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The Gamma Exponentiated Exponential–Weibull Distribution

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Abstract. A new four-parameter model called the gamma–exponentiated exponential–Weibull distribution is being introduced in this paper. The new model turns out to be quite flexible for analyzing positive data. Representations of certain statistical functions associated with this distribution are obtained. Some special cases are pointed out as well. The parameters of the proposed distribution are estimated by making use of the maximum likelihood approach. This density function is utilized to model two actual data sets. The new distribution is shown to provide a better fit than related distributions as measured by the Anderson–Darling and Cramér–von Mises goodness–of–fit statistics. The proposed distribution may serve as a viable alternative to other distributions available in the literature for modeling positive data arising in various fields of scientific investigation such as the physical and biological sciences, hydrology, medicine, meteorology and engineering.

1. Introduction

The Weibull distribution is a popular life time distribution model in reliability engineering. However, this distribution does not have a bath tub or upside–down bath tub shaped hazard rate function, which is why it cannot be utilized to model the life time of certain systems. To overcome this shortcoming, several generalizations of the classical Weibull distribution have been discussed by different authors in recent years. Many authors introduced flexible distributions for modeling complex data and obtaining a better fit. Extensions of Weibull distribution arise in different areas of research as discussed for instance in [1–12], [16], [20], [27], [29–31, 33]. Many extended Weibull models have an upside–down bath tub shaped hazard rate, which is the case of the extensions discussed by [5], [18], [24], and [30], among others.

By adding parameters to an existing distribution we obtain classes of more flexible distributions, see for instance the method by Zografos and Balakrishnan [33]. Their new distribution provides more flexibility to model various types of data. The baseline distribution has the survivor function $\overline{G}(x) = 1 - G(x)$. Then, the gamma–exponentiated extended distribution has cumulative distribution function (CDF) $F(x)$ given by

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^{-\log \overline{G}(x)} t^{\alpha-1}\, \mathrm{e}^{-t}\, \mathrm{d}t\,, \qquad \alpha > 0,\ x \in \mathbb{R}\,. \tag{1}$$

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The gamma–exponentiated extended probability density function (PDF) related to (1) can be expressed in the following form:

$$f(x) = \frac{1}{\Gamma(\alpha)} \left(-\log \overline{G}(x) \right)^{\alpha-1} g(x), \qquad \alpha > 0,\, x \in \mathbb{R}.$$

Cordeiro *et al.* [8] introduced an exponential–Weibull distribution. The CDF and PDF of their distribution are defined as follows:

$$G(x) = \left(1 - \mathrm{e}^{-\lambda x - \beta x^k}\right) \mathbf{1}_{\mathbb{R}_+}(x), \qquad \min\{\lambda, \beta, k\} > 0,$$

and

$$g(x) = \left(\lambda + \beta k x^{k-1}\right) \mathrm{e}^{-\lambda x - \beta x^k} \mathbf{1}_{\mathbb{R}_+}(x);$$

here and in what follows $\mathbf{1}_A(x)$ denotes the characteristic function of the set A, that is $\mathbf{1}_A(x) = 1$ when $x \in A$ and equals 0 else.

Now, we generalize their model by applying the gamma–exponentiated technique [33], which results in what we are referring to as the Gamma–exponentiated exponential–Weibull distribution. The new model's characterization is as follows.

In the sequel we apply the so–called *regularized gamma function*

$$Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_z^\infty t^{a-1} \mathrm{e}^{-t}\, \mathrm{d}t, \qquad \Re(a) > 0,$$

where $\Gamma(a, x)$ denotes the familiar *upper incomplete gamma function*. Both, regularized gamma and incomplete gamma, are in-built in *Mathematica* under `GammaRegularized[a, z]` and `Gamma[a,z]` respectively.

