations by applying contraction mapping principle, so the significance of our paper is clear.

Throughout the paper, $\mathbb{R}_+ := [0, \infty)$. If A is a matrix, $A > 0$: A is a symmetric positive-definite matrix, $\lambda_{\min}(A)$: the smallest eigenvalue of a symmetric matrix A , $\lambda_{\max}(A)$: the largest eigenvalue of a symmetric matrix A . For $p \in (0, \infty)$, let $L^p(\Omega; \mathbb{R}^n)$ be the family of \mathbb{R}^n -valued random variables X with $\mathbb{E}|X|^p < \infty$. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of all non-negative functions on $\mathbb{R}^n \times \mathbb{R}_+$ which are continuously twice differential in x and once in t . Let \mathcal{K} denote the family of all continuous increasing functions $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\kappa(0) = 0$ while $\kappa(u) > 0$ for $u > 0$. Let \mathcal{K}_\vee denote the family of all convex functions $\kappa \in \mathcal{K}$ while \mathcal{K}_\wedge denote the family of all concave functions $\kappa \in \mathcal{K}$ [27].

The rest of this paper is organized as follows: In Section 2 we establish some lemmas and preliminary facts which will be used in the sequel. In Section 3 the existence and uniqueness of periodic stochastic solutions to stochastic periodic differential equations are arranged. In Section 4, examples are given to illustrate our main result.

2. Preliminaries

In this section we mainly give some lemmas and preliminary facts which are used in the analysis in what follows. The obtained results and their proofs are motivated by Mao and Yuan [27], Wu and Kloeden [28]. For completeness, we also put them.

Firstly, we assume that the coefficients $f(t, x)$ and $g(t, x)$ of Eq.(1) satisfy the following assumption:

Assumption 1. (Local Lipschitz condition) For each $k = 1, 2, \dots$, and all $t \in [0, \infty)$, $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq k$, there is an $h_k > 0$ such that

$$|f(t, x) - f(t, y)| \vee |g(t, x) - g(t, y)| \leq h_k|x - y|;$$

(Linear growth condition) For all $(t, x) \in [0, \infty) \times \mathbb{R}^n$, there is an $h > 0$ such that

$$|f(t, x)| + |g(t, x)| \leq h(1 + |x|).$$

Under Assumption 1, from [27], we observe that Eq.(1) has a unique continuous solution $X(t)$ on $t \geq 0$. If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, define

$$LV(x, t) = V_t(x, t) + V_x(x, t)f(t, x) + \frac{1}{2}\text{trace}[g^T(t, x)V_{xx}(x, t)g(t, x)],$$

where $V_x = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})$, $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{n \times n}$.

The following lemma gives a criterion on the boundedness of p th moment for the solution.

Lemma 2.1. *Let Assumption 1 hold. If, in addition, there exist functions $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\kappa_1 \in \kappa_V$, $\kappa_2 \in \kappa_\wedge$ and positive numbers λ_1, β such that*

$$\kappa_1(|x|^p) \leq V(x, t) \leq \kappa_2(|x|^p) \quad (2)$$

and

$$LV(x, t) \leq -\lambda_1 V(x, t) + \beta \quad (3)$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$. Then for any initial value $X(0) \in L^p(\Omega; \mathbb{R}^n)$, the p th moment of the solution of Eq.(1) is bounded, say

$$\mathbb{E}|X(t)|^p \leq K \quad \forall t \geq 0, \quad (4)$$

where K is a positive constant.

Proof. For each integer k , define the stopping time

$$\rho_k = \inf\{t \geq 0 : |X(t)| \geq k\}.$$

Clearly, $\rho_k \rightarrow \infty$ almost surely as $k \rightarrow \infty$. Applying Itô's formula, integrating from 0 to t and taking expectations we have

$$\mathbb{E}[e^{\lambda_1(\rho_k \wedge t)} V(X(\rho_k \wedge t), \rho_k \wedge t)] = \mathbb{E}V(X(0), 0) + \mathbb{E} \int_0^{\rho_k \wedge t} e^{\lambda_1 s} LV(X(s), s) ds + \lambda_1 \mathbb{E} \int_0^{\rho_k \wedge t} e^{\lambda_1 s} V(X(s), s) ds.$$

