Filomat 30:14 (2016), 3667–3680
DOI 10.2298/FIL1614667T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New Hermite-Hadamard-Type Inequalities and Their Applications

Kuei-Lin Tseng[a], Shiow-Ru Hwang[b,*]

[a]*Department of Applied Mathematics, Aletheia University, Tamsui, New Taipei City 25103, Taiwan*
[b]*China University of Science and Technology, Nankang, Taipei, Taiwan 11522*

Abstract. In this paper, we establish some new Hermite-Hadamard-type inequalities for differentiable functions. Several applications for special means are given as well.

1. Introduction

Throughout in this paper, let $a \leq c < d \leq b$ in $\mathbb{R}$ with $a + b = c + d$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

which holds for all convex functions $f : [a, b] \to \mathbb{R}$, is known in the literature as Hermite-Hadamard inequality [7].

For some results which generalize, improve, and extend the inequality (1), see [1]-[6] and [8]-[18].

In [4], Dragomir and Agarwal established the following results connected with the second inequality in the inequality (1).

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right) \tag{2}$$

which is the trapezoid inequality provided $|f'|$ is convex on $[a, b]$.

In [12], Kirmaci and Özdemir established the following results connected with the first inequality in the inequality (1).

2010 *Mathematics Subject Classification*. Primary 26D15; Secondary 26A51
Keywords. Hermite-Hadamard Inequality, Differentiable Function, Convex Function, Concave Function, Special mean.
Received: 08 September 2014; Accepted: 29 November 2014
Communicated by Hari M. Srivastava
* Corresponding author
Email addresses: kltseng@mail.au.edu.tw (Kuei-Lin Tseng), hsru@cc.cust.edu.tw (Shiow-Ru Hwang)

Theorem 1.2. *Under the assumptions of Theorem 1.1, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right) \tag{3}$$

which is the midpoint inequality provided $\left|f'\right|$ *is convex on* $[a,b]$.

In [14], Pearce and Pečarić established the following Hermite-Hadamard-type inequalities for differentiable functions:

Theorem 1.3. *If* $f : I^o \subseteq R \to R$ *is a differentiable mapping on* I^o, $a, b \in I^o$ *with* $a < b$, $f' \in L_1[a,b]$, $q \ge 1$ *and* $\left|f'\right|^q$ *is convex on* $[a,b]$*, then the following inequalities hold:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{1/q}. \tag{4}$$

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{1/q}. \tag{5}$$

Theorem 1.4. *If* $f : I^o \subseteq R \to R$ *is a differentiable mapping on* I^o, $a, b \in I^o$ *with* $a < b$, $f' \in L_1[a,b]$, $q \ge 1$ *and* $\left|f'\right|^q$ *is concave on* $[a,b]$*, then the following inequalities hold:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{6}$$

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{7}$$

Remark 1.5. *In Theorem 1.3, let* $q = 1$*. Then Theorem 1.3 reduces to Theorems 1.1 and 1.2.*

In this paper, we establish some results which refine Hermite-Hadamard inequality (1) and generalize Theorems 1.1-1.4. Some applications for special means are given.

2. Hermite-Hadamard-Type Inequalities for Convex Functions

In this section, we establish some Hermite-Hadamard-type inequalities which refine Hermite-Hadamard inequality (1).

Theorem 2.1. (1) *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function. Then we have Hermite-Hadamard-type inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{a+b-2c}{b-a} f\left(\frac{a+b}{2}\right) + \frac{c-a}{b-a}\left[f(c)+f(d)\right] \\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\le \frac{a+b-2c}{2(b-a)}\left[f(c)+f(d)\right] + \frac{c-a}{b-a}\left[f(a)+f(b)\right] \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned} \tag{8}$$

(2) Let $f : [a,b] \to \mathbb{R}$ be a concave function. Then the inequality (8) is reversed.

Proof. (1) Suppose f is convex. It is easily observed from the convexity of f that the first and last inequalities of (8) hold.

Using simple computation, we have the following identities:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^c [f(x)+f(a+b-x)]\,dx + \frac{1}{b-a}\int_c^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\,dx. \tag{9}$$

$$c = \frac{a+b-x-c}{a+b-2x}x + \frac{c-x}{a+b-2x}(a+b-x) = \frac{d-x}{c+d-2x}x + \frac{c-x}{c+d-2x}(a+b-x) \tag{10}$$

and

$$d = \frac{a+b-x-d}{a+b-2x}x + \frac{d-x}{a+b-2x}(a+b-x) = \frac{c-x}{c+d-2x}x + \frac{d-x}{c+d-2x}(a+b-x) \tag{11}$$

where $x \in [a,c]$ with $0 \le \frac{c-x}{c+d-2x}, \frac{d-x}{c+d-2x} \le 1$.

$$\frac{a+b}{2} = \frac{1}{2}[x+(a+b-x)] \tag{12}$$

where $x \in \left[c, \frac{a+b}{2}\right]$.

