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# Complete Convergence for Weighted Sums of Weakly Negative Dependent of Random Variables

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**Abstract.** Some basic theoretical properties and complete convergence theorems for weighted sums of weakly negative dependent are provided and applied to random weighting estimate. Moreover, various examples are presented.

## 1. Introduction

The concept of complete convergence introduced by Hsu and Robbins [6] as follows. The sequence $\{X_i, i \geq 1\}$ of random variables convergence to constant $C$ completely, if $\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty$ for every $\varepsilon > 0$. The complete convergence of dependent random variables has been investigated by, for example, Wang *et al.* [16], Ko [7], Wu [17], Sung [15], Jun and Demei [11], Amini *et al.* [1] and Deng *et al.* [4]. In this paper, we derive some basic theoretical properties, complete convergence for weakly negative dependent (WND) under conditions and applied the results to random weighting estimate. The concept of WND random variables was introduced by Ranjbar *et al.* [12] in bivariate case, this class is well defined and contains a large class of multinomial distribution functions and some interesting application for WND sequence have been found. For example, for WND random variables with heavy-tails Ranjbar *et al.* [13] obtained the asymptotic behaviour of product of two random variables and also, Ranjbar *et al.* [12] studied the asymptotic behaviour of convolution. The assumption of WND for a sequence of random variables is special case of extended negative dependent (END), or negative dependent (ND). So, it is very significant to study on a limiting behaviour of this WND class.

The paper is organized as follows. Two important definitions, some basic properties and the relationship between WND, END and ND are given in Section 2. The main results are presented in Section 3 that application of them are used about random weighted estimate. Further, in Section 4 some well-known multivariate distributions possess in Definition 2.1.

## 2. Definitions and Preliminaries

This section contains main definitions and some properties of WND random variables. Also, we will provide some preliminary facts needed for the proof of our main results in next section.

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**Definition 2.1.** (*Ranjbar et al.* [12]) *The random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is said to be weakly negative dependent* (*WND*) *if there exists a constant* $M_1 \geq 1$*, such that for* $n > 1$

$$f(x_1, \ldots, x_n) \leq M_1 \prod_{i=1}^{n} f_i(x_i),$$

*where* $f(x_1, \ldots, x_n)$ *and* $f_i(x_i)$*'s are joint density and marginal densities of random vector* $\mathbf{X}$*, respectively.*

We say that $f(x_1, \ldots, x_n)$ is WND, if satisfy condition in Definition 2.1. A sequence $\{X_i, i \geq 1\}$ of random variables is said to be pairwise WND if $X_i$ and $X_j$ are WND for all $i \neq j$ and also, is said to be WND if every finite subset $X_1, \ldots X_n$ is WND.

In the following we have example that satisfy the Definition 2.1.

**Definition 2.2.** (*Liu* [9]) *The random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is said to be extended negative dependent* (*END*) *if there exists a constant* $M_2 > 0$ *such that both*

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq M_2 \prod_{i=1}^{n} P(X_i \leq x_i)$$

*and*

$$P(X_1 > x_1, \ldots, X_n > x_n) \leq M_2 \prod_{i=1}^{n} P(X_i > x_i)$$

*holds for* $n \geq 1$ *and* $x_1, \ldots, x_n \in \mathcal{R}$*. It is obvious that the sequence* $\{X_i, i \geq 1\}$ *is negative dependent* (*ND*) *if* $M_2 = 1$ *.*

**Example 2.3.** Let random vector $\mathbf{X} = (X_1, \ldots, X_n)$ have $n$-dimensional Farlie-Gumble-Morgenstern (*FGM*) distribution as

$$F_{1,\cdots,n}(x_1, \cdots, x_n) = \left(\prod_{i=1}^{n} F_i(x_i)\right)\left(1 + \sum_{1 \leq i \leq j \leq n} \alpha_{ij} \overline{F_i}(x_i)\overline{F_j}(x_j)\right),$$

where $\overline{F_i} = 1 - F_i$ , $i = 1, \cdots, n$, are corresponding marginal distributions and $\alpha_{ij}$ are real numbers chosen such that $F_{1,\cdots,n}$ is a proper $n$-dimensional distribution. It is easy to show that the $n$-variate FGM family is END with $M_2 = 1 + \sum_{1 \leq i \leq j \leq n} |\alpha_{ij}|$.

