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A Sequence of Modular Forms Associated with Higher-Order Derivatives of Weierstrass-Type Functions

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Abstract. In this article, we first determine a sequence $\{f_n(\tau)\}_{n\in\mathbb{N}}$ of modular forms with weight

$$2^n k + 4(2^{n-1} - 1) \qquad (n \in \mathbb{N};\ k \in \mathbb{N} \setminus \{1\};\ \mathbb{N} := \{1, 2, 3, \cdots\}).$$

We then present some applications of this sequence which are related to the Eisenstein series and the cusp forms. We also prove that higher-order derivatives of the Weierstrass type $\wp_{2n}$-functions are related to the above-mentioned sequence $\{f_n(\tau)\}_{n\in\mathbb{N}}$ of modular forms.

1. Introduction, Definitions and Preliminaries

Throughout this paper, we let

$$\mathbb{N} := \{1, 2, 3, \cdots\} \qquad \text{and} \qquad \mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}.$$

We also let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. We shall make use of the following definitions and notations. Let $\mathbb{H}$ denotes the right-half complex plane, that is,

$$\mathbb{H} := \{z : z \in \mathbb{C} \quad \text{and} \quad \Im(z) > 0\}.$$

Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ with $\frac{z_1}{z_2} \notin \mathbb{R}$. Then

$$\Omega = \Omega(z_1, z_2) := \{w : w = m_1 z_1 + m_2 z_2 \qquad (m_1, m_2 \in \mathbb{Z})\}$$

be a lattice in $\mathbb{C}$, $\frac{z_1}{z_2} \in \mathbb{H}$ and

$$\Omega^* = \Omega^*(z_1, z_2) = \Omega(z_1, z_2) \setminus \{0, 0\}.$$

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Then the Weierstrass $\wp$-function is defined by (see, for example, [1], [11] and [20])

$$\wp(u;w) = \frac{1}{u^2} + \sum_{w\in\Omega^*}\left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right). \tag{1.1}$$

Next, if we define the Eisenstein series $G(2k,w)$ of weight $2k$ by the series:

$$G(2k,w) = \sum_{w\in\Omega^*} w^{-2k} \qquad (k \in \mathbb{N} \setminus \{1\}), \tag{1.2}$$

the following relation involving Laurent series holds true between the Eisenstein series $G(2k,w)$ and the Weierstrass $\wp$-function $\wp(u;w)$:

$$\wp(u;w) = \frac{1}{u^2} + \sum_{k=1}^{\infty}(2k+1)G(2k+2,w)u^{2k} \tag{1.3}$$

$$(0 < |u| < \min\{|w| : w \neq 0\}).$$

In Section 2, we shall find it to be convenient to recall a family of Weierstrass-type functions which were introduced and investigated by Chang *et al.* (see, for details, [3] [4]; see also [21]). The above-defined Weierstrass $\wp$-function $\wp(u;w)$ will then turn out to be a special case of the Weierstrass-type functions. Naturally, therefore, our aim will be to find a relationship between the r th derivative of the Weierstrass-type functions and the inhomogeneous Eisenstein series $G_{N,k,a}$ for fixed a, which was defined by Schoeneberg [11]. As a matter of fact, Simsek [15] already gave a *special* relation between the $(k-2)$th derivative of the Weierstass $\wp$-function and the inhomogeneous Eisenstein series $G_{N,k,a}$ (see also [14]).

In Section 3, we derive a general formula involving functional sequences which consist of modular forms with weight

$$2^{n+1}k + 4(2^n - 1) \qquad (n \in \mathbb{N};\ k \in \mathbb{N} \setminus \{1\}).$$

Finally, in Section 4, we consider an interesting special case of the formula (derived in Section 4) which involves functional sequences consisting of modular forms with weight $4k+4$. By using this special case, we obtain Fourier series expansions of modular forms with weights 12 and 16.

2. The Derivatives of the Weierstrass-Type Functions

Following the earlier investigations by Chang *et al.* ([3] and [4]) (see also Wu *et al.* [21]), the Weierstrass-type functions $\wp_{2n}$ are defined by

$$\wp_{2n}(u;w) = \frac{1}{u^{2n}} + \sum_{w\in\Omega^*}\left(\frac{1}{(u-w)^{2n}} - \frac{1}{w^{2n}}\right) \qquad (n \in \mathbb{N}).$$

The function $\wp_{2n}(u;w)$ is a Mittag-Leffler meromorphic function in the complex u-plane and has poles of the second order at the points of Ω.