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b \tag{13}$$

and

$$a+b-x = \frac{x-a}{b-a}a + \frac{b-x}{b-a}b \tag{14}$$

where $x \in [a,c]$.

$$x = \frac{d-x}{d-c}c + \frac{x-c}{d-c}d \tag{15}$$

and

$$a+b-x = \frac{d-a-b+x}{d-c}c + \frac{a+b-x-c}{d-c}d = \frac{x-c}{d-c}c + \frac{d-x}{d-c}d \tag{16}$$

where $x \in \left[c, \frac{a+b}{2}\right]$ with $0 \le \frac{x-c}{d-c}, \frac{d-x}{d-c} \le 1$.

Now, using the above identities and the convexity of f, we have the following inequalities:

$$\begin{aligned}\frac{c-a}{b-a}[f(c)+f(d)] &= \frac{1}{b-a}\int_a^c [f(c)+f(d)]\,dx \\ &\le \frac{1}{b-a}\int_a^c \left[\frac{d-x}{c+d-2x}f(x) + \frac{c-x}{c+d-2x}f(a+b-x)\right. \\ &\quad \left. + \frac{c-x}{c+d-2x}f(x) + \frac{d-x}{c+d-2x}f(a+b-x)\right]dx \\ &= \frac{1}{b-a}\int_a^c [f(x)+f(a+b-x)]\,dx\end{aligned} \tag{17}$$

by the identities (10) and (11).

$$\frac{a+b-2c}{b-a}f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_c^{\frac{a+b}{2}} 2f\left(\frac{a+b}{2}\right)dx \leq \frac{1}{b-a}\int_c^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)\right]dx \tag{18}$$

by the identity (12).

$$\begin{aligned} &\frac{1}{b-a}\int_a^c \left[f(x)+f(a+b-x)\right]dx \\ &\leq \frac{1}{b-a}\int_a^c \left[\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)+\frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)\right]dx \\ &= \frac{1}{b-a}\int_a^c \left[f(a)+f(b)\right]dx \\ &= \frac{c-a}{b-a}\left[f(a)+f(b)\right] \end{aligned} \tag{19}$$

by the identities (13) and (14).

$$\begin{aligned} &\frac{1}{b-a}\int_c^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)\right]dx \\ &\leq \frac{1}{b-a}\int_c^{\frac{a+b}{2}} \left[\frac{d-x}{d-c}f(c)+\frac{x-c}{d-c}f(d)+\frac{x-c}{d-c}f(c)+\frac{d-x}{d-c}f(d)\right]dx \\ &= \frac{1}{b-a}\int_c^{\frac{a+b}{2}} \left[f(c)+f(d)\right]dx \\ &= \frac{a+b-2c}{2(b-a)^{\alpha}}\left[f(c)+f(d)\right] \end{aligned} \tag{20}$$

by the identities (15) and (16).

The second and third inequalities of (8) follow from the identity (9) and the inequalities (17) − (20).

(2) In case f is concave, so $-f$ is convex, we obtain that the inequality (8) is reversed.

This completes the proof. □

Remark 2.2. *In Theorem 2.1, the inequality* (8) *refines Hermite-Hadamard inequality* (1).

Corollary 2.3. *In Theorem 2.1, let* $c = (1-\beta)a+\beta b$ *and* $d = \beta a+(1-\beta)b$ *with* $0 \leq \beta < \frac{1}{2}$. *Then we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq (1-2\beta)f\left(\frac{a+b}{2}\right)+\beta\left[f((1-\beta)a+\beta b)+f(\beta a+(1-\beta)b)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leq \left(\frac{1}{2}-\beta\right)\left[f((1-\beta)a+\beta b)+f(\beta a+(1-\beta)b)\right]+\beta\left[f(a)+f(b)\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

3. Hermite-Hadamard-Type Inequalities for Differentiable Functions

In this section, we establish some results connected with the second and third inequalities in the inequality (8) .

Theorem 3.1. *Under the assumptions of Theorem 1.3, we have the following Hermite-Hadamard-type inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\left\{\frac{a+b-2c}{2(b-a)}[f(c)+f(d)]+\frac{c-a}{b-a}[f(a)+f(b)]\right\}\right| \tag{21}$$
$$\leq H(c,d)(b-a)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}$$

where

$$H(c,d):=\left(\frac{c-a}{b-a}\right)^2+\frac{1}{4}\left(\frac{d-c}{b-a}\right)^2.$$

Proof. Define

$$h_1(x)=\begin{cases} c-x, & x\in[a,c) \\ \dfrac{a+b}{2}-x, & x\in[c,d) \\ d-x, & x\in[d,b] \end{cases}.$$