**Remark 2.4.** If $\mathbf{X} = (X_1, \ldots, X_n)$ is a WND random vector then is END with $M_2 \geq 1$. In fact the WND condition presented in Definition 2.1 is a useful criterion for characterization class of END random variables.

We list some basic properties of WND based on our results. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two WND random vectors, then

$P_1$. If $h_1(\cdot), \ldots, h_n(\cdot)$ are all strictly monotone real functions then $h_1(X_1), \ldots, h_n(X_n)$ are WND.

$P_2$. If $h_1(\cdot), \ldots, h_n(\cdot)$ are non-negative real function then for all $n \geq 2$, then there exists $M_1 \geq 1$ such that

$$E(\prod_{i=1}^{n} h_i(X_i)) \leq M_1 \prod_{i=1}^{n} E(h_i(X_i)).$$

In particular, if $h_i(x_i) = e^{tX_i}$, for all $i \geq 1$ and some real $t$, then $E\left(\prod_{i=1}^{n} e^{tX_i}\right) \leq M_1 \prod_{i=1}^{n} E(e^{tX_i})$.

$P_3$. If $\mathbf{X} \perp \mathbf{Y}$ ($\mathbf{X}$ and $\mathbf{Y}$ are independent ) then random vector $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is WND.

$P_4$. If $\mathbf{X} \perp \mathbf{Y}$ and $\mathbf{Y}$ is a positive random vector then random vector $(X_1Y_1, \ldots, X_nY_n)$ is END.

$P_5$. If $\mathbf{X} \perp \mathbf{Y}$ then $(X_1 + Y_1, \ldots, X_n + Y_n)$ is END.

$P_6$. If $S_n = \sum_{i=1}^{n} X_i$ , then $f_{S_n}(s) \leq M_1 f^{(n)}$, where $f^{(n)}$ is $n$-convolution independent of random variables.

$P_7$. For all $i \neq j$, there exists $M_1 \geq 1$ such that

$$Cov(X_i, X_j) \leq (M_1 - 1) \int_R \int_R F_i(x) F_j(y) dx dy.$$

$P_8$. Let random vectors $\mathbf{X}$ and $\mathbf{Y}$ have joint densities $f_1(x_1, \ldots, x_n)$ and $f_2(x_1, \ldots, x_n)$ respectively that having fixed marginal densities. Then $f_\alpha(x_1, \ldots, x_n) = \alpha f_1(x_1, \ldots, x_n) + (1-\alpha) f_2(x_1, \ldots, x_n)$ for $0 \leq \alpha \leq 1$, is still WND.

$P_9$. Any subset of WND random variables is WND.

*Proof.* Here, we prove the property $P_{10}$ as the following. One can easily show the properties $P_1 - P_9$. Let $A$ and $B$ are partitions of $\{1, 2, 3, \ldots, n\}$, $n(B) = \#B$ and $\underline{dx_j} = dx_{j_1}, dx_{j_2}, \cdots, dx_{j_{n(B)}}$ corresponding to partition B then,

$$\begin{aligned} f(x_i, i \in A) &= \int_R \ldots \int_R f(x_i, i \in A, x_j, j \in B) \underline{dx_j} \leq M_1 \int_R \ldots \int_R \prod_{i \in A} f_i(x_i) \cdot \prod_{i \in B} f_j(x_j) \underline{dx_j} \\ &= M_1 \prod_{i \in A} f(x_i). \square \end{aligned}$$