Consider a random variable X on a standard probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, having cumulative distribution function and the probability density function given by

$$F(x) = \left(1 - Q\left(\alpha,\, \lambda x + \beta x^k\right)\right) \mathbf{1}_{\mathbb{R}_+}(x), \tag{2}$$

$$f(x) = \frac{1}{\Gamma(\alpha)} \left(\lambda + \beta k x^{k-1}\right) \mathrm{e}^{-\lambda x - \beta x^k} \left(\lambda x + \beta x^k\right)^{\alpha-1} \mathbf{1}_{\mathbb{R}_+}(x), \tag{3}$$

respectively, where the baseline survivor function $\overline{G}(x) = 1 - G(x) = \exp\{-(\lambda x + \beta x^k)\}$. Then we said that X has *Gamma–exponentiated exponential–Weibull*, writing this $X \sim \mathrm{GEEW}(\theta), \theta = (\lambda, \beta, k, \alpha)$, where the four–parameter vector θ is assumed to be strictly positive. Also the related hazard rate function becomes

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\left(\lambda + \beta k x^{k-1}\right) \mathrm{e}^{-\lambda x - \beta x^k} \left(\lambda x + \beta x^k\right)^{\alpha-1}}{\Gamma\left(\alpha,\, \lambda x + \beta x^k\right)} \mathbf{1}_{\mathbb{R}_+}(x). \tag{4}$$

The values appearing in (2), (3), that is in (4) can be evaluated numerically using computational packages such as *Mathematica*, *Maple*, *MATLAB* and *R*.

Graphical representations of the parameter effects are included in Section 2. Representations of certain statistical functions are provided in Section 3. The parameter estimation technique described in Section 4 is utilized in connection with the modeling of two actual data sets originating from the engineering and biological sciences in Section 5, where the new model is compared with several related distributions.

2. Some Graphical Representations of the GEEW Distribution

Graphs of the PDF (3) and the hazard rate function (4) are presented in this section for certain values of parameters. It is manifest that the parameter k and α influence the shape of the hazard function.

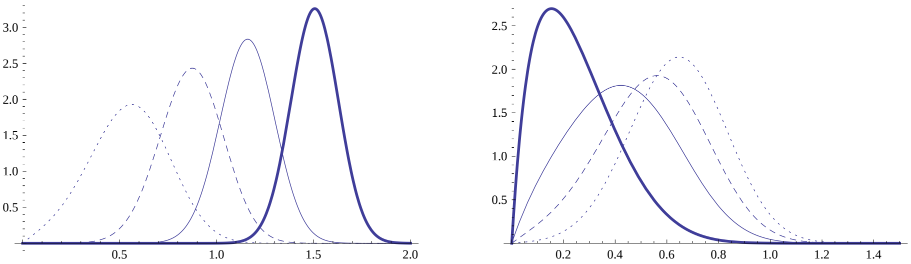

Figure 1: The GEEW PDF. Left panel: $\lambda = 1.5$, $\beta = 5.1$, $k = 3$ and $\alpha = 2$ (dotted line) $\alpha = 5$ (dashed line), $\alpha = 10$ (solid line), $\alpha = 20$ (thick line). Right panel: $\beta = 5.1$, $k = 3$, $\alpha = 2$ and $\lambda = 0.5$ (dotted line) $\lambda = 1.5$ (dashed line), $\lambda = 3$ (solid line), $\lambda = 7$ (thick line).

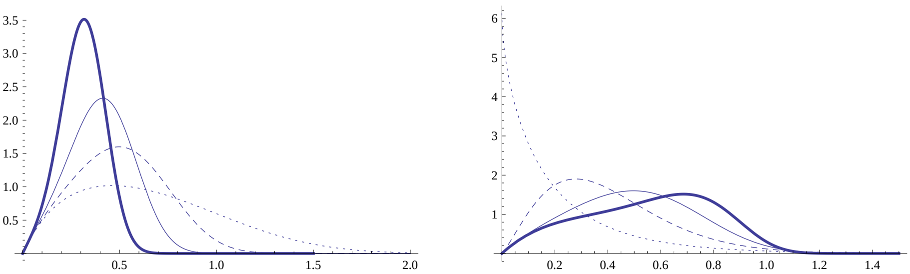

Figure 2: The GEEW PDF. Left panel: $\lambda = 2.5$, $k = 3$, $\alpha = 5$, and $\beta = 0.5$ (dotted line), $\beta = 3.5$ (dashed line), $\beta = 10$ (solid line), $\beta = 30$ (thick line). Right panel: $\lambda = 2.5$, $\beta = 3.5$, $\alpha = 2$, and $k = 0.5$, (dotted line) $k = 1.5$ (dashed line), $k = 3$ (solid line), $k = 5$ (thick line).