Using the integration by parts, we have the following identities:

$$\frac{1}{b-a}\int_a^b h_1(x)f'(x)\,dx=\frac{1}{b-a}\int_a^b f(x)\,dx-\left\{\frac{a+b-2c}{2(b-a)}[f(c)+f(d)]+\frac{c-a}{b-a}[f(a)+f(b)]\right\}. \tag{22}$$

$$\begin{aligned} P_1 &:= \int_a^c \frac{(c-x)(b-x)}{b-a}|f'(a)|^q\,dx+\int_d^b \frac{(x-d)(b-x)}{b-a}|f'(a)|^q\,dx \\ &= \int_a^c \frac{(c-x)(b-x)}{b-a}|f'(a)|^q\,dx+\int_a^c \frac{(c-x)(x-a)}{b-a}|f'(a)|^q\,dx \\ &= |f'(a)|^q\int_a^c (c-x)\,dx=\frac{(c-a)^2}{2}|f'(a)|^q. \end{aligned} \tag{23}$$

$$\begin{aligned} P_2 &:= \int_a^c \frac{(c-x)(x-a)}{b-a}|f'(b)|^q\,dx+\int_d^b \frac{(x-d)(x-a)}{b-a}|f'(b)|^q\,dx \\ &= \int_a^c \frac{(c-x)(x-a)}{b-a}|f'(b)|^q\,dx+\int_a^c \frac{(c-x)(b-x)}{b-a}|f'(b)|^q\,dx \\ &= \int_a^c (c-x)\,dx\,|f'(b)|^q=\frac{(c-a)^2}{2}|f'(b)|^q. \end{aligned} \tag{24}$$

$$\begin{aligned} P_3 &:= \int_c^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\frac{b-x}{b-a}|f'(a)|^q\,dx+\int_{\frac{a+b}{2}}^d\left(x-\frac{a+b}{2}\right)\frac{b-x}{b-a}|f'(a)|^q\,dx \\ &= \int_c^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\frac{b-x}{b-a}|f'(a)|^q\,dx+\int_c^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\frac{x-a}{b-a}|f'(a)|^q\,dx \\ &= \int_c^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)dx\,|f'(a)|^q \\ &= \frac{(a+b-2c)^2}{8}|f'(a)|^q=\frac{(d-c)^2}{8}|f'(a)|^q. \end{aligned} \tag{25}$$

$$
\begin{aligned}
P_4 &:= \int_c^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)\frac{x-a}{b-a}\left|f'(b)\right|^q dx + \int_{\frac{a+b}{2}}^{d}\left(x-\frac{a+b}{2}\right)\frac{x-a}{b-a}\left|f'(b)\right|^q dx \qquad (26)\\
&= \int_c^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)\frac{x-a}{b-a}\left|f'(b)\right|^q dx + \int_c^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\frac{b-x}{b-a}\left|f'(b)\right|^q dx\\
&= \int_c^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)dx\left|f'(b)\right|^q\\
&= \frac{(a+b-2c)^2}{8}\left|f'(b)\right|^q = \frac{(d-c)^2}{8}\left|f'(b)\right|^q .
\end{aligned}
$$

$$
\int_a^b |h_1(x)|\,dx = (c-a)^2 + \frac{(d-c)^2}{4}. \qquad (27)
$$

Now, using Power mean inequality, the convexity of $\left|f'\right|^q$ and the identities (15) and (23) – (27) , we have the inequality

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b h_1(x) f'(x)\,dx\right| \qquad (28)\\
&\le \frac{1}{b-a}\int_a^b |h_1(x)|\left|f'(x)\right| dx\\
&\le \frac{1}{b-a}\left[\int_a^b |h_1(x)|\,dx\right]^{\frac{q-1}{q}}\left[\int_a^b |h_1(x)|\left|f'(x)\right|^q dx\right]^{\frac{1}{q}}\\
&= \frac{1}{b-a}\left[\int_a^b |h_1(x)|\,dx\right]^{\frac{q-1}{q}}\left[\int_a^c (c-x)\left|f'(x)\right| dx + \int_c^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\left|f'(x)\right|^q dx\right.\\
&\left.+\int_{\frac{a+b}{2}}^{d}\left(x-\frac{a+b}{2}\right)\left|f'(x)\right|^q dx + \int_d^b (x-d)\left|f'(x)\right|^q dx\right]^{\frac{1}{q}}\\
&\le \frac{1}{b-a}\left[\int_a^b |h_1(x)|\,dx\right]^{\frac{q-1}{q}}(P_1+P_2+P_3+P_4)^{\frac{1}{q}}\\
&= \frac{1}{b-a}\left[(c-a)^2+\frac{(d-c)^2}{4}\right]^{\frac{q-1}{q}}\left[\left(\frac{(c-a)^2}{2}+\frac{(d-c)^2}{8}\right)\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\\
&= \frac{1}{b-a}\left[(c-a)^2+\frac{(d-c)^2}{4}\right]\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\\
&= H(c,d)(b-a)\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}
$$

The inequality (21) follows from the identity (22) and the inequality (28) .
This completes the proof. □

Remark 3.2. *In Theorem 3.1, let $c = a$ and $d = b$. Then the inequality* (21) *reduces to the inequality* (4) .