**Theorem 2.5.** *Let $\{\mathbf{X}_n, n \geq 1\}$ be a absolutely continuous sequence of WND r-dimensional random vectors with sequence of density functions $\{f_n, n \geq 1\}$ such that there exists constant $M > 0$, $\forall n : f_n \leq M$ a.e. and $f_n \to f$ a.e., where f is density function of random r-dimensional vector $\mathbf{X}$. Then $\mathbf{X}$ is END.*

*Proof.* Consider random vector $\mathbf{X_n} = (X_{1n}, X_{2n}, \ldots, X_{rn})$ and, $f_{in}$ and $f_i$ be density functions random variable $X_{in}$ and $X_i$ for $1 \leq i \leq r$ respectively, we can write

$$\begin{aligned} \lim_{n \to \infty} f_{in}(x_i) &= \lim_{n \to \infty} \int_R \ldots \int_R f_n(x_1, \ldots, x_r) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_r \\ &= \int_R \ldots \int_R f(x_1, \ldots, x_r) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_r = f_i(x_i), \ (\textit{by dominated convergence theorem}). \end{aligned}$$

Therefore exists $M_1 \geq 1$ such that

$$f(x_1, \ldots, x_r) = \lim_{n \to \infty} f_n(x_1, \ldots, x_r) \leq M_1 \prod_{i=1}^{r} \lim_{n \to \infty} f_{in}(x_i) = M_1 \prod_{i=1}^{r} f_i(x_i).$$

So, random vector $\mathbf{X}$ is WND, hence by Remark 2.4 $\mathbf{X}$ is END. □

**Definition 2.6.** *A sequence $\{X_i, i \geq 1\}$ of random variables is said to be stochastically dominated by a random variable $X$ if there exists a positive constant $C$ such that $P(|X_i| > x) \leq CP(|X| > x)$, for all $x \geq 0$ and $n \geq 1$.*

**Lemma 2.7.** *Let $\{X_i, i \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable $X$. For any $\beta > 0$ and $b > 0$, the following two statements hold:*

$$\begin{aligned} E|X_i|^\beta I(|X_i| \leq b) &\leq C_1[E|X|^\beta I(|X| \leq b) + b^\beta P(|X| > b)], \\ E|X_i|^\beta I(|X_i| > b) &\leq C_2 E|X|^\beta I(|X| > b), \end{aligned}$$

*where $C_1$ and $C_2$ are positive constants. Consequently, $E|X_i|^\beta \leq CE|X|^\beta$, where $C$ is a positive constant.*

*Proof.* The proof can be found in Lemma 4.1.6 of Wu [18]. So the details are omitted. □

## 3. Complete Convergence Theorems

In this section, we derive the complete convergence for weighted sums of WND random variables under some suitable conditions.Throughout this part, the symbol $C$ denotes constant which is not necessarily the same one in each appearance.

The following theorem concerning the rate of complete convergence of weighted sum $\sum_{i=1}^{n} a_{ni}X_i$ under some suitable conditions on the coefficients.

**Theorem 3.1.** *Let $\{X_i, i \geq 1\}$ be a sequence of WND random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers with $\sum_{i=1}^{n} a_{ni}^2 = O(n)$. If $E|X|^\alpha < \infty$, $0 < \alpha < 2$. When $1 \leq \alpha < 2$, assumes that $EX_i = 0$, $i \geq 1$. Then for $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\right) < \infty.$$

*Proof.* Let $A_n = \left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\right]$ and similarly $A_n^* = \left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i^*\right| > \varepsilon n^{1/\alpha}\right]$ in which $X_1^*, X_2^*, \ldots, X_n^*$ are independent random variables with $X_i \overset{st}{=} X_i^*$ (means, equal in distribution) for each $i$. Then

$$\begin{aligned} P(A_n) &= \int \cdots \int_{A_n} f(x_1, \ldots, x_n) dx_1 \ldots dx_n \\ &\leq M_1 \int \cdots \int_{A_n} f_1(x_1) \ldots f_n(x_n) dx_1 \ldots dx_n \quad (by\ Definition\ 2.1) \\ &= M_1 \int \cdots \int_{A_n} f_1^*(x_1) \ldots f_n^*(x_n) dx_1 \ldots dx_n \\ &= M_1 P(A_n^*), \end{aligned}$$

where enough to show that $\sum_{n=1}^{\infty} n^{-1} P(A_n^*) < \infty$.