Recently, Aygunes and Simsek [2] investigated the behavior of the Weierstrass-type functions $\wp_{2n}$ under the Hecke operators.

We now compute the first-, the second- and the third-order derivatives of $\wp_{2n}(u;w)$ as follows:

$$\wp_{2n}^{'}(u;w) = -\frac{2n}{u^{2n+1}} - \sum_{w\in\Omega^*}\frac{2n}{(u-w)^{2n+1}},$$

$$\wp_{2n}^{''}(u;w) = \frac{2n(2n+1)}{u^{2n+2}} + \sum_{w\in\Omega^*}\frac{2n(2n+1)}{(u-w)^{2n+2}}$$

and

$$\wp_{2n}'''(u;w) = -\frac{2n(2n+1)(2n+2)}{u^{2n+3}} - \sum_{w\in\Omega^*} \frac{2n(2n+1)(2n+2)}{(u-w)^{2n+3}},$$

respectively. Here, as usual, the rth derivative of $\wp_{2n}(u;w)$ with respect to the argument u is denoted by $\wp_{2n}^{(r)}(u;w)$. By induction, we obtain the following theorem.

Theorem 1. *Let $n, r \in \mathbb{N} \setminus \{1\}$. Then*

$$\wp_{2n}^{(r)}(u;w) = (-1)^r \frac{(2n+r-1)!}{(2n-1)!} \sum_{w\in\Omega^*} \frac{1}{(u-w)^{2n+r}}. \tag{2.1}$$

Let N and k be natural numbers with $N \geqq 1$ and $k \geqq 3$. Following the notations of Schoeneberg [11], suppose also that $\mathbf{m}$, $\mathbf{a}$ and $\mathbf{z}$ are matrices given by

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

where

$$m_1, m_2, a_1, a_2 \in \mathbb{Z} \quad \text{and} \quad z_1, z_2 \in \mathbb{C}.$$

Also let

$$\mathbf{m}^T = \begin{bmatrix} m_1 & m_2 \end{bmatrix},$$

so that, obviously,

$$w = \mathbf{m}^T\mathbf{z} = \begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = m_1 z_1 + m_2 z_2.$$

Thus, as a function of u, the above definition of the Weierstrass-type function $\wp_{2n}(u;w)$ can be rewritten as follows:

$$\wp_{2n}(u;w) = \frac{1}{u^{2n}} + \sum_{m_1,m_2\in\mathbb{Z}\times\mathbb{Z}}^{*} \left(\frac{1}{(u-\mathbf{m}^T\mathbf{z})^{2n}} - \frac{1}{(\mathbf{m}^T\mathbf{z})^{2n}} \right),$$

where

$$\frac{z_1}{z_2} \in \mathbb{H}$$

and the prime (*) on the above summation sign indicates that the term corresponding to

$$\mathbf{m}^T\mathbf{z} = 0$$

is to be omitted.

For a fixed $\mathbf{a}$, the homogeneous Eisenstein-type series $G_{N,k,\mathbf{a}}$ can be defined by

$$G_{N,k,\mathbf{a}}(\mathbf{z}) = \sum_{\substack{\mathbf{m}\equiv\mathbf{a}(N) \\ (\mathbf{m}\neq\mathbf{0})}} \frac{1}{(\mathbf{m}^T\mathbf{z})^k} \qquad (N \in \mathbb{N};\ k \in \mathbb{N} \setminus \{1,2\}), \tag{2.2}$$

where we have used the notation $\mathbf{m} \equiv \mathbf{a}(N)$ in the following sense:

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \equiv \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \pmod N$$

or, equivalently,

$$m_1 \equiv a_1 \pmod N \qquad \text{and} \qquad m_2 \equiv a_2 \pmod N.$$

Schoeneberg [11, p. 155, Theorem 1] proved that, for natural numbers $N \geqq 1$ and $k \geqq 3$, the Eisenstein series $G_{N,k,\mathbf{a}}(\mathbf{z})$ defined by (2.2) is an entire modular form of level N and weight k. Moreover, the series in (2.2) converges absolutely for all $\mathbf{z}$ (see, for details, [11]). If, in the equation (2.1), we replace u by $\frac{\mathbf{a}^T\mathbf{z}}{N}$ with

$$\mathbf{a} \not\equiv \mathbf{0}(N) \qquad \left(\mathbf{0} := \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right),$$

then we find that

$$\wp_{2n}^{(r)}\left(\frac{\mathbf{a}^T\mathbf{z}}{N}; w\right) = (-1)^r \frac{(2n+r-1)!}{(2n-1)!} \sideset{}{^*}\sum_{m_1,m_2 \in \mathbb{Z}\times\mathbb{Z}} \frac{1}{\left[\frac{a_1z_1+a_2z_2}{N} - (m_1z_1+m_2z_2)\right]^{2n+r}}.$$

Hence we have

$$\wp_{2n}^{(r)}\left(\frac{\mathbf{a}^T\mathbf{z}}{N}; w\right) = (-1)^r N^{2n+r} \frac{(2n+r-1)!}{(2n-1)!} \cdot \sideset{}{^*}\sum_{m_1,m_2 \in \mathbb{Z}\times\mathbb{Z}} \frac{1}{[(a_1+m_1N)z_1 + (a_2+m_2N)z_2]^{2n+r}}.$$

The prime (*) on each of the above the above summation signs indicates that the term corresponding to

$$(a_1+m_1N)z_1 + (a_2+m_2N)z_2 = 0$$

is to be omitted.

By setting

$$\ell_1 = a_1 + m_1N \qquad \text{and} \qquad \ell_2 = a_2 + m_2N$$

in the above equation, we get

$$\wp_{2n}^{(r)}\left(\frac{\mathbf{a}^T\mathbf{z}}{N}; w\right) = (-1)^r N^{2n+r} \frac{(2n+r-1)!}{(2n-1)!} \sum_{w \in \Omega^*} \frac{1}{w^{2n+r}},$$

which readily yields

$$\wp_{2n}^{(r)}\left(\frac{\mathbf{a}^T\mathbf{z}}{N}; w\right) = (-1)^r N^{2n+r} \frac{(2n+r-1)!}{(2n-1)!} \sum_{\mathbf{L}\equiv\mathbf{a}(N)} \frac{1}{(\mathbf{L}^T\mathbf{z})^{2n+r}},$$

where, for convenience,

$$\mathbf{L} := \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{L}^T = \begin{bmatrix} \ell_1 & \ell_2 \end{bmatrix}.$$

We are thus led to the following theorem.

Theorem 2. *Let $n, r \in \mathbb{N} \setminus \{1\}$. If*

$$u = \frac{\mathbf{a}^T\mathbf{z}}{N} \qquad \textit{and} \qquad \mathbf{a} \not\equiv \mathbf{0}(N),$$

then

$$\wp_{2n}^{(r)}\left(\frac{\mathbf{a}^T\mathbf{z}}{N}; w\right) = (-1)^r N^{2n+r} \frac{(2n+r-1)!}{(2n-1)!} G_{N,2n+r,\mathbf{a}}(\mathbf{z}) \qquad (2n+r \in \mathbb{N} \setminus \{1,2\}). \tag{2.3}$$

Remark 1. If we take $n = 1$ in the assertion (2.3) of Theorem 2, then we arrive at the following result (*cf.* [11, p. 157, Eq. (6)]; see also [15]):

$$\wp^{(r)}\left(\frac{\mathbf{a}^T\mathbf{z}}{N};w\right) = (-1)^r(r+1)!N^{r+2}G_{N,r+2,\mathbf{a}}(\mathbf{z}).$$

3. The Set of Main Theorems

By using the derivative of modular forms, it is possible to obtain several useful formulas which can then be applied in order to derive various other modular forms. In recent years, many authors have investigated and studied this subject-area (see, for example, [7], [10] and [16]). In particular, Koblitz [7] gave a formula which leads to the modular form of weight $k+2$ for the following modular group:

$$\Gamma(1) = SL_2(\mathbb{Z})$$

as asserted by Theorem 3 below (see [7] and [16]).