Corollary 3.3. *In Theorem 3.1, let* $c = (1-\beta)a + \beta b$ *and* $d = \beta a + (1-\beta)b$ *with* $0 \le \beta < \frac{1}{2}$. *Then we have the inequality*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \left\{\frac{1-2\beta}{2}\left[f((1-\beta)a+\beta b)\right.\right.\right. \\ &\qquad \left.\left.\left. + f(\beta a + (1-\beta)b)\right] + \beta\left[f(a)+f(b)\right]\right\}\right| \\ \le\ & \left(2\beta^2 - \beta + \frac{1}{4}\right)(b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \end{aligned}$$

Remark 3.4. *In Corollary 3.3, let* $\beta = 0$ *and* $q = 1$. *Then Corollary 3.3 reduces to Theorem 1.1.*

Theorem 3.5. *Under the assumptions of Theorem 1.3, we have the following Hermite-Hadamard-type inequality:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \left\{\frac{a+b-2c}{b-a} f\left(\frac{a+b}{2}\right) + \frac{c-a}{b-a}\left[f(c)+f(d)\right]\right\}\right| \qquad (29) \\ &\le H(c,d)(b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} \end{aligned}$$

where $H(c,d)$ *is defined as in Theorem 3.1.*

Proof. Define

$$h_2(x) = \begin{cases} a-x, & x \in [a,c) \\ c-x, & x \in \left[c, \frac{a+b}{2}\right) \\ d-x, & x \in \left[\frac{a+b}{2}, d\right) \\ b-x, & x \in [d,b] \end{cases}.$$

Using the integration by parts, we have the following identities:

$$\begin{aligned} &\frac{1}{2(b-a)^{\alpha}}\int_a^b h_2(x) f'(x)\,dx \qquad (30) \\ =\ & \frac{1}{b-a}\int_a^b f(x)\,dx - \left\{\frac{a+b-2c}{b-a} f\left(\frac{a+b}{2}\right) + \frac{c-a}{b-a}\left[f(c)+f(d)\right]\right\}. \end{aligned}$$

$$\begin{aligned} Q_1 :=& \int_a^c \frac{(x-a)(b-x)}{b-a}|f'(a)|^q\,dx + \int_d^b \frac{(b-x)^2}{b-a}|f'(a)|^q\,dx \qquad (31) \\ =& \int_a^c \frac{(x-a)(b-x)}{b-a}|f'(a)|^q\,dx + \int_a^c \frac{(x-a)^2}{b-a}|f'(a)|^q\,dx \\ =& |f'(a)|^q \int_a^c (x-a)\,dx = \frac{(c-a)^2}{2}|f'(a)|^q. \end{aligned}$$

$$\begin{aligned} Q_2 :=& \int_a^c \frac{(x-a)^2}{b-a}|f'(b)|^q\,dx + \int_d^b \frac{(b-x)(x-a)}{b-a}|f'(b)|^q\,dx \qquad (32) \\ =& \int_a^c \frac{(x-a)^2}{b-a}|f'(b)|^q\,dx + \int_a^c \frac{(x-a)(b-x)}{b-a}|f'(b)|^q\,dx \\ =& \int_a^c (x-a)\,dx\,|f'(b)|^q = \frac{(c-a)^2}{2}|f'(b)|^q. \end{aligned}$$

$$Q_3 := \int_c^{\frac{a+b}{2}} \frac{(x-c)(b-x)}{b-a} \left|f'(a)\right|^q dx + \int_{\frac{a+b}{2}}^d \frac{(d-x)(b-x)}{b-a} \left|f'(a)\right|^q dx \tag{33}$$

$$= \int_c^{\frac{a+b}{2}} \frac{(x-c)(b-x)}{b-a} \left|f'(a)\right|^q dx + \int_c^{\frac{a+b}{2}} \frac{(x-c)(x-a)}{b-a} \left|f'(a)\right|^q dx$$

$$= \left|f'(a)\right|^q \int_c^{\frac{a+b}{2}} (x-c)\,dx = \frac{(d-c)^2}{8} \left|f'(a)\right|^q.$$

$$Q_4 := \int_c^{\frac{a+b}{2}} \frac{(x-c)(x-a)}{b-a} \left|f'(b)\right|^q dx + \int_{\frac{a+b}{2}}^d \frac{(d-x)(x-a)}{b-a} \left|f'(b)\right|^q dx \tag{34}$$

$$= \int_c^{\frac{a+b}{2}} \frac{(x-c)(x-a)}{b-a} \left|f'(b)\right|^q dx + \int_c^{\frac{a+b}{2}} \frac{(x-c)(b-x)}{b-a} \left|f'(b)\right|^q dx$$

$$= \left|f'(b)\right|^q \int_c^{\frac{a+b}{2}} (x-c)\,dx = \frac{(d-c)^2}{8} \left|f'(b)\right|^q.$$

$$\int_a^b |h_2(x)|\,dx = (c-a)^2 + \frac{(d-c)^2}{4}. \tag{35}$$

Now, using Power mean inequality, the convexity of $\left|f'\right|^q$ and the identities (15) and (31) – (35) , we have the inequality