For any $i \geq 1$, define

$$X_i^{*(n)} = X_i^* I(|X_i^*| \leq n^{1/\alpha}) + n^{1/\alpha} I(X_i^* > n^{1/\alpha}) - n^{1/\alpha} I(X_i^* < -n^{1/\alpha}),$$

$$T_k^{*(n)} = n^{-1/\alpha} \sum_{i=1}^{k} a_{ni}(X_i^{*(n)} - EX_i^{*(n)}).$$

It's easy to show that

$$\left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i^*\right| > n^{1/\alpha}\varepsilon\right] \subset \left[\bigcup_{i=1}^{n} |X_i^*| > n^{1/\alpha}\right] \cup \left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i^{*(n)}\right| > n^{1/\alpha}\varepsilon\right].$$

Then for every $\varepsilon > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i^*\right| > n^{1/\alpha}\varepsilon\right) &\leq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(|X_i^*| > n^{1/\alpha}) \\ &\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n}\left|T_k^{*(n)}\right| > \varepsilon - \max_{1\leq k\leq n} n^{-1/\alpha}\left|\sum_{i=1}^{k} a_{ni}EX_i^{*(n)}\right|\right) \\ &= I + II. \end{aligned}$$

First we show that $n^{-1/\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}EX_i^{*(n)}\right|\to 0.$

(i) When $0<\alpha<1$, then

$$\begin{aligned}
n^{-1/\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k}Ea_{ni}X_i^{*(n)}\right| &\le n^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}|E|X_i^{*(n)}|\\
&\le n^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}|\left\{E|X_i^*|I(|X_i^*|\le n^{1/\alpha})+n^{1/\alpha}P(|X_i^*|>n^{1/\alpha})\right\}\\
&\le Cn^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}|\left\{E|X|I(|X|\le n^{1/\alpha})+n^{1/\alpha}P(|X|>n^{1/\alpha})+n^{1/\alpha}P(|X|>n^{1/\alpha})\right\}\\
&\quad (\text{ by Lemma } 2.7\ )\\
&= Cn^{1-1/\alpha}\left\{E|X|I(|X|\le n^{1/\alpha})+2n^{1/\alpha}P(|X|>n^{1/\alpha})\right\}\\
&\quad (\text{ by Hölder inequality } \sum_{i=1}^{n}|a_{ni}|=O(n)\ )\\
&\le Cn^{1-1/\alpha}\sum_{k=1}^{n}E|X|I((k-1)^{1/\alpha}<|X|\le k^{1/\alpha})+2CnP(|X|>n^{1/\alpha}).
\end{aligned}$$

Since

$$\sum_{k=1}^{\infty}k^{1-1/\alpha}E|X|I((k-1)^{1/\alpha}<|X|\le k^{1/\alpha})\le\sum_{k=1}^{\infty}kP((k-1)^{1/\alpha}<|X|\le k^{1/\alpha})\le E|X|^{1/\alpha}<\infty.$$

Therefore, by Kronecker's lemma, we can conclude that

$$n^{1-1/\alpha}\sum_{k=1}^{n}E|X|I((k-1)^{1/\alpha}<|X|\le k^{1/\alpha})\to 0 \text{ as } n\to\infty,$$

and by Lebesgue dominated convergence theorem, we have

$$nP(|X|>n^{1/\alpha})=nEI(|X|^\alpha>n)\le E|X|^\alpha I(|X|^\alpha>n)\to 0 \text{ as } n\to\infty.$$