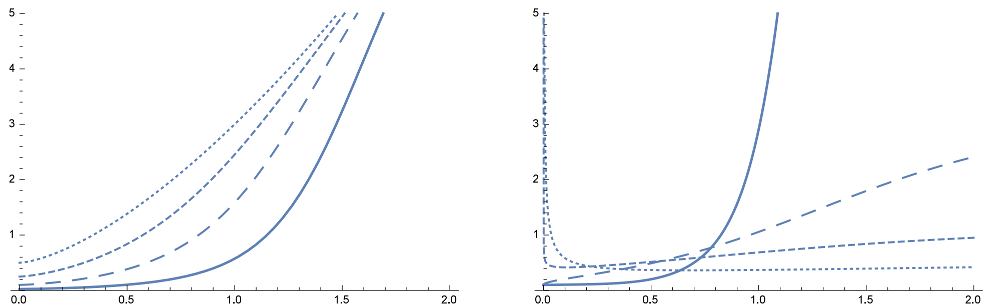

Figure 3: The GEEW hazard rate function. Left panel: $\lambda = 2.5$, $\beta = 3.5$, $k = 1.5$ and $\alpha = 1.5$, (dotted line) $\alpha = 2.5$ (short dashes), $\alpha = 5.5$ (long dashes), $\alpha = 10$ (solid line), $\alpha = 20$ (thick line). Right panel: $\lambda = 2.5$, $\beta = 2.5$, $k = 2.2$ and $\alpha = 1.2$, (dotted line) $\alpha = 2$ (short dashes), $\alpha = 3.5$ (long dashes), $\alpha = 6$ (solid line), $\alpha = 12$ (thick line).

Figures 1 and 2 indicate how the four of the parameters including new scale parameter α affect the GEEW(θ) density. These graphs illustrate the versatility of the GEEW distribution and indicate that the new parameter α has a shifting effect in addition to a noticeable effect on the skewness and kurtosis of this distribution. As can be seen from Figure 3, depending on the value of α, the hazard rate function can assume a variety of shapes.

3. Statistical Functions of the GEEW distribution

Here, we derive computational sum–representations of general order moments associated with the the rv $X \sim$ GEEW(θ) and obtain the explicit form of the quantile function in general case and two related special cases are obtained. The resulting expressions can be evaluated exactly or numerically with symbolic computational packages such as *Mathematica*, *MATLAB* or *Maple*. In numerical applications, infinite sum can be truncated whenever convergence is observed.

3.1. Moments

To derive the r^{th} raw moment of the rv $X \sim \text{GEEW}(\theta), \theta = (\lambda, \beta, k, \alpha) > 0$, we need some auxiliary tools is the form of a definite integral which integrand is related to the PDF (3) of X.

Lemma 3.1. *For all positive (μ, a, ν, ρ), for which $\rho + \nu^{-1}(\ell + \mu) \notin \mathbb{N}$ when $\ell \in \mathbb{N}$, we have*

$$\begin{aligned} I_\mu(a, \nu, \rho) &= \int_0^\infty x^{\mu-1}(1 + ax^\nu)^\rho\, \mathrm{e}^{-x}\, \mathrm{d}x \\ &= \sum_{n\geq 0} \frac{(-1)^n\, (-\rho)_n}{n!}\, \Big\{a^n\, \gamma\Big(\mu + \nu\, n, a^{-1/\nu}\Big) + a^{\rho-n}\, \Gamma\Big(\mu + \nu(\rho - n), a^{-1/\nu}\Big)\Big\}, \end{aligned} \tag{5}$$

where $\gamma(a, z) = \Gamma(a) - \Gamma(a, z), \Re(a) > 0$ signifies the lower incomplete gamma function.

