



## Fractional Hermite-Hadamard Inequalities for Differentiable $s$ -Godunova-Levin Functions

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**Abstract.** In this paper, we consider the class of  $s$ -Godunova-Levin functions. We derive a new fractional integral identity for differentiable function. Using this new identity, we establish some new fractional Hermite-Hadamard type inequalities for the class of differentiable  $s$ -Godunova-Levin functions.

### 1. Introduction

Theory of convex functions plays an important role in various fields of pure and applied sciences. In recent years, classical concept of convex functions has been generalized and extended in different directions using novel and innovative ideas, see [1–6, 12, 13]. Dragomir [4] introduced the concept of  $s$ -Godunova-Levin functions which generalizes the class of  $P$ -functions [5] and Godunova-Levin functions [6].

Theory of convex functions has also a close relationship with theory of inequalities. Many famous inequalities known to us are proved for convex functions, for example an extensively studied inequality that is Hermite-Hadamard inequality which is due to Hermite and Hadamard independently. This result provides necessary and sufficient condition for a function to be convex. Recently several new generalizations of Hermite-Hadamard's inequality have been proposed, see [2, 3, 5, 8–13].

In this paper, we establish a new fractional integral identity for differentiable functions. Using this fractional integral identity as an auxiliary result we derive some new Hermite-Hadamard type of inequalities for the class of  $s$ -Godunova-Levin functions. This is the main motivation of this paper.

Throughout this paper, let  $\mathbb{R}$  be set of real numbers and  $\mathbb{N}^* = \{1, 2, 3, \dots\}$  be the set of natural numbers.

### 2. Preliminary Results

In this section, we recall some previously known results.

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**Definition 2.1 ([4]).** A function  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is said to be  $s$ -Godunova-Levin functions of second kind, if

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}, \quad \forall x, y \in I, t \in (0, 1), s \in [0, 1]. \quad (2.1)$$

It is obvious that for  $s = 0$ ,  $s$ -Godunova-Levin convex functions of second kind reduces to the definition of  $P$ -functions and for  $s = 1$  it reduces to the definition of Godunova-Levin convex functions.

**Theorem 2.2 (Hermite-Hadamard's Inequality).** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a convex function, where  $a, b \in I$  with  $a < b$ . Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (2.2)$$

**Definition 2.3 ([7]).** Let  $f \in L_1[a, b]$ . Then Riemann-Liouville integrals  $J_{a^+}^\alpha f$  and  $J_{b^-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

and

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt,$$

is the well known Gamma function.

We now give the definition of hypergeometric series which will be used in the obtaining some integrals.

**Definition 2.4 ([7]).** For the real or complex numbers  $a, b, c$ , other than  $0, -1, -2, \dots$ , the hypergeometric series is defined by

$${}_2F_1[a, b, c; z] = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}.$$

Here  $(\phi)_m$  is the Pochhammer symbol, which is defined by

$$(\phi)_m = \begin{cases} 1 & m = 0, \\ \phi(\phi+1)\dots(\phi+m-1), & m > 0, \end{cases}$$

which has the integral form:

$${}_2F_1[a, b; c; z] = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt$$

where  $|z| < 1, c > b > 0$  and

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

is Euler function Beta with

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Now we prove the following auxiliary result which plays an important role in proving our main results.

**Lemma 2.5.** *Let  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be differentiable function such that  $f' \in L[a, b]$ . Then for  $n \in \mathbb{N}^*$  and  $\alpha > 0$ , we have*

$$G(\alpha; n; a, x, b)(f) = \frac{(x-a)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) dt - \int_0^1 \frac{t^\alpha}{2} f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) dt \right\} \\ - \frac{(b-x)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) dt - \int_0^1 \frac{t^\alpha}{2} f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) dt \right\},$$

where

$$G(\alpha; n; a, x, b)(f) = \frac{n+1}{2} \left[ \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} f(x) + \frac{(x-a)^\alpha f(a) + (b-x)^\alpha f(b)}{b-a} \right] \\ - \frac{(n+1)^{\alpha+1} \Gamma(\alpha+1)}{2(b-a)} \left[ J_{x^-}^\alpha f \left( \frac{n}{n+1}x + \frac{1}{n+1}a \right) + J_{a^+}^\alpha f \left( \frac{1}{n+1}x + \frac{n}{n+1}a \right) \right. \\ \left. + J_{x^+}^\alpha f \left( \frac{n}{n+1}x + \frac{1}{n+1}b \right) + J_{b^-}^\alpha f \left( \frac{1}{n+1}x + \frac{n}{n+1}b \right) \right].$$