(ii) When $1\le\alpha<2$, we have by $E|X|^\alpha<\infty$ that

$$\begin{aligned}
n^{-1/\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k}Ea_{ni}X_i^{*(n)}\right| &\le n^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}||EX_i^{*(n)}|\\
&\le n^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}|\left\{|EX_i^*I(|X_i^*|\le n^{1/\alpha})|+n^{1/\alpha}P(|X_i^*|>n^{1/\alpha})\right\}\\
&\le n^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}|\left\{E|X_i^*|I(|X_i^*|>n^{1/\alpha})+n^{1/\alpha}P(|X_i^*|>n^{1/\alpha})\right\}\ (\text{ by } EX_i=0\ )\\
&\le Cn^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}|\left\{E|X|I(|X|>n^{1/\alpha})+n^{1/\alpha}P(|X|>n^{1/\alpha})\right\}\ (\text{ by Lemma } 2.7\ )\\
&\le Cn^{1-1/\alpha}E|X|I(|X|>n^{1/\alpha})+nP(|X|>n^{1/\alpha})\\
&\le Cn^{1-1/\alpha}E|X|I(|X|>n^{1/\alpha})+Cn^{1-1/\alpha}E|X|I(|X|>n^{1/\alpha})\\
&= 2Cn^{1-1/\alpha}E|X|I(|X|>n^{1/\alpha})\\
&= 2Cn^{1-1/\alpha}E|X|^\alpha|X|^{1-\alpha}I(|X|>n^{1/\alpha})\\
&\le 2CE|X|^\alpha I(|X|>n^{1/\alpha})\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence, (i) and (ii) follow that for large enough n and $\varepsilon > 0$

$$n^{-1/\alpha} \max_{1\le k\le n} \left| \sum_{i=1}^{k} E a_{ni} X_i^{*(n)} \right| < \varepsilon/2.$$

Thus, we need only to prove $I < \infty$ and $II < \infty$.

By $E|X|^\alpha < \infty$, we can get that

$$I = \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(|X_i^*| > n^{1/\alpha}) \le C \sum_{n=1}^{\infty} P(|X| > n^{1/\alpha}) \le CE|X|^\alpha < \infty.$$

For $0 < \alpha < 2$ and by setting $Y_i = X_i^{*(n)} - EX_i^{*(n)}$, we have

$$\begin{aligned}
II \le \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le k\le n} \left|T_k^{*(n)}\right| > \frac{\varepsilon}{2} n^{1/\alpha}\right) &\le \frac{4}{\varepsilon^2} \sum_{n=1}^{\infty} n^{-1-2/\alpha} Var\left(\sum_{n=1}^{n} a_{ni} Y_i\right) \ (by\ Kolmogorov\ inequality)\\
&\le \frac{4}{\varepsilon^2} \sum_{n=1}^{\infty} n^{-1-2/\alpha} E\left(\sum_{n=1}^{n} a_{ni} Y_i\right)^2\\
&= \frac{4}{\varepsilon^2} \sum_{n=1}^{\infty} n^{-1-2/\alpha} \sum_{i=1}^{n} a_{ni}^2 E|X_i^{*(n)}|^2\\
&\le \sum_{n=1}^{\infty} n^{-1-2/\alpha} \sum_{i=1}^{n} a_{ni}^2 \left\{E|X_i^*|^2 I(|X_i^*| \le n^{1/\alpha}) + n^{2/\alpha} P(|X_i^*| > n^{1/\alpha})\right\}\\
&\le \sum_{n=1}^{\infty} n^{-1-2/\alpha} \sum_{i=1}^{n} a_{ni}^2 \left\{E|X|^2 I(|X| \le n^{1/\alpha}) + 2n^{2/\alpha} P(|X| > n^{1/\alpha})\right\}\\
&\quad (\ by\ Lemma\ 2.7\ )\\
&\le \sum_{n=1}^{\infty} n^{-2/\alpha} \left\{E|X|^2 I(|X| \le n^{1/\alpha}) + 2n^{2/\alpha} P(|X| > n^{1/\alpha})\right\}\\
&\le \sum_{n=1}^{\infty} n^{-2/\alpha} \sum_{k=1}^{n} E|X|^2 I((k-1)^{1/\alpha} < |X| \le k^{1/\alpha}) + 2E|X|^\alpha\\
&\le \sum_{k=1}^{\infty} E|X|^2 I((k-1)^{1/\alpha} < |X| \le k^{1/\alpha}) \sum_{n=k}^{\infty} n^{-2/\alpha} + 2E|X|^\alpha\\
&\le C \sum_{k=1}^{\infty} k^{1-2/\alpha} E|X|^2 I((k-1)^{1/\alpha} < |X| \le k^{1/\alpha}) + 2E|X|^\alpha\\
&\quad (\ by \sum_{n=k}^{\infty} n^{-2/\alpha} < \infty \Leftrightarrow \int_{k-1}^{\infty} x^{-\frac{2}{\alpha}} dx = Ck^{1-2/\alpha} < \infty\ )\\
&\le \sum_{k=1}^{\infty} kP((k-1)^{1/\alpha} < |X| \le k^{1/\alpha}) + 2E|X|^\alpha\\
&\le CE|X|^\alpha < \infty.
\end{aligned}$$