Theorem 3. *Let $f(z)$ be a modular form of weight k for the modular group*

$$\Gamma(1) = SL_2(\mathbb{Z}).$$

If

$$h(z) = \frac{1}{2\pi \mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}\{f(z)\} - \frac{k}{12}E_2(z)f(z),$$

then $h(z)$ is a modular form of weight $k+2$ for the modular group $\Gamma(1)$, where $E_2(z)$ is an Eisenstein series.

Rankin [10] proved the following result.

Theorem 4. (*cf.* [10, p. 114, Theorem 3]) *For an even integer $k \in \mathbb{N}\setminus\{1\}$,*

$$\left(\frac{k+3}{k+1}\right)G_{k+2}(\tau) = 2\frac{\mathrm{d}}{\mathrm{d}\tau}\{G_k(\tau)\} + 2\sum_{v=1}^{\frac{k}{2}}\binom{k}{2v-1}G_{2v}(\tau)G_{k+2-2v}(\tau).$$

Remark 2. For even integer $k \geqq 6$, we get (*cf.* [1] and [10])

$$\left(\frac{(k+3)\,(k-4)}{12k(k-1)}\right)G_{k+2}(\tau) = \sum_{v=1}^{\frac{k}{2}-2}\binom{k-2}{2v}G_{2v+2}(\tau)G_{k-2v}(\tau), \tag{3.1}$$

which can indeed be proven by means of the differential equation satisfied by the Weierstrass $\wp$-function (see, for details, [1] and [10]).

Theorem 5. (*cf.* [10, p. 114, Theorem 3]) *If the integer k is even and greater than 2, then*

$$\begin{aligned}\left(\frac{1}{k+1}\right)&G_k''(\tau) + 4G_2(\tau)G_k'(\tau) - 2kG_2'(\tau)G_k(\tau)\\ &= \frac{1}{3}G_{k-2}'''(\tau) + 2\sum_{1<v<\frac{k}{2}}\binom{k}{2v-1}G_{2v-2}''(\tau)G_{k-2v+2}(\tau) - \sum_{0<v<\frac{k}{2}}\binom{k}{2v}G_{2v}'(\tau)G_{k-2v}'(\tau).\end{aligned}$$

Rankin's form of the *normalized* Eisenstein series can be written as follows (see [10]):

$$G_k(\tau) = -\frac{B_k}{2k}\, E_k(\tau) \qquad (k \in \mathbb{N}), \tag{3.2}$$

where B_k denotes the Bernoulli number of order k (see, for details, [19]; see also the recent investigations [17] and [18]). It is easily seen from (3.2) that

$$E_{2k}(\tau) = \left(2(-1)^k\, \frac{(2\pi)^{2k}}{(2k-1)!}\right) \cdot \frac{1}{2\zeta(2k)}\, G_{2k}(\tau) \qquad (k \in \mathbb{N}) \tag{3.3}$$

in terms of the Riemann zeta function $\zeta(s)$ given by (see, for details, [19])

$$\zeta(s) := \begin{cases} \sum\limits_{n=1}^{\infty} \frac{1}{n^s} = \dfrac{1}{1-2^{-s}} \sum\limits_{n=1}^{\infty} \dfrac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\[2ex] \dfrac{1}{1-2^{1-s}} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big) \end{cases}$$

and

$$\zeta(2k) = \left(\frac{(-1)^{k+1}(2\pi)^{2k}}{2 \cdot (2k)!}\right) B_{2k} \qquad (k \in \mathbb{N}). \tag{3.4}$$

It should be remarked in passing that the factor

$$\left(2(-1)^k\, \frac{(2\pi)^{2k}}{(2k-1)!}\right)$$

is dropped from the right-hand side of (3.3) in the normalization of the Eisenstein series used by many authors including (for example) Apostol [1], Eichler and Zagier [6], Koblitz [7] and Silverman [13].

In this section, we need the following definitions and theorems which will be used to prove Theorem 6. Our Theorem 6 provides us with a general formula answering a question of Silverman [13, p. 90, Exercise 1.20 (a)].