$$\begin{aligned}
\left|\frac{1}{b-a}\int_a^b h_2(x) f'(x)\,dx\right| &\leq \frac{1}{b-a}\int_a^b |h_2(x)| \left|f'(x)\right| dx \\
&\leq \frac{1}{b-a}\left[\int_a^b |h_2(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^b |h_2(x)| \left|f'(x)\right|^q dx\right]^{\frac{1}{q}} \\
&= \frac{1}{b-a}\left[\int_a^b |h_2(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^c (x-a)\left|f'(x)\right| dx + \int_c^{\frac{a+b}{2}} (x-c)\left|f'(x)\right|^q dx\right. \\
&\quad \left. + \int_{\frac{a+b}{2}}^d (d-x)\left|f'(x)\right|^q dx + \int_d^b (b-x)\left|f'(x)\right|^q dx\right]^{\frac{1}{q}} \\
&\leq \frac{1}{b-a}\left[\int_a^b |h_1(x)|\,dx\right]^{\frac{q-1}{q}} (Q_1+Q_2+Q_3+Q_4)^{\frac{1}{q}} \\
&= \frac{1}{b-a}\left[(c-a)^2 + \frac{(d-c)^2}{4}\right]^{\frac{q-1}{q}} \left[\left(\frac{(c-a)^2}{2} + \frac{(d-c)^2}{8}\right)\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)\right]^{\frac{1}{q}} \\
&= \frac{1}{b-a}\left[(c-a)^2 + \frac{(d-c)^2}{4}\right]\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}} \\
&= H(c,d)(b-a)\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned} \tag{36}$$

The inequality (29) follows from the identity (30) and the inequality (36) .
This completes the proof. □

Remark 3.6. *In Theorem 3.5, let $c = a$ and $d = b$. Then the inequality* (29) *reduces to the inequality* (5) *.*

Remark 3.7. *In Theorems 3.1 and 3.5, let $c = a$ and $d = b$. Then Theorems 3.1 and 3.5 reduce to Theorem 1.3.*

Corollary 3.8. *In Theorem 3.5, let $c = (1-\beta)a + \beta b$ and $d = \beta a + (1-\beta)b$ with $0 \leq \beta < \frac{1}{2}$. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \left\{(1-2\beta)\left[f\left(\frac{a+b}{2}\right) + \beta\left[f((1-\beta)a+\beta b) + f(\beta a + (1-\beta)b)\right]\right\}\right|$$
$$\leq \left(2\beta^2 - \beta + \frac{1}{4}\right)(b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}.$$

Remark 3.9. *In Corollary 3.8, let $\beta = 0$ and $q = 1$. Then Corollary 3.8 reduces to Theorem 1.2.*

Remark 3.10. *In Corollaries 3.3 and 3.8, let $\beta = \frac{1}{4}$. Then we obtain the following inequalities:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{4}\left[f(a) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f(b)\right]\right| \leq \frac{b-a}{8}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{4}\left[f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+b}{2}\right) + f\left(\frac{a+3b}{4}\right)\right]\right| \leq \frac{b-a}{8}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}$$

which are similar extensions of Theorem 1.3.

Theorem 3.11. *Under the assumptions of Theorem 1.4, we have the following Hermite-Hadamard-type inequalities:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \left\{\frac{a+b-2c}{2(b-a)}\left[f(c)+f(d)\right] + \frac{c-a}{b-a}\left[f(a)+f(b)\right]\right\}\right| \leq H(c,d)(b-a)\left|f'\left(\frac{a+b}{2}\right)\right| \tag{37}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \left\{\frac{a+b-2c}{b-a}f\left(\frac{a+b}{2}\right) + \frac{c-a}{b-a}\left[f(c)+f(d)\right]\right\}\right| \leq H(c,d)(b-a)\left|f'\left(\frac{a+b}{2}\right)\right| \tag{38}$$

hold where $H(c,d)$ is defined as in Theorem 3.1.

Proof. We observe that $|f'|^q$ is concave on $[a,b]$ implies $|f'| = \left(|f'|^q\right)^{\frac{1}{q}}$ is also concave on $[a,b]$. Using the identities (22), (30) and Jensen integral inequality, we have the inequalities

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \left\{\frac{a+b-2c}{2(b-a)}\left[f(c)+f(d)\right] + \frac{c-a}{b-a}\left[f(a)+f(b)\right]\right\}\right| \tag{39}$$
$$\leq \frac{1}{b-a}\int_a^b |h_1(x)|\,|f'(x)|\,dx$$
$$\leq \frac{1}{b-a}\left|f'\left(\frac{\int_a^b |h_1(x)|\,x\,dx}{\int_a^b |h_1(x)|\,dx}\right)\right|\int_a^b |h_1(x)|\,dx$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \left\{\frac{a+b-2c}{b-a}f\left(\frac{a+b}{2}\right) + \frac{c-a}{b-a}\left[f(c)+f(d)\right]\right\}\right| \tag{40}$$
$$\leq \frac{1}{b-a}\int_a^b |h_2(x)|\,|f'(x)|\,dx$$
$$\leq \frac{1}{b-a}\left|f'\left(\frac{\int_a^b |h_2(x)|\,x\,dx}{\int_a^b |h_2(x)|\,dx}\right)\right|\int_a^b |h_2(x)|\,dx.$$

In Theorems 3.1 and 3.5, we have

$$|h_1(x)| = \begin{cases} c - x, & x \in [a, c) \\ \frac{a+b}{2} - x, & x \in \left[c, \frac{a+b}{2}\right) \\ x - \frac{a+b}{2}, & x \in \left[\frac{a+b}{2}, d\right) \\ x - d, & x \in [d, b] \end{cases}$$

and

$$|h_2(x)| = \begin{cases} x - a, & x \in [a, c) \\ x - c, & x \in \left[c, \frac{a+b}{2}\right) \\ d - x, & x \in \left[\frac{a+b}{2}, d\right) \\ b - x, & x \in [d, b] \end{cases}.$$