Moreover, when $\rho \in \mathbb{N}_0$, $(\mu, a, \nu) > 0$, we have

$$I^*_\mu(a, \nu, \rho) = \Gamma(\mu) \sum_{n=0}^{\rho} (-\rho)_n\, (\mu)_{\nu n}\, \frac{(-a)^n}{n!}\,. \tag{6}$$

Proof. The binomial series of $(1 + ax^\nu)^\rho$ converges only for $|x| < a^{-1/\nu} = a_0$, therefore we have to split the integral into two parts

$$I_\mu(a, \nu, \rho) = \left(\int_0^{a_0} + \int_{a_0}^\infty\right) x^{\mu-1}(1 + ax^\nu)^\rho\, \mathrm{e}^{-x}\, \mathrm{d}x =: I_1 + I_2\,.$$

Consequently, by the legitimate exchange of the order of integration and summation, we have

$$I_1 = \sum_{n\geq 0} \binom{\rho}{n} a^n \int_0^{a_0} x^{\mu+\nu\, n-1}\, \mathrm{e}^{-x}\, \mathrm{d}x = \sum_{n\geq 0} \frac{(-\rho)_n\, (-a)^n}{n!}\, \gamma(\mu + \nu\, n, a_0)\,.$$

On the other hand for $x > a^{-1/\nu} = a_0$ transforming the binomial term in the integrand we have

$$\begin{aligned} I_2 &= a^\rho \int_{a_0}^\infty x^{\mu+\nu\rho-1}\Big(1 + a^{-1}x^{-\nu}\Big)^\rho\, \mathrm{e}^{-x}\, \mathrm{d}x = \sum_{n\geq 0} \binom{\rho}{n} a^{\rho-n} \int_{a_0}^\infty x^{\mu+\nu(\rho-n)-1}\, \mathrm{e}^{-x}\, \mathrm{d}x \\ &= \sum_{n\geq 0} \frac{(-\rho)_n\, (-a)^{\rho-n}}{n!}\, \Gamma(\mu + \nu(\rho - n), a_0). \end{aligned}$$

The sum of I_1 and I_2 gives the value of the considered integral.

In the case when $\rho \in \mathbb{N}_0$, the binomial term becomes a polynomial in x^k which causes that the integral turns out to be a polynomial in $-a$ of degree ρ with generalized Pochhammer symbol coefficients. So the expression (6). □

Theorem 3.2. *Let $X \sim$ GEEW$(\theta), \theta = (\lambda, \beta, k, \alpha)$ a rv having* PDF

$$f(x) = \frac{1}{\Gamma(\alpha)}(\lambda + \beta k x^{k-1})(\lambda x + \beta x^k)^{\alpha-1}\, \mathrm{e}^{-(\lambda x+\beta x^k)}\, 1_{\mathbb{R}_+}(x)\,.$$

Then, for all $r + \alpha > 0$ and $\alpha - 1 + (\ell + r + \alpha)/(k - 1) \notin \mathbb{N}$ when $\ell \in \mathbb{N}$, we have

$$\begin{aligned} \mathsf{E}X^r = \frac{1}{\lambda^r} \sum_{m,n\geq 0} \Bigg\{ &\left(\frac{\beta}{\lambda^k}\right)^n \Big[q\Big(r + \alpha + km + (k - 1)n, a_0\Big) + \frac{\beta k}{\lambda^k}\, q\Big(r + \alpha + k(m + 1) + (k - 1)n - 1, a_0\Big)\Big] \\ &+ \left(\frac{\beta}{\lambda^k}\right)^{\alpha-n-1} \Big[Q\Big(r + k(\alpha - 1) + 1 + km - (k - 1)n, a_0\Big) \end{aligned}$$

$$+ \frac{\beta k}{\lambda^k}\, Q\Big(r + k\alpha + km - (k-1)n, a_0\Big)\Big]\bigg\} \frac{(-\beta\lambda^{-k})^m}{m!}\, \frac{(-1)^n\,(1-\alpha)_n}{n!}\,, \tag{7}$$

where $a_0 = \left(\beta\lambda^{-k}\right)^{-1/(k-1)}$ *and* $q(a,z) = \gamma(a,z)/\Gamma(a), \Re(a) > 0$ *denotes the regularized lower incomplete gamma function.*