Definition 1. (see [13, p. 24]) Let $k \in \mathbb{Z}$ and let $f(\tau)$ be a function defined on $\mathbb{H}$. We say that f is *weakly modular of weight* $2k$ [for $\Gamma(1)$] if the following two conditions are satisfied:
(i) The function $f(\tau)$ is meromorphic on $\mathbb{H}$;

(ii) The function $f(\tau)$ is such that

$$f(\mathbf{A}\tau) = (c\tau + d)^{2k} f(\tau),$$

where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1), \quad \tau \in \mathbb{H} \qquad \text{and} \qquad A\tau = \frac{a\tau + b}{c\tau + d}.$$

Definition 2. (see [13, p. 24]) A weakly modular function that is meromorphic at ∞ is called a *modular function*. Furthermore, a modular function which is everywhere holomorphic on $\mathbb{H}$ (including also at ∞) is called a modular form. If, in addition, $f(\infty) = 0$, then the function f is called a *cusp form*.

Remark 3. According to the work of Silverman [13, p. 24, Remark 3.3], one can see that

$$f(\tau + 1) = f(\tau) \qquad \text{and} \qquad f\left(-\frac{1}{\tau}\right) = \tau^{2k} f(\tau).$$

Therefore, f is a function of $q := e^{2\pi i\tau}$. This function f is meromorphic in

$$\left\{q : 0 < \left|q\right| < 1\right\}$$

and f has the following Fourier expansion:

$$\sum_{n=-\infty}^{\infty} a_n q^n.$$

Remark 4. If the function f is meromorphic at ∞, then it has the following Fourier expansion:

$$\sum_{n=-n_0}^{\infty} a_n q^n,$$

where n_0 is a positive integer. If the function f is holomorphic at ∞, then it has the following Fourier expansion:

$$\sum_{n=0}^{\infty} a_n q^n.$$

In this section, we are interested in addressing the following question which is raised in the book of Silverman [13, p. 90, Exercise 1.20 (a)]:

Let $f_1(\tau)$ be a modular form of weight $2k$. The prove that

$$f_2(\tau) = (2k+1)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\{f_1(\tau)\}\right)^2 - 2kf_1(\tau)\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\{f_1(\tau)\} \tag{3.5}$$

is a modular form of weight $4k+4$.

Motivated essentially by Silverman's question (3.5), we give our main result in this section, which is stated as Theorem 6 below. For some closely-related earlier works on this subject, the interested reader may be referred to the works by (for example) Cohen [5] and Rankin ([8] *and* [9]). Indeed, by suitably specializing Theorem 6 below, we deduce a number of interesting consequences including (3.2). We also give some examples and corollaries related to Theorem 6.

The *main theorem* of our present investigation can now be stated as follows.

Theorem 6. *Let $n \in \mathbb{N}$ and $\tau \in \mathbb{H}$. Suppose also that $f_1(\tau)$ be a modular form with weight $2k$. Then the sequence $(f_n(\tau))_{n\in\mathbb{N}}$ of modular forms satisfies the following recurrence relation:*

$$f_{n+1}(\tau) = \left[2^n k + 4(2^{n-1}-1)+1\right][f_n'(\tau)]^2 - \left[2^n k + 4(2^{n-1}-1)\right] f_n(\tau) f_n''(\tau),$$

where

$$f_n'(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\{f_n(\tau)\} \qquad \textit{and} \qquad f_n''(\tau) = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\{f_n(\tau)\}.$$

Proof. Since $f_n(\tau)$ is a modular form with weight

$$2^n k + 4(2^{n-1}-1) \qquad (n \in \mathbb{N}),$$

we have

$$f_n\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2^n k + 4(2^{n-1}-1)} f_n(\tau).$$

We now compute the derivative of each member of this last equation with respect to τ to prove that $f_{n+1}(\tau)$ is a modular forms with the following weight:

$$2^{n+1}k + 4(2^n - 1).$$

We thus find that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{f_n\left(\frac{a\tau+b}{c\tau+d}\right)\right\} = \frac{a(c\tau+d)-c(a\tau+b)}{(c\tau+d)^2} f_n'\left(\frac{a\tau+b}{c\tau+d}\right)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{f_n\left(\frac{a\tau+b}{c\tau+d}\right)\right\} = \left[2^n k + 4(2^{n-1}-1)\right] c(c\tau+d)^{2^n k+4(2^{n-1}-1)-1} f_n(\tau) \\ + (c\tau+d)^{2^n k+4(2^{n-1}-1)} f_n'(\tau).$$