By combining conditions (2) and (3), it follows that

$$\mathbb{E}[e^{\lambda_1(\rho_k \wedge t)} \kappa_1(|X(\rho_k \wedge t)|^p)] \leq \mathbb{E}(\kappa_2(|X(0)|^p)) + \mathbb{E} \int_0^{\rho_k \wedge t} e^{\lambda_1 s} \beta ds.$$

Letting $k \rightarrow \infty$ results in

$$\mathbb{E}[e^{\lambda_1 t} \kappa_1(|X(t)|^p)] \leq \mathbb{E}(\kappa_2(|X(0)|^p)) + \frac{\beta}{\lambda_1} [e^{\lambda_1 t} - 1].$$

Using Jensen's inequality yields

$$\kappa_1(\mathbb{E}|X(t)|^p) \leq e^{-\lambda_1 t} \kappa_2(\mathbb{E}|X(0)|^p) + \frac{\beta}{\lambda_1} [1 - e^{-\lambda_1 t}].$$

Hence

$$\limsup_{t \rightarrow \infty} \mathbb{E}|X(t)|^p \leq \kappa_1^{-1}\left(\frac{\beta}{\lambda_1}\right).$$

So there is a $M > 0$ such that $\mathbb{E}|X(t)|^p \leq 1.5\kappa_1^{-1}(\frac{\beta}{\lambda_1})$ for all $t \geq M$. At the same time, by the continuity of $\mathbb{E}|X(t)|^p$, it is clearly that there is a $K_0 > 0$ such that $\mathbb{E}|X(t)|^p \leq K_0$ for $t \leq M$. Denote $K = \max\{1.5\kappa_1^{-1}(\frac{\beta}{\lambda_1}), K_0\}$, then we have for all $t \geq 0$, $\mathbb{E}|X(t)|^p \leq K$. This completes the proof.

□

Now we consider the difference between two solutions of Eq.(1) starting from different values, that is

$$X^{x_1}(t) - X^{x_2}(t) = x_1 - x_2 + \int_0^t [f(s, X^{x_1}(s)) - f(s, X^{x_2}(s))] ds + \int_0^t [g(s, X^{x_1}(s)) - g(s, X^{x_2}(s))] dB_s, \quad (5)$$

where $X^{x_1}(t)$ and $X^{x_2}(t)$ denote the solutions of Eq.(1) starting from initial values x_1 and x_2 , respectively. The following lemma will show that $\mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p$ is uniformly continuous on $[0, \infty)$, which will be used later. And the idea for our proof comes from [27, 29].

Lemma 2.2. Suppose all the conditions of Lemma 2.1 hold and $p \geq 2$. Then $\mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p$ is uniformly continuous on the entire $t \in [0, \infty)$.

Proof. By Lemma 2.1, for any $p \geq 2$, we can show that there exist constants K_1, K_2 such that $\mathbb{E}|X^{x_1}(t)|^p \leq K_1$, $\mathbb{E}|X^{x_2}(t)|^p \leq K_2$. By the Itô's formula,

$$\begin{aligned} & \mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p - \mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p \\ &= \mathbb{E} \int_s^t \left[p|X^{x_1}(u) - X^{x_2}(u)|^{p-2} (X^{x_1}(u) - X^{x_2}(u))^T (f(u, X^{x_1}(u)) - f(u, X^{x_2}(u))) \right. \\ & \quad + \frac{p}{2} |X^{x_1}(u) - X^{x_2}(u)|^{p-2} |g(u, X^{x_1}(u)) - g(u, X^{x_2}(u))|^2 \\ & \quad \left. + \frac{p(p-2)}{2} |X^{x_1}(u) - X^{x_2}(u)|^{p-4} |(X^{x_1}(u) - X^{x_2}(u))^T (g(u, X^{x_1}(u)) - g(u, X^{x_2}(u)))|^2 \right] du. \end{aligned}$$