Proof. Firstly, we split the PDF $f(x)$ in (3) into two addends as follows

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\left(1 + \frac{\beta}{\lambda} x^{k-1}\right)^{\alpha-1} \mathrm{e}^{-(\lambda x + \beta x^k)}\, 1_{\mathbb{R}_+}(x) + \frac{\lambda^{\alpha-1}\beta k}{\Gamma(\alpha)} x^{k+\alpha-2}\left(1 + \frac{\beta}{\lambda} x^{k-1}\right)^{\alpha-1} \mathrm{e}^{-(\lambda x + \beta x^k)}\, 1_{\mathbb{R}_+}(x).$$

Obviously, the moment $\mathsf{E}X^r$ is equal to the linear combination of two integrals $I_\mu(a,\nu,\rho)$ described in Lemma 1. Indeed, expanding the the exponential terms $\exp\{-\beta x^k\}$ into Maclaurin series, then substituting $x \mapsto \lambda x$, exchanging the order of integration and summation, finally applying Lemma 1, we arrive at

$$\mathsf{E}X^r = \frac{1}{\lambda^r\,\Gamma(\alpha)} \sum_{m\geq 0} \frac{(-\beta)^m}{m!} \left\{ I_{r+\alpha+km}(\beta\lambda^{-k}, k-1, \alpha-1) + \frac{\beta k}{\lambda^k}\, I_{r+\alpha+k(m+1)-1}(\beta\lambda^{-k}, k-1, \alpha-1) \right\}.$$

The rest is straightforward *via* (5). □

Certain attractive special cases of the Theorem 1, which are evidently not so obvious corollaries of Lemma 1 and (7) are listed in the sequel. First we need the definition of the unified confluent Fox–Wright hypergeometric function ${}_1\Psi_0^*$, see Appendix A, then the Meijer G function, see Appendix B.

Theorem 3.3. *Let* $X \sim \mathrm{GEEW}(\theta)$. *Then, for all* $\alpha \in \mathbb{N}$; $(\lambda, \beta, k) > 0$ *and for all* $r > \max\{-\alpha, 1-\alpha-k\}$, *we have*

$$\begin{aligned} \mathsf{E}X^r &= \frac{(r)_\alpha}{\lambda^r} \sum_{n=0}^{\alpha-1} \frac{(1-\alpha)_n (r+\alpha)_{kn}}{n!}\, {}_1\Psi_0^*\left[\begin{matrix} (r+\alpha+(k-1)n, k) \\ - \end{matrix} \,\Big|\, -\frac{\beta}{\lambda^k} \right] \left(-\frac{\beta}{\lambda^k}\right)^n \\ &\quad + \frac{(r+k-1)_\alpha}{\lambda^{r+k}} \sum_{n=0}^{\alpha-1} \frac{(1-\alpha)_n (r+\alpha+k-1)_{kn}}{n!}\, {}_1\Psi_0^*\left[\begin{matrix} (r+\alpha-1+k(n+1), k) \\ - \end{matrix} \,\Big|\, -\frac{\beta}{\lambda^k} \right] \left(-\frac{\beta}{\lambda^k}\right)^n . \end{aligned}$$

Proof. By direct calculation we expand the exponential term $\exp\{-\beta x^k\}$ into Maclaurin series, then we transform the binomial in the integrand, following the derivation steps used in Lemma 1 we deduce

$$\begin{aligned} \mathsf{E}X^r &= \sum_{m\geq 0} \frac{(-\beta)^m}{m!} \left\{ \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{r+\alpha-1}\left(1 + \frac{\beta}{\lambda} x^{k-1}\right)^{\alpha-1} \mathrm{e}^{-\lambda x}\,\mathrm{d}x \right. \\ &\qquad \left. + \frac{\lambda^{\alpha-1}\beta k}{\Gamma(\alpha)} \int_0^\infty x^{r+\alpha+k-2}\left(1 + \frac{\beta}{\lambda} x^{k-1}\right)^{\alpha-1} \mathrm{e}^{-\lambda x}\,\mathrm{d}x \right\} \\ &= \frac{1}{\lambda^r\Gamma(\alpha)} \sum_{m\geq 0} \left\{ I^*_{r+\alpha+km}(\beta\lambda^{-k}, k-1, \alpha-1) \right. \\ &\qquad \left. + \frac{\beta k}{\lambda^k}\, I^*_{r+\alpha+k-1+km}(\beta\lambda^{-k}, k-1, \alpha-1) \right\} \frac{\left(-\frac{\beta}{\lambda^k}\right)^m}{m!}\,. \end{aligned} \tag{8}$$