Using simple computation, we obtain that

$$\int_a^c |h_1(x)|\, dx = \int_a^c (c - x)\, dx = \int_d^b (x - d)\, dx = \int_d^b |h_1(x)|\, dx,$$

$$\int_c^{\frac{a+b}{2}} |h_1(x)|\, dx = \int_c^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) dx = \int_{\frac{a+b}{2}}^d \left(x - \frac{a+b}{2}\right) dx = \int_{\frac{a+b}{2}}^d |h_1(x)|,$$

$$\int_a^c |h_2(x)|\, dx = \int_a^c (x - a)\, dx = \int_d^b (b - x)\, dx = \int_d^b |h_2(x)|\, dx$$

and

$$\int_c^{\frac{a+b}{2}} |h_2(x)|\, dx = \int_c^{\frac{a+b}{2}} (x - c)\, dx = \int_{\frac{a+b}{2}}^d (d - x)\, dx = \int_{\frac{a+b}{2}}^d |h_2(x)|\, dx$$

which provide

$$\begin{aligned} &\int_a^b |h_1(x)|\, x dx \qquad (41) \\ &= \int_a^c (c - x)\, x dx + \int_c^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) x dx + \int_{\frac{a+b}{2}}^d \left(x - \frac{a+b}{2}\right) x dx + \int_d^b (x - d)\, x dx \\ &= \int_a^c (c - x)\, x dx + \int_c^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) x dx + \int_c^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)(a + b - x)\, dx + \int_a^c (c - x)(a + b - x)\, dx \\ &= (a + b)\left[\int_a^c (c - x)\, dx + \int_c^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) dx\right] \\ &= (a + b) \int_a^{\frac{a+b}{2}} |h_1(x)|\, dx = \frac{a+b}{2} \int_a^b |h_1(x)|\, dx \end{aligned}$$

and

$$\int_a^b |h_2(x)|\, x dx = \int_a^c (x - a)\, x dx + \int_c^{\frac{a+b}{2}} (x - c)\, x dx + \int_{\frac{a+b}{2}}^d (d - x)\, x dx + \int_d^b (b - x)\, x dx \qquad (42)$$

$$\begin{aligned} &= \int_a^c (x - a)\, x dx + \int_c^{\frac{a+b}{2}} (x - c)\, x dx \\ &= \int_c^{\frac{a+b}{2}} (x - c)(a + b - x)\, dx + \int_a^c (x - a)(a + b - x)\, dx \end{aligned}$$

$$= (a+b)\left[\int_a^c (x-a)\,dx + \int_c^{\frac{a+b}{2}} (x-c)\,dx\right]$$
$$= (a+b)\int_a^{\frac{a+b}{2}} |h_2(x)|\,dx = \frac{a+b}{2}\int_a^b |h_2(x)|\,dx.$$

The inequalities (37) and (38) follow from the identities (27), (35), (41), (42) and the inequalities (39), (40). This completes the proof. □

Remark 3.12. *In Theorems 3.1 and 3.5, let $c = a$ and $d = b$. Then Theorem 3.11 reduces to Theorem 1.4.*

4. Applications for Special Means

Throughout this section, let $H(c,d)$ *is defined as in Theorem 3.1.*

Let us recall the following special means of the two non-negative number u and v with $\alpha \in [0,1]$:

1. The weighted arithematic mean
$$A(u,v;\alpha) := \alpha u + (1-\alpha)v,\ u,v \ge 0.$$
2. The arithmetic mean
$$A(u,v) := \frac{u+v}{2},\ u,v \ge 0.$$
3. The geometric mean
$$G(u,v) := \sqrt{uv},\ u,v > 0.$$
4. The power mean
$$M_p(u,v) = \left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}},\ u,v>0 \text{ and } p \in (-\infty,\infty)\setminus\{0\}.$$
5. The logarithmic mean
$$L(u,v) := \begin{cases} \frac{v-u}{\ln v - \ln u} & \text{if } u \ne v \\ u & \text{if } u = v \end{cases},\ u,v > 0.$$
6. The identric mean
$$I = I(u,v) := \begin{cases} \frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}} & \text{if } u \ne v \\ u & \text{if } u = v \end{cases},\ u,v > 0.$$
7. The p-logarithmic mean
$$L_p(u,v) := \begin{cases} \left[\frac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right]^{\frac{1}{p}} & \text{if } u \ne v \\ u & \text{if } u = v \end{cases},\ u,v>0 \text{ and } p \in (-\infty,\infty)\setminus\{0,-1\}.$$

Using Theorems 2.1-3.11, we have the following propositions:

Proposition 4.1. *Let $r \in (-\infty,1] \cup \left[1+\frac{1}{q},\infty\right) \setminus \{0,-1\}$ and $[a,b] \subset (0,\infty)$. Then the following ineqality holds in Theorem 3.1:*

$$\begin{aligned} & \left| L_r^r(a,b) - A\left(M_r^r(a,b), M_r^r(c,d); \frac{2c-2a}{b-a}\right)\right| \\ \leq \quad & |r| H(c,d)(b-a) M_q\left(a^{r-1}, b^{r-1}\right). \end{aligned} \tag{43}$$