Next, since $ad - bc = 1$, we have

$$f_n'\left(\frac{a\tau+b}{c\tau+d}\right) = \left[2^n k + 4(2^{n-1}-1)\right] c(c\tau+d)^{2^n k+4(2^{n-1}-1)+1} f_n(\tau) \\ + (c\tau+d)^{2^n k+4(2^{n-1}-1)+2} f_n'(\tau). \tag{3.6}$$

On the other hand, we easily see that

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{f_n\left(\frac{a\tau+b}{c\tau+d}\right)\right\} = -\frac{2c}{(c\tau+d)^3} f_n'\left(\frac{a\tau+b}{c\tau+d}\right) + \frac{1}{(c\tau+d)^4} f_n''\left(\frac{a\tau+b}{c\tau+d}\right).$$

We thus obtain

$$\begin{aligned}&\frac{d^2}{d\tau^2}\left\{f_n\left(\frac{a\tau+b}{c\tau+d}\right)\right\} \\ &\quad= \left[2^n k + 4(2^{n-1}-1)\right]\left[2^n k + 4(2^{n-1}-1) - 1\right] c^2(c\tau+d)^{2^n k+4(2^{n-1}-1)-2} f_n(\tau) \\ &\qquad + 2c\left[2^n k + 4(2^{n-1}-1)\right](c\tau+d)^{2^n k+4(2^{n-1}-1)-1} f_n'(\tau) \\ &\qquad + (c\tau+d)^{2^n k+4(2^{n-1}-1)} f_n''(\tau).\end{aligned}$$

or, equivalently,

$$\begin{aligned}&-2c(c\tau+d) f_n'\left(\frac{a\tau+b}{c\tau+d}\right) + f_n''\left(\frac{a\tau+b}{c\tau+d}\right) \\ &\quad= \left[2^n k + 4(2^{n-1}-1)\right]\left[2^n k + 4(2^{n-1}-1) - 1\right] c^2(c\tau+d)^{2^n k+4(2^{n-1}-1)+2} f_n(\tau) \\ &\qquad + 2c\left[2^n k + 4(2^{n-1}-1)\right](c\tau+d)^{2^n k+4(2^{n-1}-1)+3} f_n'(\tau) \\ &\qquad + (c\tau+d)^{2^n k+4(2^{n-1}-1)+4} f_n''(\tau).\end{aligned}$$

By using (3.6), we have

$$\begin{aligned}f_n''\left(\frac{a\tau+b}{c\tau+d}\right) &= \left[2^n k + 4(2^{n-1}-1)\right]\left[2^n k + 4(2^{n-1}-1) + 1\right] c^2(c\tau+d)^{2^n k+4(2^{n-1}-1)+2} f_n(\tau) \\ &\quad + 2c\left[2^n k + 4(2^{n-1}-1) + 1\right](c\tau+d)^{2^n k+4(2^{n-1}-1)+3} f_n'(\tau) \\ &\quad + (c\tau+d)^{2^n k+4(2^{n-1}-1)+4} f_n''(\tau).\end{aligned} \tag{3.7}$$

Moreover, by using (3.6) and (3.7), we get

$$\begin{aligned}
&\left[2^n k + 4(2^{n-1} - 1) + 1\right]\left[f_n'\left(\frac{a\tau + b}{c\tau + d}\right)\right]^2 \\
&\quad = \left[2^n k + 4(2^{n-1} - 1) + 1\right]\{ \left[2^n k + 4(2^{n-1} - 1)\right]^2 c^2 (c\tau + d)^{2^{n+1}k + 8(2^{n-1}-1)+2} \left[f_n(\tau)\right]^2 \\
&\qquad + 2c\left[2^n k + 4(2^{n-1} - 1)\right](c\tau + d)^{2^{n+1}k + 8(2^{n-1}-1)+3} f_n(\tau) f_n'(\tau) \\
&\qquad + (c\tau + d)^{2^{n+1}k + 8(2^{n-1}-1)+4} \left[f_n'(\tau)\right]^2 \}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&-\left[2^n k + 4(2^{n-1} - 1)\right] f_n\left(\frac{a\tau + b}{c\tau + d}\right) f_n''\left(\frac{a\tau + b}{c\tau + d}\right) \\
&\quad = -\left[2^n k + 4(2^{n-1} - 1)\right]\{ \left[2^n k + 4(2^{n-1} - 1)\right]\left[2^n k + 4(2^{n-1} - 1) + 1\right] c^2 (c\tau + d)^{2^{n+1}k + 8(2^{n-1}-1)+2} \left[f_n(\tau)\right]^2 \\
&\qquad + 2c\left[2^n k + 4(2^{n-1} - 1) + 1\right](c\tau + d)^{2^{n+1}k + 8(2^{n-1}-1)+3} f_n(\tau) f_n'(\tau) \\
&\qquad + (c\tau + d)^{2^{n+1}k + 8(2^{n-1}-1)+4} f_n(\tau) f_n''(\tau) \}.
\end{aligned} \tag{3.9}$$