Rewriting (6) into

$$I^*_{r+\alpha+km}(\beta\lambda^{-k}, k-1, \alpha-1) = \Gamma(r+\alpha) \sum_{n=0}^{\alpha-1} (1-\alpha)_n\, (r+\alpha+(k-1)n)_{km} (r+\alpha)_{(k-1)n}\, \frac{\left(-\frac{\beta}{\lambda^k}\right)^n}{n!}\,,$$

we get

$$\begin{aligned}\mathsf{E}X^r &= \frac{(r)_\alpha}{\lambda^r}\sum_{n=0}^{\alpha-1}(1-\alpha)_n\,(r+\alpha)_{(k-1)n}\frac{\left(-\frac{\beta}{\lambda^k}\right)^n}{n!}\sum_{m\geq 0}(r+\alpha+(k-1)n)_{km}\,\frac{\left(-\frac{\beta}{\lambda^k}\right)^m}{m!}\\ &+\frac{(r+k-1)_\alpha}{\lambda^{r+k}}\sum_{n=0}^{\alpha-1}(1-\alpha)_n\,(r+\alpha+k-1)_{(k-1)n}\frac{\left(-\frac{\beta}{\lambda^k}\right)^n}{n!}\\ &\times\sum_{m\geq 0}(r+\alpha+k-1+(k-1)n)_{km}\,\frac{\left(-\frac{\beta}{\lambda^k}\right)^m}{m!}\,,\end{aligned}$$

which is equivalent to the stated formula. □

Theorem 3.4. *Let $X \sim \mathrm{GEEW}(\lambda,(u\lambda)^{-2},3,\alpha)$. Then for all $\lambda > 0, u > 0, \alpha > -1$ and for all $r > -1$, we have*

$$\begin{aligned}\mathsf{E}X^r &= \frac{u^{r+\alpha}\lambda^r}{2\sqrt{\pi}\,\Gamma(\alpha)\Gamma(1-\alpha)}\sum_{n\geq 0}\left\{G^{31}_{13}\left(\frac{(u\lambda)^2}{4}\,\middle|\,\begin{matrix}1-\frac{r+\alpha+3n}{2}\\ 1-\frac{r-\alpha+3n}{2},\,0,\,\frac12\end{matrix}\right)\right.\\ &\left.+\frac{3}{\lambda^3}\,G^{31}_{13}\left(\frac{(u\lambda)^2}{4}\,\middle|\,\begin{matrix}-\frac{r+\alpha+3n}{2}\\ \frac{\alpha-r-3n}{2},\,0,\,\frac12\end{matrix}\right)\right\}\frac{(-u\lambda^{-2})^n}{n!}\,.\end{aligned}$$

Proof. Consider the formula which connects the I^*–type integrals which occur in Lemma 1 and the Meijer's G function [17, p. 347, Eq. **3.389** 2^7]:

$$\int_0^\infty x^{2\mu-1}(u^2+x^2)^{\alpha-1}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x = \frac{u^{2(\mu+\alpha-1)}}{2\sqrt{\pi}\,\Gamma(1-\alpha)}\,G^{31}_{13}\left(\frac{(u\lambda)^2}{4}\,\middle|\,\begin{matrix}1-\mu\\ 1-\alpha-\mu,\,0,\,\frac12\end{matrix}\right),$$

where $|\arg(u)| < \frac{\pi}{2}, \Re(\lambda) > 0, \Re(\mu) > 0$. The parameters fit the theorem's constraints, so

$$\int_0^\infty x^{2\mu-1}(u^2+x^2)^{\alpha-1}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x = \frac{u^{2\alpha-2}}{\lambda^{2\mu}}\,I^*_{2\mu}((u\lambda)^{-2},2,\alpha-1)\,,$$

therefore

$$I^*_{2\mu}((u\lambda)^{-2},2,\alpha-1) = \frac{(u\lambda)^{2\mu}}{2\sqrt{\pi}\,\Gamma(1-\alpha)}\,G^{31}_{13}\left(\frac{(u\lambda)^2}{4}\,\middle|\,\begin{matrix}1-\mu\\ 1-\alpha-\mu,\,0,\,\frac12\end{matrix}\right).$$