*Proof.* It suffices to show that

$$\int_0^1 \frac{t^\alpha}{2} f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) dt = \frac{n+1}{2(x-a)} f(x) - \frac{(n+1)\Gamma(\alpha+1)}{2(x-a)} \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} f \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) dt \\ = \frac{n+1}{2(x-a)} f(x) - \frac{(n+1)^{\alpha+1} \Gamma(\alpha+1)}{2(x-a)^{\alpha+1}} \frac{1}{\Gamma(\alpha)} \int_{\frac{n}{n+1}x + \frac{1}{n+1}a}^x \left( u - \frac{n}{n+1}x - \frac{1}{n+1}a \right)^{\alpha-1} f(u) du \\ = \frac{n+1}{2(x-a)} f(x) - \frac{(n+1)^{\alpha+1} \Gamma(\alpha+1)}{2(x-a)^{\alpha+1}} J_{x^-}^\alpha f \left( \frac{n}{n+1}x + \frac{1}{n+1}a \right). \tag{2.3}$$

Similarly

$$\int_0^1 \frac{t^\alpha}{2} f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) dt = -\frac{n+1}{2(x-a)} f(a) + \frac{(n+1)^{\alpha+1} \Gamma(\alpha+1)}{2(x-a)^{\alpha+1}} J_{a^+}^\alpha f \left( \frac{1}{n+1}x + \frac{n}{n+1}a \right). \tag{2.4}$$

$$\int_0^1 \frac{t^\alpha}{2} f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) dt = -\frac{n+1}{2(b-x)} f(x) + \frac{(n+1)^{\alpha+1} \Gamma(\alpha+1)}{2(b-x)^{\alpha+1}} J_{x^+}^\alpha f \left( \frac{n}{n+1}x + \frac{1}{n+1}b \right), \tag{2.5}$$

and

$$\int_0^1 \frac{t^\alpha}{2} f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) dt = \frac{n+1}{2(b-x)} f(b) - \frac{(n+1)^{\alpha+1} \Gamma(\alpha+1)}{2(b-x)^{\alpha+1}} J_{b^-}^\alpha f \left( \frac{1}{n+1}x + \frac{n}{n+1}b \right). \tag{2.6}$$

After suitable rearrangements the proof is complete.  $\square$

**Remark 2.6.** We would like to remark that for  $n = 1$  Lemma 2.5 reduces to Lemma 1 [9].

### 3. Main Results

In this section, we derive our main results.

**Theorem 3.1.** Let  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be differentiable function such that  $f' \in L[a, b]$  and  $n \in \mathbb{N}^*$ ,  $\alpha \geq 0$ . If  $|f'|$  is  $s$ -Godunova-Levin function where  $0 < s < 1$ , then

$$|G(\alpha; n; a, x, b)(f)| \leq \frac{(n+1)^s(A+B)}{2(b-a)} \left[ (x-a)^{\alpha+1} |f'(a)| + (b-x)^{\alpha+1} |f'(b)| \right],$$

where

$$A = \frac{\Gamma(1-s)\Gamma(1+\alpha)}{\Gamma(2-s+\alpha)}, \tag{3.1}$$

$$B = \frac{n^{-s} {}_2F_1 \left[ s, 1+\alpha, 2+\alpha, -\frac{1}{n} \right]}{1+\alpha}. \tag{3.2}$$

*Proof.* Using Lemma 2.5, taking modulus and the fact that  $|f'|$  is  $s$ -Godunova-Levin function, we have:

$$\begin{aligned} |G(\alpha; n; a, x, b)(f)| &\leq \frac{(x-a)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt + \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \right\} \\ &\quad + \frac{(b-x)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt + \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \right\} \\ &\leq \frac{(x-a)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} \left[ \left( \frac{n+t}{n+1} \right)^{-s} |f'(x)| + \left( \frac{1-t}{n+1} \right)^{-s} |f'(a)| \right] dt + \int_0^1 \frac{t^\alpha}{2} \left[ \left( \frac{1-t}{n+1} \right)^{-s} |f'(x)| + \left( \frac{n+t}{n+1} \right)^{-s} |f'(a)| \right] dt \right\} \\ &\quad + \frac{(b-x)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} \left[ \left( \frac{n+t}{n+1} \right)^{-s} |f'(x)| + \left( \frac{1-t}{n+1} \right)^{-s} |f'(b)| \right] dt + \int_0^1 \frac{t^\alpha}{2} \left[ \left( \frac{1-t}{n+1} \right)^{-s} |f'(x)| + \left( \frac{n+t}{n+1} \right)^{-s} |f'(b)| \right] dt \right\} \\ &= \frac{(n+1)^s(x-a)^{\alpha+1}}{2(b-a)} \left\{ \left[ \frac{n^{-s} {}_2F_1 \left[ s, 1+\alpha, 2+\alpha, -\frac{1}{n} \right]}{1+\alpha} |f'(x)| + \frac{\Gamma(1-s)\Gamma(1+\alpha)}{\Gamma(2-s+\alpha)} |f'(a)| \right] \right. \\ &\quad \left. + \left[ \frac{n^{-s} {}_2F_1 \left[ s, 1+\alpha, 2+\alpha, -\frac{1}{n} \right]}{1+\alpha} |f'(a)| + \frac{\Gamma(1-s)\Gamma(1+\alpha)}{\Gamma(2-s+\alpha)} |f'(x)| \right] \right\} \\ &\quad + \frac{(n+1)^s(b-x)^{\alpha+1}}{2(b-a)} \left\{ \left[ \frac{n^{-s} {}_2F_1 \left[ s, 1+\alpha, 2+\alpha, -\frac{1}{n} \right]}{1+\alpha} |f'(x)| + \frac{\Gamma(1-s)\Gamma(1+\alpha)}{\Gamma(2-s+\alpha)} |f'(b)| \right] \right. \\ &\quad \left. + \left[ \frac{n^{-s} {}_2F_1 \left[ s, 1+\alpha, 2+\alpha, -\frac{1}{n} \right]}{1+\alpha} |f'(b)| + \frac{\Gamma(1-s)\Gamma(1+\alpha)}{\Gamma(2-s+\alpha)} |f'(x)| \right] \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(n+1)^s(x-a)^{\alpha+1}}{2(b-a)}(A+B)|f'(a)| + \frac{(n+1)^s(b-x)^{\alpha+1}}{2(b-a)}(A+B)|f'(b)| \\
 &= \frac{(n+1)^s(A+B)}{2(b-a)}[(x-a)^{\alpha+1}|f'(a)| + (b-x)^{\alpha+1}|f'(b)|].
 \end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.2.** Let  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be differentiable function such that  $f' \in L[a, b]$  and  $n \in \mathbb{N}^*$ ,  $\alpha \geq 0$ . If  $|f'|^q$  is  $s$ -Godunova-Levin function where  $0 < s < 1$  and  $\frac{1}{p} + \frac{1}{q} = 1, q > 1$ , then

$$\begin{aligned}
 |G(\alpha; n; a, x, b)(f)| &\leq \frac{(n+1)^{\frac{s}{q}}}{2(b-a)} \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \left\{ (x-a)^{\alpha+1} \left\{ (\mu_1|f'(x)|^q + \mu_2|f'(a)|^q)^{\frac{1}{q}} \right. \right. \\
 &\quad \left. \left. + (\mu_2|f'(x)|^q + \mu_1|f'(a)|^q)^{\frac{1}{q}} \right\} + (b-x)^{\alpha+1} \left\{ (\mu_1|f'(x)|^q + \mu_2|f'(b)|^q)^{\frac{1}{q}} \right. \right. \\
 &\quad \left. \left. + (\mu_2|f'(x)|^q + \mu_1|f'(b)|^q)^{\frac{1}{q}} \right\} \right\},
 \end{aligned}$$

where

$$\mu_1 = \frac{(n(1+n))^{-s}(-n^s(1+n) + n(1+n)^s)}{s-1} \text{ and } \mu_2 = \frac{1}{1-s}.$$

*Proof.* Using Lemma 2.5, Hölder’s inequality and the fact that  $|f'|^q$  is  $s$ -Godunova-Levin function, we have

$$\begin{aligned}
 |G(\alpha; n; a, x, b)(f)| &\leq \frac{(x-a)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt + \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \right\} \\
 &\quad + \frac{(b-x)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt + \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \right\} \\
 &\leq \frac{(x-a)^{\alpha+1}}{b-a} \left( \int_0^1 \left( \frac{t^\alpha}{2} \right)^p dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right|^q dt \right)^{\frac{1}{q}} \right] \\
 &\quad - \frac{(b-x)^{\alpha+1}}{b-a} \left( \int_0^1 \left( \frac{t^\alpha}{2} \right)^p dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} \right] \\
 &\leq \frac{(x-a)^{\alpha+1}(n+1)^{\frac{s}{q}}}{2(b-a)} \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \left\{ \left( \int_0^1 [(n+t)^{-s}|f'(x)|^q + (1-t)^{-s}|f'(a)|^q] dt \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left( \int_0^1 [(1-t)^{-s}|f'(x)|^q + (n+t)^{-s}|f'(a)|^q] dt \right)^{\frac{1}{q}} \right\} \\
 &\quad + \frac{(b-x)^{\alpha+1}(n+1)^{\frac{s}{q}}}{2(b-a)} \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \left\{ \left( \int_0^1 [(n+t)^{-s}|f'(x)|^q + (1-t)^{-s}|f'(b)|^q] dt \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left( \int_0^1 [(1-t)^{-s}|f'(x)|^q + (n+t)^{-s}|f'(b)|^q] dt \right)^{\frac{1}{q}} \right\}
 \end{aligned}$$