Hence using elementary inequality $(a + b)^p \leq 2^p(|a|^p + |b|^p)$ for $p > 0$ and the linear growth condition we obtain that

$$\begin{aligned} & |\mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p - \mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p| \\ & \leq \mathbb{E} \int_s^t \left[p|X^{x_1}(u) - X^{x_2}(u)|^{p-1} |f(u, X^{x_1}(u)) - f(u, X^{x_2}(u))| \right. \\ & \quad \left. + \frac{p(p-1)}{2} |X^{x_1}(u) - X^{x_2}(u)|^{p-2} |g(u, X^{x_1}(u)) - g(u, X^{x_2}(u))|^2 \right] du \\ & \leq \mathbb{E} \int_s^t \left[hp2^{p-1} (|X^{x_1}(u)|^{p-1} + |X^{x_2}(u)|^{p-1}) (2 + |X^{x_1}(u)| + |X^{x_2}(u)|) \right. \\ & \quad \left. + 2h^2 p(p-1) 2^{p-2} (|X^{x_1}(u)|^{p-2} + |X^{x_2}(u)|^{p-2}) (2 + |X^{x_1}(u)|^2 + |X^{x_2}(u)|^2) \right] du. \end{aligned}$$

From Young inequality we derive that

$$|\mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p - \mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p| \leq C_1 \int_s^t [1 + \mathbb{E}|X^{x_1}(u)|^p + \mathbb{E}|X^{x_2}(u)|^p] du \leq C_1(1 + K_1 + K_2)(t - s),$$

where C_1 is a constant dependent of only p and h . This implies that $\mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p$ is uniformly continuous on the entire $[0, \infty)$. The proof is complete. \square

For a given function $U \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, we define an operator $\mathcal{L}U : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ associated with Eq.(5) by

$$\mathcal{L}U(x, y, t) = U_t(x - y, t) + U_x(x - y, t)[f(t, x) - f(t, y)] + \frac{1}{2} \text{trace}[(g(t, x) - g(t, y))^T U_{xx}(x - y, t)(g(t, x) - g(t, y))].$$

Lemma 2.3. Let the conditions of Lemma 2.1 hold and $p \geq 2$. Assume further that there are functions $U \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\kappa_3 \in \mathcal{K}_\wedge$ and $\kappa_4 \in \mathcal{K}_\vee$ such that

$$U(x, t) \leq \kappa_3(|x|^p) \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \tag{6}$$

and

$$\mathcal{L}U(x, y, t) \leq -\kappa_4(|x - y|^p) \quad \forall (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+. \tag{7}$$

If initial values x_1 and x_2 belong to $L^p(\Omega; \mathbb{R}^n)$, then

$$\lim_{t \rightarrow \infty} \mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p = 0. \tag{8}$$

Proof. Let N be a positive number. Define the stopping time

$$\tau_N = \inf\{t \geq 0 : |X^{x_1}(t) - X^{x_2}(t)| \geq N\}.$$

Clearly, $\tau_N \rightarrow \infty$. Setting $t_N = \tau_N \wedge t$ and applying Itô's formula to $U(X^{x_1}(t_N) - X^{x_2}(t_N), t_N)$ yields

$$\mathbb{E}U(X^{x_1}(t_N) - X^{x_2}(t_N), t_N) = \mathbb{E}U(x_1 - x_2, 0) + \mathbb{E} \int_0^{t_N} \mathcal{L}U(X^{x_1}(s), X^{x_2}(s), s) ds.$$

So by conditions (6), (7) and then letting $N \rightarrow \infty$, we have

$$0 \leq \mathbb{E}(\kappa_3(|x_1 - x_2|^p)) - \mathbb{E} \int_0^t \kappa_4(|X^{x_1}(s) - X^{x_2}(s)|^p) ds.$$

Using Jensen's inequality results in

$$\int_0^t \kappa_4(\mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p) ds \leq \kappa_3(\mathbb{E}|x_1 - x_2|^p) < \infty. \quad (9)$$