Consequently, by applying these last results (3.6) to (3.9), we arrive at the desired assertion given by

$$\begin{aligned}
f_{n+1}\left(\frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^{2^{n+1}k + 8(2^{n-1}-1)+4} \{ \left[2^n k + 4(2^{n-1} - 1) + 1\right]\left[f_n'(\tau)\right]^2 - \left[2^n k + 4(2^{n-1} - 1)\right] f_n(\tau) f_n''(\tau) \} \\
&= (c\tau + d)^{2^{n+1}k + 4(2^n - 1)} f_{n+1}(\tau).
\end{aligned}$$

□

Remark 5. By replacing n by $n + 1$ in the weight of the modular form $f_n(\tau)$ in Theorem 6, we readily see that the modular form $f_{n+1}(\tau)$ has the following weight:

$$2^{n+1}k + 4(2^n - 1).$$

4. Applications of Theorem 6

Theorem 6 can be shown to have many applications in the theory of the elliptic modular forms. Here we give a few applications related to Theorem 6.

Upon setting $n = 1$ in Theorem 6, we get the following corollary, which was given by Silverman [13, p. 90] and (more recently) also by Sebbar and Sebbar [12, p. 408].

Corollary 1. *Let $f_1(\tau)$ be a modular form with weight $2k$. Then $f_2(\tau)$ defined by*

$$f_2(\tau) = (2k + 1)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\{f(\tau)\}\right)^2 - 2kf(\tau)\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\{f(\tau)\} \tag{4.1}$$

is a modular form with weight $4k + 4$.

Remark 6. If we put

$$f_1(\tau) = E_4(\tau)$$

in (4.1), we have

$$f_2(\tau) = 5\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\{E_4(\tau)\}\right)^2 - 4E_4(\tau)\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\{E_4(\tau)\}. \tag{4.2}$$

Next, by substituting the following well-known derivative formulas for $E_4(\tau)$ (*cf.* [6], [10] and [12]):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{E_4(\tau)\right\} = \frac{2\pi\mathrm{i}}{3}\left[E_2(\tau)E_4(\tau) - E_6(\tau)\right]$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{E_4(\tau)\right\} = -\frac{5\pi^2}{9}\left[\left\{E_2(\tau)\right\}^2 E_4(\tau) + \left\{E_4(\tau)\right\}^2 - 2E_2(\tau)E_6(\tau)\right]$$

into the equation (4.2), one can easily see that $f_2(\tau)$ is a cusp form for the function $f_1(\tau)$ given by

$$f_1(\tau) = G_4(\tau),$$

that is, we have (*cf.* [13, p. 90, Exercise 1.20 (c)] and [12])

$$f_2(\tau) = \frac{\Delta(\tau)}{2^4 \cdot 3^3 \cdot 5^2 \cdot \pi^2}.$$

Similarly, if $f_2(\tau)$ is a cusp form for the function $f_1(\tau)$ given by

$$f_1(\tau) = E_4(\tau),$$

then we can readily find that (*cf.* [13, p. 90, Exercise 1.20 (c)] and [12])

$$f_2(\tau) = \frac{5\pi^2}{3888}\Delta(\tau).$$

Thus, if $f_1(\tau)$ is a modular form, then $f_2(\tau)$ is a cusp form (*cf.* [13, p. 90, Exercise 1.20 (b)]).