Inserting in (8) the appropriate $I^*_{2\mu}$ for $2\mu = r+\alpha+3m$ and $2\mu = r+\alpha+2+3m$, respectively, we finish the proof of the statement. □

Remark 3.5. *Finally, in the special case of* Theorem 1, *that is, for the rv $X \sim \mathrm{GEEW}(\lambda,(b\lambda)^{-1},2,\alpha)$, when $\lambda > 0, b > 0, \alpha > -1$ and $r > -1$, the moments $\mathsf{E}\,X^r$ are expressible in terms of the Whittaker function of the second kind $W_{a,b}(z)$, introduced e.g. in* [21, p. 134]. *This case is also of interest since Whittaker function is implemented in Mathematica as* `WhittakerW[k, m, z].`

3.2. Quantile Function

The next statistical function, the *quantile function* $\mathscr{Q}_X$ for the rv $X \sim F(x)$, say

$$\mathscr{Q}_X(p) = \inf\{x \in \mathbb{R} \colon p \leq F(x)\}\,, \qquad p \in (0,1)\,;$$

it consists form the generalized inverse of the CDF for a fixed probability p.

Theorem 3.6. *Let $X \sim \mathrm{GEEW}(\theta)$, $\theta = (\lambda, \beta, k, \alpha) > 0$. Then the related quantile function $\mathcal{Q}_X(p)$ is the unique positive solution of the equation*

$$\lambda x + \beta x^k = Q^{-1}\Big(\alpha,\, 1 - p\Big), \qquad x = \mathcal{Q}_X(p),\, p \in (0,1)\,,$$

where Q^{-1} stands for the inverse of the function Q.

Proof. The quantile function we derive by inverting (2). So, fixing $p \in (0,1)$ fixed, solving the equation $1 - Q\Big(\alpha,\, \lambda x + \beta x^k\Big) = F(x) = p$ with respect to the regularized upper incomplete Gamma–function Q, we get $Q\Big(\alpha,\, \lambda x + \beta x^k\Big) = 1 - p$. Because $\Gamma'(a, z) = -z^{a-1}\mathrm{e}^{-z} < 0$, the function $\Gamma(a, z) = \Gamma(\alpha)\, Q(a, z)$ is monotone in z, therefore Q possesses an unique inverse Q^{-1}. Thus we yield

$$\lambda x + \beta x^k = Q^{-1}\Big(\alpha,\, 1 - p\Big),$$

which finishes the proof. □

Corollary 3.7. *Let $X \sim \mathrm{GEEW}(\lambda, (u\lambda)^{-2}, 3, \alpha)$. Then for all $p \in (0,1)$ we have*

$$\mathcal{Q}_X(p) = \sqrt[3]{-\frac{(u\lambda)^2}{2}Q^{-1} + \sqrt{\frac{(u\lambda)^4}{4}(Q^{-1})^2 + \frac{u^4\lambda^6}{27}}} + \sqrt[3]{-\frac{(u\lambda)^2}{2}Q^{-1} - \sqrt{\frac{(u\lambda)^4}{4}(Q^{-1})^2 + \frac{u^4\lambda^6}{27}}},$$

where $Q^{-1} = Q^{-1}\Big(\alpha,\, 1 - p\Big)$ stands for the inverse of the function Q.

Moreover, when $X \sim \mathrm{GEEW}(\lambda, (b\lambda)^{-1}, 2, \alpha)$, we have

$$\mathcal{Q}_X(p) = \frac{b\lambda}{2}\left(\sqrt{\lambda^2 + 4\frac{Q^{-1}(\alpha, 1-p)}{b\lambda}} - \lambda\right), \qquad p \in (0,1)\,.$$

4. Parameter Estimation

In this section, we will make use of the GEEW(θ), extended Weibull (ExtW) [26], exponential–Weibull (EW) [8], gamma exponentiated exponential (GEE) distribution [28], two parameter Weibull (Weibull) and two parameter gamma (Gamma) distributions to model two well–known real data sets, namely the 'Carbon fibres' [25] and the 'Cancer patients' [20] data sets. The parameters of the GEEW distribution can be estimated from the loglikelihood of the samples in conjunction with the *NMaximize* command in the symbolic computational package *Mathematica*. Additionally, two goodness-of-fit measures are proposed to compare the density estimates.