We now claim $\lim_{t \rightarrow \infty} \mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p = 0$. If this assertion is not true, then there is some $\varepsilon > 0$ and a sequence $\{t_n\}_{n \geq 1}$ satisfying $0 \leq t_n \leq t_{n+1} \leq t_{n+1} + 1 \leq t_{n+1}$ such that

$$\mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p \geq \varepsilon, \quad n \geq 1.$$

From Lemma 2.2, there is a positive constant C such that $|\mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p - \mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p| \leq C(t - s)$. Let $\delta = 1 \wedge (\varepsilon/2C)$, then, for $t_n \leq s \leq t_n + \delta$, we can get

$$\begin{aligned} \mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p &\geq \mathbb{E}|X^{x_1}(t_n) - X^{x_2}(t_n)|^p - |\mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p - \mathbb{E}|X^{x_1}(t_n) - X^{x_2}(t_n)|^p| \\ &\geq \varepsilon - C(s - t_n) \geq \varepsilon - C\delta \geq \frac{\varepsilon}{2}. \end{aligned}$$

Consequently

$$\int_0^\infty \kappa_4(\mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p) ds \geq \sum_{n=1}^\infty \int_{t_n}^{t_n+\delta} \kappa_4(\mathbb{E}|X^{x_1}(s) - X^{x_2}(s)|^p) ds \geq \sum_{n=1}^\infty \int_{t_n}^{t_n+\delta} \kappa_4\left(\frac{\varepsilon}{2}\right) ds = \infty.$$

But this is in contradiction with (9). So

$$\lim_{t \rightarrow \infty} \mathbb{E}|X^{x_1}(t) - X^{x_2}(t)|^p = 0.$$

□

Lemma 2.4. [30] For $X_1, X_2 \in L^p(\Omega; \mathbb{R}^n)$ ($p \geq 2$), define a metric

$$d(X_1, X_2) = (\mathbb{E}|X_1 - X_2|^p)^{\frac{1}{p}},$$

then $L^p(\Omega; \mathbb{R}^n)$ is a complete metric space.

3. Stochastic Periodic Solution

In this section, we present and prove our main theorem. First we state the definition of periodic stochastic process and stochastic periodic solution.

Definition 3.1. [26] Stochastic process $f(t)$ ($t \geq 0$) is said to be T -periodic stochastic process, if stochastic processes $g(t) := f(t + T)$ ($t \geq 0$) and $f(t)$ ($t \geq 0$) have the same finite-dimensional distributions.

Definition 3.2. [26] If $X(t)$ is a solution of Eq.(1) and $X(t)$ is a T -periodic stochastic process, then $X(t)$ is said to be a stochastic periodic solution with period T of Eq.(1).

Theorem 3.3. Assume that the conditions of Lemmas 2.1 and 2.3 are all satisfied, then Eq.(1) admits a unique T -periodic stochastic periodic solution.

Proof. For arbitrary $x_1, x_2 \in L^p(\Omega; \mathbb{R}^n)$ ($p \geq 2$), from Lemma 2.1, we get that for any $t \in [0, \infty)$, the solutions $X^{x_1}(t), X^{x_2}(t) \in L^p(\Omega; \mathbb{R}^n)$.

Define a mapping $P : L^p(\Omega; \mathbb{R}^n) \rightarrow L^p(\Omega; \mathbb{R}^n)$ as follows: $Px = X^x(T)$, then from Lemma 2.3, for any $\varepsilon \in (0, 1)$, there is a constant $M > 0$ such that for any $m > M$

$$d^p(P^m x_1, P^m x_2) = \mathbb{E}|X^{x_1}(mT) - X^{x_2}(mT)|^p < \varepsilon^p \mathbb{E}|x_1 - x_2|^p = \varepsilon^p d^p(x_1, x_2),$$

that is

$$d(P^m x_1, P^m x_2) < \varepsilon d(x_1, x_2).$$

Therefore, P^m is a contraction mapping on the complete metric space $L^p(\Omega; \mathbb{R}^n)$, and so there exists a unique fixed point $x^* \in L^p(\Omega; \mathbb{R}^n)$ such that $x^* = Px^* = X^{x^*}(T)$.