If we set

$$f_1(\tau) = E_6(\tau)$$

in the equation (4.1), we have

$$f_2(\tau) = 7\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{E_6(\tau)\right\}\right)^2 - 6E_6(\tau)\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{E_6(\tau)\right\}. \tag{4.3}$$

By substituting the following well-known derivative formulas for $E_6(\tau)$ (*cf.* [6], [10] and [12]):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{E_6(\tau)\right\} = \pi\mathrm{i}\left[E_2(\tau)E_6(\tau) - \left\{E_4(\tau)\right\}^2\right]$$

and (*cf.* [6])

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{E_6(\tau)\right\} = -\frac{7\pi^2}{6}\left[\left\{E_2(\tau)\right\}^2 E_6(\tau) + E_4(\tau)E_6(\tau) - 2E_2(\tau)\left\{E_4(\tau)\right\}^2\right]$$

into the equation (4.3), we arrive at the following results.

Corollary 2. *It is asserted that*

$$f_2(\tau) = 7\pi^2\left[\left\{E_6(\tau)\right\}^2 E_4(\tau) - \left\{E_4(\tau)\right\}^4\right].$$

Corollary 3. *Let $n, r \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{1,2\}$. Suppose also that*

$$\mathbf{a} \not\equiv \mathbf{0}(N) \qquad \textit{and} \qquad N \in \mathbb{N} \setminus \{1\}.$$

If

$$2m + r = 2^n k + 4(2^{n-1} - 1),$$

then the sequence

$$\left\{\wp_{2m}^{(r)}\left(\frac{\mathbf{a}^T\mathbf{z}}{N}; w\right)\right\}_{m\in\mathbb{N}},$$

which is a modular form with weight $2m + r$, satisfies Theorem 6.

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References

[1] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Second Edition, Springer-Verlag, Berlin, Heidelberg and New York, 1997.

[2] A. A. Aygunes and Y. Simsek, The action of Hecke operators to families of Weierstrass-type functions and Weber-type functions and their applications, *Appl. Math. Comput.* **218** (2011), 678–682.

[3] C.-H. Chang and H. M. Srivastava, A note on Bernoulli identities associated with the Weierstrass $\wp$-function, *Integral Transforms Spec. Funct.* **18** (2007), 245–253.

[4] C.-H. Chang, H. M. Srivastava and T.-C. Wu, Some families of Weierstrass-type functions and their applications, *Integral Transforms Spec. Funct.* **19** (2008), 621–632.

[5] H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.* **217** (1975), 271–285.

[6] M. Eichler and D. Zagier, On the zeros of the Weierstrass $\wp$-function, *Math. Ann.* **258** (1982), 399–407.

[7] N. I. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg and New York, 1993.

[8] R. A. Rankin, The construction of automorphic forms from the derivatives of a given form, *J. Indian Math. Soc.* (*New Ser.*) **20** (1956), 103–116.

[9] R. A. Rankin, The construction of automorphic forms from the derivatives of given forms, *Michigan Math. J.* **4** (1957), 181–186.

[10] R. A. Rankin, Elementary proofs of relations between Eisenstein series, *Proc. Royal Soc. Edinburgh Sect. A* **76** (1976), 107–117.

[11] B. Schoeneberg, *Elliptic Modular Forms*: *An Introduction*, Springer-Verlag, Berlin, Heidelberg and New York, 1974.

[12] A. Sebbar and A. Sebbar, Eisenstein series and modular differential equations, *Canad. Math. Bull.* **55** (2012), 400–409.

[13] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg and New York, 1991.

[14] Y. Simsek, Relations between theta-functions Hardy sums Eisenstein series and Lambert series in the transformation formula of $\log \eta_{g,h}(z)$, *J. Number Theory* **99** (2003), 338–360.

[15] Y. Simsek, On Weierstrass $\wp(z)$-function, Hardy sums and Eisenstein series, *Proc. Jangjeon Math. Soc.* **7** (2004), 99–108.

[16] Y. Simsek, On normalized Eisenstein series and new theta functions, *Proc. Jangjeon Math. Soc.* **8** (2005), 25–34.

[17] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inform. Sci.* **5** (2011), 390–444.

[18] H. M. Srivastava, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, *SpringerPlus* **2** (2013), Article ID 2:67, 1–14.

[19] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[20] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *with an Account of the Principal Transcendental Functions*, Fourth Edition (Reprinted), Cambridge University Press, Cambridge, London and New York, 1962.

[21] T.-C. Wu, C.-H. Chang and H. M. Srivastava, A unified presentation of identities involving Weierstrass-type functions, *Appl. Math. Lett.* **23** (2010), 864–870.