The proof of Theorem 3.1 has completed. □

**Corollary 3.2.** *Let $\{X_i, i \geq 1\}$ be a sequence of WND random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers with $\sum_{i=1}^{n} a_i^2 = O(n)$. If $E|X|^\alpha < \infty$, $0 < \alpha < 2$. When $1 \leq \alpha < 2$, assumes that $EX_i = 0$, $i \geq 1$. Therefore*

$$\lim_{n\to\infty} \frac{\left|\sum_{i=1}^{n} a_i X_i\right|}{n^{1/\alpha}} = 0 \qquad a.s.$$

*Proof.* Let $a_{ni} = a_i$ for $i \leq n$ and $a_{ni} = 0$ for $i > n$ in Theorem 3.1. Hence, we have for all $\varepsilon > 0$

$$\begin{aligned}
\infty > \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_i X_i\right| > \varepsilon n^{1/\alpha}\right) &= \sum_{r=0}^{\infty}\sum_{n=2^r}^{2^{r+1}-1} n^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_i X_i\right| > \varepsilon n^{1/\alpha}\right) \\
&\geq \sum_{r=0}^{\infty}\sum_{n=2^r}^{2^{r+1}-1} (2^{r+1}-1)^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_i X_i\right| > \varepsilon 2^{(r+1)/\alpha}\right) \\
&\geq \frac{1}{2}\sum_{r=1}^{\infty} P\left(\max_{1\leq k\leq 2^r}\left|\sum_{i=1}^{k} a_i X_i\right| > \varepsilon 2^{(r+1)/\alpha}\right).
\end{aligned}$$

It follows from Borel-Cantelli Lemma,

$$\lim_{r\to\infty} \frac{\max_{1\leq k\leq 2^r}\left|\sum_{i=1}^{k} a_i X_i\right|}{2^{(r+1)/\alpha}} = 0. \ a.s.$$

For all positive integers n, there exists a non-negative integer $r_0$ such that $2^{r_0-1} \leq n < 2^{r_0}$. Thus

$$\begin{aligned}
\frac{1}{n^{1/\alpha}}\left|\sum_{i=1}^{n} a_i X_i\right| &\leq \max_{2^{r_0-1}\leq n<2^{r_0}} \frac{1}{n^{1/\alpha}}\left|\sum_{i=1}^{n} a_i X_i\right| \leq \frac{1}{2^{(r_0-1)/\alpha}} \max_{2^{r_0-1}\leq n<2^{r_0}}\left|\sum_{i=1}^{n} a_i X_i\right| \\
&\leq 2^{\frac{2}{\alpha}} \frac{\max_{1\leq r<2^{r_0}}\left|\sum_{i=1}^{r} a_i X_i\right|}{2^{(r_0+1)/\alpha}} \to 0 \ \ as \ \ r_0 \to \infty.
\end{aligned}$$