Proof. In Theorem 3.1, let $f : [a,b] \to (0,\infty)$ with $f(x) = x^r$ $(x \in [a,b])$. Then the inequality (21) deduces the inequality (43). This completes the proof. □

Remark 4.2. *In Proposition 4.1, let* $r \in (-\infty, 0) \cup \left[1 + \frac{1}{q}, \infty\right) \setminus \{-1\}$. *Then* f *is convex on* $[a,b]$. *By the inequality* (43) *and Theorem 2.1, we get the inequality*

$$\begin{aligned} 0 \quad \leq \quad & A\left(M_r^r(a,b), M_r^r(c,d); \frac{2c-2a}{b-a}\right) - L_r^r(a,b) \\ \leq \quad & |r| H(c,d)(b-a) M_q\left(a^{r-1}, b^{r-1}\right). \end{aligned}$$

Remark 4.3. *In Proposition 4.1, let* $0 < r \leq 1$. *Then* f *is concave on* $[a,b]$. *By the inequality* (43) *and Theorem 2.1, we get the inequality*

$$\begin{aligned} 0 \quad \leq \quad & L_r^r(a,b) - A\left(M_r^r(a,b), M_r^r(c,d); \frac{2c-2a}{b-a}\right) \\ \leq \quad & rH(c,d)(b-a) M_q\left(a^{r-1}, b^{r-1}\right). \end{aligned}$$

Proposition 4.4. *Let* $r \in (-\infty, 1] \cup \left[1 + \frac{1}{q}, \infty\right) \setminus \{0, -1\}$ *and* $[a,b] \subset (0,\infty)$. *Then the following ineqality holds in Theorem 3.5:*

$$\left| L_r^r(a,b) - A\left(M_r^r(c,d), A^r(a,b); \frac{2c-2a}{b-a}\right)\right| \leq |r| H(c,d)(b-a) M_q\left(a^{r-1}, b^{r-1}\right). \tag{44}$$

Proof. In Theorem 3.5, let $f : [a,b] \to (0,\infty)$ with $f(x) = x^r$ $(x \in [a,b])$. Then the inequality (29) deduces the inequality (44). This completes the proof. □

Remark 4.5. *In Proposition 4.4, let* $r \in (-\infty, 0) \cup \left[1 + \frac{1}{q}, \infty\right) \setminus \{-1\}$. *Then* f *is convex on* $[a,b]$. *By the inequality* (43) *and Theorem 2.1, we get the inequality*

$$\begin{aligned} 0 \quad \leq \quad & L_r^r(a,b) - A\left(M_r^r(c,d), A^r(a,b); \frac{2c-2a}{b-a}\right) \\ \leq \quad & |r| H(c,d)(b-a) M_q\left(a^{r-1}, b^{r-1}\right). \end{aligned}$$

Remark 4.6. *In Proposition 4.4, let* $0 < r \leq 1$. *Then* f *is concave on* $[a,b]$. *By the inequality* (43) *and Theorem 2.1, we get the inequality*

$$\begin{aligned} 0 \quad \leq \quad & A\left(M_r^r(c,d), A^r(a,b); \frac{2c-2a}{b-a}\right) - L_r^r(a,b) \\ \leq \quad & rH(c,d)(b-a) M_q\left(a^{r-1}, b^{r-1}\right). \end{aligned}$$

Proposition 4.7. *Let* $r \in \left(1, 1 + \frac{1}{q}\right)$ *and* $[a,b] \subset (0,\infty)$. *Then the following ineqalities hold:*

$$\begin{aligned} 0 \quad \leq \quad & A\left(M_r^r(a,b), M_r^r(c,d); \frac{2c-2a}{b-a}\right) - L_r^r(a,b) \\ \leq \quad & rH(c,d)(b-a) A^{r-1}(a,b). \end{aligned} \tag{45}$$

$$\begin{aligned} 0 \quad \leq \quad & L_r^r(a,b) - A\left(M_r^r(c,d), A^r(a,b); \frac{2c-2a}{b-a}\right) \\ \leq \quad & rH(c,d)(b-a) A^{r-1}(a,b). \end{aligned} \tag{46}$$

Proof. In Theorem 3.11, let $f : [a,b] \to [0,\infty)$ with $f(x) = x^r$ $(x \in [a,b])$. Then f is convex and $|f'|^q$ is concave on $[a,b]$. By the inequalities (37) – (38) and Theorem 2.1, we get the inequalities (45) and (46). This completes the proof. □

Proposition 4.8. *Let $[a,b] \subset (0,\infty)$ in Theorem 3.1. Then the following ineqality holds:*

$$\begin{aligned} 0 &\le \ln I(a,b) - A\left(\ln G(a,b), \ln G(c,d); \frac{2c-2a}{b-a}\right) \\ &\le H(c,d)(b-a) M_q\left(a^{-1}, b^{-1}\right). \end{aligned} \tag{47}$$