Now we are in the position to prove that $X^{x^*}(t)$ is the unique T -periodic stochastic periodic solution of Eq.(1). By Assumption 1, $X^{x^*}(t)$ satisfy the following SDE:

$$X^{x^*}(t) = x^* + \int_0^t f(s, X^{x^*}(s))ds + \int_0^t g(s, X^{x^*}(s))dB_s, \quad t \geq 0. \quad (10)$$

Let $t = T$ and $t = t + T$ in equation(10) respectively, we get

$$\begin{aligned} X^{x^*}(T) &= x^* + \int_0^T f(s, X^{x^*}(s))ds + \int_0^T g(s, X^{x^*}(s))dB_s, \\ X^{x^*}(t + T) &= x^* + \int_0^{t+T} f(s, X^{x^*}(s))ds + \int_0^{t+T} g(s, X^{x^*}(s))dB_s. \end{aligned}$$

Let $s = r + T, \tilde{B}_t = B_{t+T} - B_t$, the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is fixed, then

$$\begin{aligned} X^{x^*}(t + T) &= X^{x^*}(T) + \int_T^{t+T} f(s, X^{x^*}(s))ds + \int_T^{t+T} g(s, X^{x^*}(s))dB_s \\ &= X^{x^*}(T) + \int_0^t f(r + T, X^{x^*}(r + T))dr + \int_0^t g(r + T, X^{x^*}(r + T))d\tilde{B}_r. \end{aligned}$$

Note

$$f(t + T, x) = f(t, x), \quad g(t + T, x) = g(t, x),$$

therefore,

$$X^{x^*}(t + T) = X^{x^*}(T) + \int_0^t f(r, X^{x^*}(r + T))dr + \int_0^t g(r, X^{x^*}(r + T))d\tilde{B}_r, \quad t \geq 0. \quad (11)$$

Hence $(B_t, \{X^{x^*}(t)\}_{t \geq 0})$ and $(\tilde{B}_t, \{X^{x^*}(t + T)\}_{t \geq 0})$ are two weak solutions of Eq.(1) on the same complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$. By the strong property of Brownian motion, $\{B_t\}_{t \geq 0}$ and $\{\tilde{B}_t\}_{t \geq 0}$ have the same distribution. Notice that the solution for SDE (10) is a pathwise unique strong solution, moreover SDEs (10) and (11) have the same formation, hence there is a measurable function F such that

$$X^{x^*}(t) = F(B_s; \forall s \leq t), \quad \mathcal{P} - a.s.$$

This means that for (11) we must also have

$$X^{x^*}(t+T) = F(\widetilde{B}_s; \forall s \leq t), \mathcal{P} - a.s.$$

So $\forall A \in (\mathbb{R}^n)^{\otimes d}$, and $\forall t_1, \dots, t_d \in [0, \infty)$,

$$\mathcal{P}((X^{x^*}(t_1), \dots, X^{x^*}(t_d)) \in A) = \mathcal{P}((X^{x^*}(t_1+T), \dots, X^{x^*}(t_d+T)) \in A),$$

moreover, initial values $X^{x^*}(T)$ and x^* have the same distribution, hence we can get that $X^{x^*}(t)$ and $X^{x^*}(t+T)$ have the same finite-dimensional distributions. Since x^* is unique, we know that $X^{x^*}(t)$ is the unique T -periodic stochastic periodic solution of Eq.(1). The proof of Theorem 3.3 is complete. \square

Remark 3.4. From the proof of Theorem 3.3, it is easy to see that if the solutions of Eq.(1) has the property (4) and (8), then Eq.(1) admits a unique stochastic periodic solution.

Let us now consider the following special case of Eq.(1):

$$dX(t) = A(t)X(t)dt + g(t, X(t))dB_t, \quad (12)$$

where $A(t)$ is a $n \times n$ continuous matrix, $A(t+T) = A(t)$, and $g(t+T, x) = g(t, x)$. Denote $\mu(A(t)) = \lambda_{\max}(A(t) + A^T(t))$, then $\mu(A(t))$ is a continuous T -periodic function.