This completes the proof. □

Random weighting is an emerging computing method in statistics, it has been put forward by Zheng [19]. Since this method has many advantages, it drew much attention of statisticians. Gao *et al.* [5] studied law of large numbers for sample mean of random weighting estimate for i.i.d random variables. In this section we determine almost sure convergence for sample mean of random weighting estimate for WND random variables.

**Theorem 3.3.** *The random weighting estimate of sample mean $\bar{X}_n$ is $T_n = \sum_{i=1}^{n} \beta_i X_i$ such that $\sum_{i=1}^{n} \beta_i = 1$ for random variables $0 \leq \beta_i \leq 1$ a.s., under the conditions of Corollary 3.2,*

$$\frac{|T_n - \overline{X}_n|}{n^{1/\alpha}} \to 0 \ \ as \ \ n \to \infty,$$

*where $\Theta_i = \beta_i - \frac{1}{n}$ a.s.*

*Proof.* We can write $T_n - \bar{X} = \sum_{i=1}^{n} \Theta_i X_i$ and $\sum_{i=1}^{n} \Theta_i^2 = O(n)$ a.s. hence conditions Corollary 3.2 hold when $\Theta_i = a_i$ a.s. for $i \geq 1$ and $0 < \alpha < 2$. For that reason, this completes the proof of Theorem 3.3. □

**Remark 3.4.** Theorem 3.3 extends and improves of the Theorem 2 obtained by Gao *et al.* [5].

## 4. Examples

In this section, some useful examples are discussed. In each case, Definition 2.1 is provided.

A copula is a function that links univariate marginal distributions to the full multivariate distribution and it's a general tool for assessing the dependence structure of random variables. For details see Nelsen [10].

**Definition 4.1.** *An n-dimensional copula is a function C with domain* $[0,1]^n$ *if*

(*i*) $C(\boldsymbol{u}) = 0$ *whenever* $\boldsymbol{u} \in [0,1]^n$ *has at least one component equal to 0,*

(*ii*) $C(\boldsymbol{u}) = u_i$ *whenever* $\boldsymbol{u} \in [0,1]^n$ *has all components equal to 1 except the i-th one, which is equal to* $u_i$*,*

(*iii*) *C is n-increasing,i.e., for each n-box* $B = \times_{i=1}^n [u_i, v_i]$ *in* $[0,1]^n$ *with* $u_i \le v_i$ *for each* $i \in 1, \cdots, n$*,*

$$V_C(B) = \sum_{z \in \times_{i=1}^n [u_i, v_i]} (-1)^{N(z)} C(z) \ge 0,$$

*where* $N(\boldsymbol{z}) = card\{k | z_k = u_k\}$*.*

Some well-known examples of $n$-copulas are the functions $\Pi$, $M$ and $W$ respectively defined by

$$\Pi(x_1, x_2, \ldots, x_n) = x_1.x_2 \ldots .x_n$$
$$M(x_1, x_2, \ldots, x_n) = Min(x_1, x_2, \ldots, x_n)$$
$$W(x_1, x_2, \ldots, x_n) = Max(x_1 + x_2 + \ldots + x_n - n + 1, 0)$$

and while $\Pi$ and $M$ are always $n$-copulas, $W$ is a copula when $n = 2$.

More formally, Sklar's theorem states that any continuous the joint distribution can be uniquely represented by a copula

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)),$$

where $F$ is the joint distribution to the marginal distribution functions $F_1(x_1), F_2(x_2), \ldots, F_n(x_n)$.