Proof. In Theorem 3.1, let $f : [a,b] \to (0,\infty)$ with $f(x) = -\ln x$ $(x \in [a,b])$. Then by the inequality (21) and Theorem 2.1, we get the inequality (47). This completes the proof. □

Proposition 4.9. *Let $[a,b] \subset (0,\infty)$ in Theorem 3.5. Then the following ineqality holds:*

$$\begin{aligned} 0 &\le A\left(\ln G(c,d), \ln A(a,b); \frac{2c-2a}{b-a}\right) - \ln I(a,b) \\ &\le H(c,d)(b-a) M_q\left(a^{-1}, b^{-1}\right). \end{aligned} \tag{48}$$

Proof. In Theorem 3.5, let $f : [a,b] \to (0,\infty)$ with $f(x) = -\ln x$ $(x \in [a,b])$. Then by the inequality (29) and Theorem 2.1, we get the inequality (48). This completes the proof. □

Proposition 4.10. *The following ineqality holds in Theorem 3.1:*

$$\begin{aligned} 0 &\le A\left(A\left(e^a, e^b\right), A\left(e^c, e^d\right); \frac{2c-2a}{b-a}\right) - \frac{e^b - e^a}{b-a} \\ &\le H(c,d)(b-a) M_q\left(e^a, e^b\right). \end{aligned} \tag{49}$$

Proof. In Theorem 3.1, let $f : [a,b] \to (0,\infty)$ with $f(x) = e^x$ $(x \in [a,b])$. Then f and $|f'|^q$ are convex on $[a,b]$. By the inequality (21) and Theorem 2.1, we get the inequality (49). This completes the proof. □

Proposition 4.11. *The following ineqality holds in Theorem 3.5:*

$$\begin{aligned} 0 &\le \frac{e^b - e^a}{b-a} - A\left(A\left(e^c, e^d\right), e^{A(a,b)}; \frac{2c-2a}{b-a}\right) \\ &\le H(c,d)(b-a) M_q\left(e^a, e^b\right). \end{aligned} \tag{50}$$

Proof. In Theorem 3.5, let $f : [a,b] \to (0,\infty)$ with $f(x) = e^x$ $(x \in [a,b])$. Then f and $|f'|^q$ are convex on $[a,b]$. By the inequality (29) and Theorem 2.1, we get the inequality (50). This completes the proof. □

References

[1] M. Alomari and M. Darus, On the Hadamard's inequality for log-convex functions on the coordinates, *J. Ineq. Appl.* Article ID 283147 (2009), 13 pages.

[2] S. S. Dragomir, Two Mappings in Connection to Hadamard's Inequalities, *J. Math. Anal. Appl.* 167 (1992), 49-56.

[3] S. S. Dragomir, On the Hadamard's Inequality for Convex on the Co-ordinates in a Rectangle from the Plane, *Taiwanese J. Math.*, 5 (4) (2001), 775-788.

[4] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.* 11 (5) (1998), 91-95.

[5] S. S. Dragomir, Y.-J. Cho and S.-S. Kim, Inequalities of Hadamard's type for Lipschitzian Mappings and Their Applications, *J. Math. Anal. Appl.* 245 (2000),489-501.

[6] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss.* 24 (1906), 369-390 (In Hungarian).

[7] J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.* 58 (1893), 171-215.

[8] S.-R. Hwang, K.-C. Hsu and K.-L. Tseng, Hadamard-type Inequalities for Lipschitzian Functions in One and Two Variables with Their Applications, *J. Math. Anal. Appl.* 405 (2013), 546-554.
[9] S.-R. Hwang, K.-L. Tseng and K.-C. Hsu, New Inequalities for Fractional Integrals and Their Applications. (Submitted)
[10] S.-R. Hwang, K.-L. Tseng and K.-C. Hsu, New Hermite-Hadamard-type Inequalities for Fractional Integrals and Their Applications. (Submitted)
[11] S.-R. Hwang, S.-Y. Yeh and K.-L. Tseng, Refinements and Similar Extensions of Hermite-Hadamard Inequality for Fractional Integrals and Their Applications, *Appl. Math. Comp.* 249 (2014), 103-113.
[12] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comp.* 147 (2004), 137-146.
[13] U. S. Kirmaci, M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Appl. Math. Comp.* 153 (2004), 361-368.
[14] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formula, *Appl. Math. Lett.* **13** (2), 51-55, (2000).
[15] H. M. Srivastava, Z.-H. Zhang and Y.-D. Wu, Some further refinements and extensions of the Hermite–Hadamard and Jensen inequalities in several variables, *Math. Comput. Model.*, 54 (2011), 2709–2717
[16] G.-S. Yang and K.-L. Tseng, On Certain Integral Inequalities Related to Hermite-Hadamard Inequalities, *J. Math. Anal. Appl.* 239 (1999), 180-187.
[17] G.-S. Yang and K.-L. Tseng, Inequalities of Hadamard's Type for Lipschitzian Mappings, *J. Math. Anal. Appl.* 260 (2001), 230-238.
[18] C. Zhu, M. Fečkan and J.-R. Wang, Fractional Integral Inequalities for Differentiable Convex Mappings and Applications to Special Means and a Midpoint Formula, *JAMSI* 8 (2) (2012), 21-28.