Corollary 3.5. Assume that for $x, y \in \mathbb{R}^n$,

$$\begin{aligned} |g(t, x)|^2 &\leq K_0|x|^2 + \alpha, \\ |g(t, x) - g(t, y)|^2 &\leq K_0|x - y|^2, \\ \max_{[0, T]} \mu(A(t)) + K_0 &< 0, \end{aligned}$$

where α is a positive constant. Then Eq.(12) has a unique T -periodic stochastic periodic solution.

Proof. Define $V(x, t) = U(x, t) = q|x|^2$, $q > 0$. Combining the conditions, we compute the operator $LV(x, t)$ associated with Eq.(12) as follows:

$$\begin{aligned} LV(x, t) &= 2qx^T A(t)x + q|g(t, x)|^2 = q(x^T(A(t) + A^T(t))x) + q|g(t, x)|^2 \\ &\leq q\lambda_{\max}(A(t) + A^T(t))|x|^2 + qK_0|x|^2 + q\alpha = q(\mu(A(t)) + K_0)|x|^2 + q\alpha \leq q(\max_{[0, T]} \mu(A(t)) + K_0)|x|^2 + q\alpha. \end{aligned}$$

We see that the conditions of Lemma 2.1 hold.

Moreover, compute the operator $\mathcal{L}U$:

$$\begin{aligned} \mathcal{L}U(x, t) &= 2q(x - y)^T A(t)(x - y) + q|g(t, x) - g(t, y)|^2 \\ &\leq q(\max_{[0, T]} \mu(A(t)) + K_0)|x - y|^2, \end{aligned}$$

so the conditions of Lemma 2.3 are satisfied. Hence Eq.(12) admits a unique T -periodic stochastic periodic solution from Theorem 3.3. \square

4. Examples and Numerical Simulations

To illustrate our main theorem we consider two simple examples in this section.

Example 1. We examine the existence and uniqueness of stochastic periodic solutions for the following stochastic differential equation

$$dX(t) = -(2 + \sin t)X(t)dt + \frac{1}{2} \sin(t/10)dB_t, \quad t \geq 0, \quad (13)$$

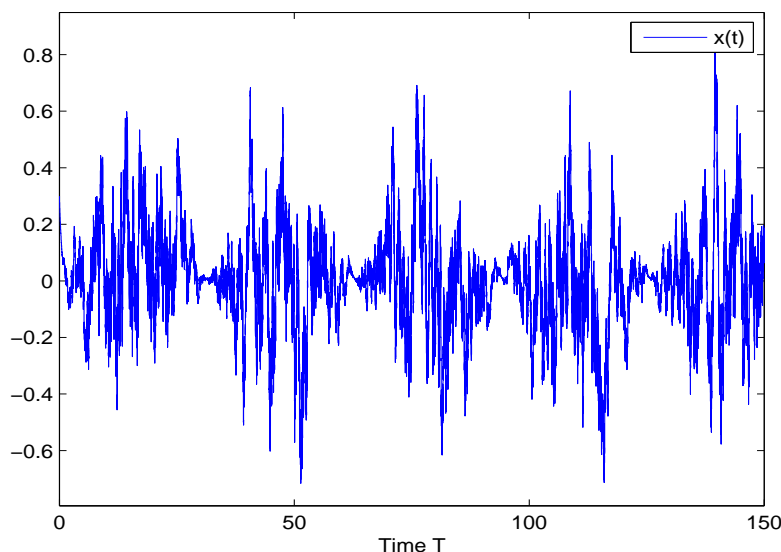


Figure 1: The solutions of Eq.(13)

$f(t, x) = -(2 + \sin t)x$, $g(t, x) = 0.5 \sin(t/10)$ are periodic functions with period 20π and satisfy Assumption 1. Define $V(x, t) = U(x, t) = c|x|^2$, $c > 0$. Compute $LV(x, t)$ associated with Eq.(13) as

$$\begin{aligned} LV(x, t) &= 2cx f(t, x) + cg^2(t, x) = -2c(2 + \sin t)x^2 + \frac{1}{4}c \sin^2(t/10) \\ &\leq -2cx^2 + \frac{1}{4}c. \end{aligned}$$

Similarly, compute $\mathcal{L}U(x, y, t)$ associated with Eq.(13)

$$\begin{aligned} \mathcal{L}U(x, y, t) &= 2c(x - y)(f(t, x) - f(t, y)) + c(g(t, x) - g(t, y))^2 = -2c(2 + \sin t)(x - y)^2 \\ &\leq -2c(x - y)^2. \end{aligned}$$

So we get that the conditions of Lemma 2.1 and Lemma 2.3 are satisfied, an application of Theorem 3.3 yields that Eq.(13) has a unique stochastic periodic solution. Fig. 1 describes the trajectory of the solution to Eq.(13).