Recall that the copula density $c$ is the derivative of $C$ with respect to each of its arguments as:

$$c(u_1, u_2, \ldots, u_n) = \frac{\partial^n}{\partial u_1 \ldots, \partial u_n} C(u_1, \ldots, u_n).$$

By using the chain rule, the density function $f(x_1, x_2, \ldots, x_n)$ corresponding to the copula $C(x_1, x_2, \ldots, x_n)$ is,

$$f(x_1, x_2, \ldots, x_n) = c(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) \prod_{i=1}^n f_i(x_i)$$

where $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$ are the marginal densities of $f$.

**Lemma 4.2.** *Let* $\boldsymbol{X} = (X_1, \ldots, X_n)$ *be a random vector having to joint distribution function* $F(x_1, \ldots, x_n)$ *and absolute continuous marginals* $F_1, \ldots, F_n$*, with corresponding copula function* $C(u_1, \cdots, u_n)$*. If* $c(u_1, \ldots, u_n) \le M_1$ *for some* $M_1 \ge 1$*, then* $\boldsymbol{X}$ *is WND.*

*Proof.* The proof of the lemma is standard, so we omit the details. □

**Example 4.3.** Suppose that $C(u_1, \ldots, u_n) = (\prod_{i=1}^n u_i) \exp\{\beta \prod_{i=1}^n (1 - u_i)\}$ for $\beta \in [-1, 1]$. Then $C$ is a absolutely continuous copula which was introduced by Čelebioğlu [2] and also Cuadras [3] has studied some of it's properties. It's easy to check that if $\beta \in [-1, 1]$ there exists $M_1 = \exp(|\beta|) \ge 1$ such that Lemma 4.2 holds, for $n \ge 2$. Therefore, all of the distributions with given margins $F_1, F_2, \ldots, F_n$ generated by this copula are WND.

**Example 4.4.** (The Farlie-Gumble-Morgenstern $n$-copulas). Let $(X_1, \ldots, X_n)$ be random vector with the following joint distribution:

$$F(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} F_i(x_i) \left[ 1 + \sum_{l=2}^{n} \sum_{1 \le j_1 < \ldots < j_l \le n} \theta_{j_1 j_2 \ldots, j_l} \bar{F}_{j_1}(x_{j_1}) \bar{F}_{j_2}(x_{j_2}) \ldots \bar{F}_{j_l}(x_{j_k}) \right] (|\theta_{j_1 j_2 \ldots j_l}| \le 1).$$

Where $\bar{F}_{j_k} = 1 - F_{j_k}$ for $l = 2, \ldots, n$. Via Sklar's Theorem, it's easy to show that, the copula density is

$$c(u_1, u_2, \ldots, u_n) = 1 + \sum_{l=2}^{n} \sum_{1 \le j_1 < \ldots < j_l \le n} \theta_{j_1 j_2 \ldots j_l} (1 - 2u_{j_1})(1 - 2u_{j_2}) \ldots (1 - 2u_{j_l}).$$

It is easy to show that $c(u_1, u_2, \ldots, u_n) \le 1 + \sum_{l=2}^{n} \sum_{1 \le j_1 < \ldots < j_l \le n} |\theta_{j_1 j_2 \ldots j_l}|$, so there exists $M_1 \ge 1$ such that Lemma 4.2 holds. Therefore, all of the distributions with given margins $F_1, F_2, \ldots, F_n$ generated by this copula are WND.

**Example 4.5.** (The Dirichlet distribution) Let $(X_1, \ldots, X_n)$ denotes random vector with the following joint density function:

$$f(x_1, \ldots, x_n) = \frac{\Gamma(\sum_{i=1}^{n} \gamma_i)}{\prod_{i=1}^{n} \Gamma(\gamma_i)} \prod_{i=1}^{n} x_i^{\gamma_i - 1}, \; x_i \ge 0, \; \sum_{i=1}^{n} x_i = 1.$$

If $0 \le \gamma_i \le 1$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \gamma_i \le 1$ then $\mathbf{X}$ is WND.

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