$$= \frac{(n+1)^{\frac{s}{q}}}{2(b-a)} \left( \frac{1}{p\alpha+1} \right)^{\frac{1}{p}} \left\{ (x-a)^{\alpha+1} \left\{ (\mu_1|f'(x)|^q + \mu_2|f'(a)|^q)^{\frac{1}{q}} \right. \right. \\ \left. \left. + (\mu_2|f'(x)|^q + \mu_1|f'(a)|^q)^{\frac{1}{q}} \right\} + (b-x)^{\alpha+1} \left\{ (\mu_1|f'(x)|^q + \mu_2|f'(b)|^q)^{\frac{1}{q}} \right. \right. \\ \left. \left. + (\mu_2|f'(x)|^q + \mu_1|f'(b)|^q)^{\frac{1}{q}} \right\} \right\}.$$

This completes the proof.  $\square$

**Theorem 3.3.** Let  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be differentiable function such that  $f' \in L[a, b]$  and  $n \in \mathbb{N}^*$ ,  $\alpha \geq 0$ . If  $|f'|^q$  is  $s$ -Godunova-Levin function where  $0 < s < 1$  and  $q > 1$ , then

$$|G(\alpha; n; a, x, b)(f)| \leq \frac{(n+1)^{\frac{s}{q}}}{2(b-a)} \left( \frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left\{ (x-a)^{\alpha+1} \left\{ (B|f'(x)|^q + A|f'(a)|^q)^{\frac{1}{q}} \right. \right. \\ \left. \left. + (A|f'(x)|^q + B|f'(a)|^q)^{\frac{1}{q}} \right\} + (b-x)^{\alpha+1} \left\{ (B|f'(x)|^q + A|f'(b)|^q)^{\frac{1}{q}} \right. \right. \\ \left. \left. + (A|f'(x)|^q + B|f'(b)|^q)^{\frac{1}{q}} \right\} \right\},$$

where  $A$  and  $B$  are given by (3.1) and (3.2).

*Proof.* Using Lemma 2.5, Power’s mean inequality and the fact that  $|f'|^q$  is  $s$ -Godunova-Levin function, we have

$$|G(\alpha; n; a, x, b)(f)| \leq \frac{(x-a)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt + \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \right\} \\ + \frac{(b-x)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt + \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \right\} \\ \leq \frac{(x-a)^{\alpha+1}}{b-a} \left( \int_0^1 \frac{t^\alpha}{2} dt \right)^{1-\frac{1}{q}} \left[ \left( \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right|^q dt \right)^{\frac{1}{q}} \right] \\ + \frac{(b-x)^{\alpha+1}}{b-a} \left( \int_0^1 \frac{t^\alpha}{2} dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 \frac{t^\alpha}{2} \left| f' \left( \frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} \right] \\ \leq \frac{(x-a)^{\alpha+1}(n+1)^{\frac{s}{q}}}{2(b-a)} \left( \frac{1}{\alpha+1} \right)^{\frac{1}{p}} \left\{ \left( \int_0^1 t^\alpha [(n+t)^{-s}|f'(x)|^q + (1-t)^{-s}|f'(a)|^q] dt \right)^{\frac{1}{q}} \right. \\ \left. + \left( \int_0^1 t^\alpha [(1-t)^{-s}|f'(x)|^q + (n+t)^{-s}|f'(a)|^q] dt \right)^{\frac{1}{q}} \right\} \\ + \frac{(b-x)^{\alpha+1}(n+1)^{\frac{s}{q}}}{2(b-a)} \left( \frac{1}{\alpha+1} \right)^{\frac{1}{p}} \left\{ \left( \int_0^1 t^\alpha [(n+t)^{-s}|f'(x)|^q + (1-t)^{-s}|f'(b)|^q] dt \right)^{\frac{1}{q}} \right. \\ \left. + \left( \int_0^1 t^\alpha [(1-t)^{-s}|f'(x)|^q + (n+t)^{-s}|f'(b)|^q] dt \right)^{\frac{1}{q}} \right\}$$

$$= \frac{(n+1)^{\frac{s}{q}}}{2(b-a)} \left( \frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left\{ (x-a)^{\alpha+1} \left\{ (|Bf'(x)|^q + |Af'(a)|^q)^{\frac{1}{q}} \right. \right. \\ \left. \left. + (|Af'(x)|^q + |Bf'(a)|^q)^{\frac{1}{q}} \right\} + (b-x)^{\alpha+1} \left\{ (|Bf'(x)|^q + |Af'(b)|^q)^{\frac{1}{q}} + (|Af'(x)|^q + |Bf'(b)|^q)^{\frac{1}{q}} \right\} \right\}.$$

This completes the proof.  $\square$

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