Example 2. Consider the following 2-dimensional stochastic periodic differential equation

$$\begin{aligned} dx(t) &= (-5x(t) + y(t) - 0.5)dt + 0.1 \sin(t/8)dB(t), \\ dy(t) &= (x(t) - 8y(t) + 0.1)dt + 0.1 \cos(t/3)dB(t). \end{aligned} \quad (14)$$

Let $x^*(t) = x(t) + 0.1$, then Eq.(14) can be written as

$$\begin{aligned} dx^*(t) &= (-5x^*(t) + y(t))dt + 0.1 \sin(t/8)dB(t), \\ dy(t) &= (x^*(t) - 8y(t))dt + 0.1 \cos(t/3)dB(t). \end{aligned} \quad (15)$$

Denote $A = \begin{pmatrix} -5 & 1 \\ 1 & -8 \end{pmatrix}$, and $g = (0.1 \sin(t/8), 0.1 \cos(t/3))^T$ is a 48π -periodic continuous function. All the eigenvalues of A are negative, according to Corollary 3.5, we choose $K_0 = 0$, $\alpha = 1$, then all the conditions of Corollary 3.5 hold, and Eq.(15) has a unique stochastic periodic solution with period 48π , so Eq.(14) has a unique stochastic periodic solution. We plot the trajectory of the solution to Eq.(14) in Fig. 2.

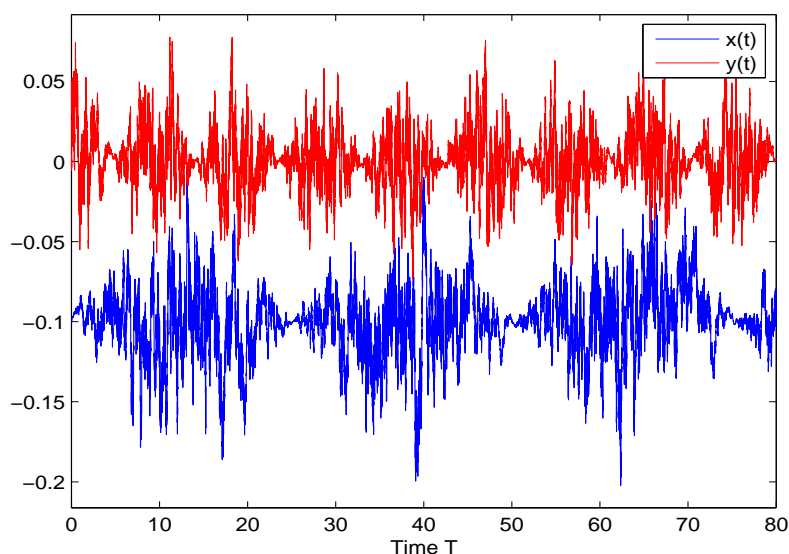


Figure 2: The trajectory of the solutions to Eq.(14) for $x(0) = -0.1$, $y(0) = 0$.

5. Conclusions

In this paper, we study the stochastic periodic solutions of stochastic differential equations with periodic coefficients by applying Lyapunov's second method and contraction mapping principle. From above analysis, we can see that if properties (4) and (8) are satisfied, Eq.(1) has a unique stochastic periodic solution. In section 2, we discuss the sufficient conditions for properties (4) and (8). As we mentioned in the introduction section, investigating periodic solutions for differential equations is an important problem. So it is necessary to study the stochastic periodic solutions to various of stochastic periodic population systems, and we will report our findings in our following papers.

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