4.1. Maximum Likelihood Estimation

In order to estimate the parameters of the proposed GEEW density function as defined in Equation (7), the loglikelihood of the sample is maximized with respect to the parameters. Given the data $\boldsymbol{x} = (x_1, \cdots, x_n)$, the loglikelihood function is

$$\ell(\theta) = -n\,\log\Gamma(\alpha) - \sum_{i=1}^{n}\Big(\lambda\, x_i + \beta\, x_i^k\Big) + (\alpha - 1)\,\sum_{i=1}^{n}\Big(\lambda\, x_i + \beta\, x_i^k\Big) + \sum_{i=1}^{n}\log\Big(\lambda + \beta\, k\, x_i^{k-1}\Big)\,,$$

where $f(x)$ is as given in (3). The associated nonlinear loglikehood system $\frac{\partial\ell(\theta)}{\partial\theta} = 0$ for MLE estimator derivation reads as follows:

$$\frac{\partial\ell(\theta)}{\partial\lambda} = \sum_{j=1}^{n}\frac{\lambda\alpha + (1 + k(\alpha - 1))\beta x_j^{k-1}}{(\lambda + \beta k x_j^{k-1})(\lambda + \beta x_j^{k-1})} - \sum_{j=1}^{n} x_j = 0$$

$$\frac{\partial \ell(\theta)}{\partial \beta} = \sum_{j=1}^{n} \frac{x_j^{k-1}[(\lambda(k+\alpha-1)+\alpha k\beta x_j^{k-1}]}{(\lambda+\beta k x_j^{k-1})(\lambda+\beta x_j^{k-1})} - \sum_{j=1}^{n} x_j^k = 0$$

$$\frac{\partial \ell(\theta)}{\partial k} = \beta \sum_{j=1}^{n} \frac{(1+k\ln x_j)x_j^{k-1}}{\lambda+\beta k x_j^{k-1}} + (\alpha-1)\beta \sum_{j=1}^{n} \frac{x_j^k \ln x_j}{\lambda+\beta x_j^{k-1}} - \sum_{j=1}^{n} x_j^k \ln x_j = 0$$

$$\frac{\partial \ell(\theta)}{\partial \alpha} = -n\,\psi(\alpha) + \sum_{j=1}^{n} \ln(\lambda x_j + \beta x_j^k) = 0\,, \tag{9}$$

where $\psi(x) = [\ln\Gamma(x)]'$ denotes the familiar *digamma* (or differentiated Gamma) function. The system (9) we solve numerically.

4.2. Goodness–of–Fit Statistics

The Anderson-Darling [1] and the Cramér-von Mises [14, 32] statistics determine how closely a specific distribution whose associated CDF fits the empirical distribution associated with a given data set. These statistics one define

$$A_0^* = -\left(\frac{9}{4n^2}+\frac{3}{4n}+1\right)\left[n+\frac{1}{n}\sum_{j=1}^{n}(2j-1)\,\log\left(z_i\,(1-z_{n-i+1})\right)\right]$$

$$W_0^* = \left(\frac{1}{2n}+1\right)\left[\sum_{j=1}^{n}\left(z_j-\frac{2j-1}{2n}\right)^2+\frac{1}{12n}\right],$$

respectively, where $z_j = F(y_j)$, and the y_j values being the *ordered* observations. The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of these goodness–of–fit statistics were tabulated e.g. in [25].

5. Applications

In this section, we present two applications where the GEEW model is compared with other related models, namely extended Weibull (ExtW) [26], exponential–Weibull (EW) [8], gamma exponentiated exponential (GEE) distribution [28], two parameter Weibull (Weibull) and two parameter gamma (Gamma) distributions. We make use of two data sets: first, the uncensored real data set on the breaking stress of carbon fibres (in Gba) as reported in [8] and, secondly, the remission times (in months) of a random sample of 128 bladder cancer patients as reported in [20].

- The classical gamma (Gamma) distribution with density function

$$f(x) = \frac{x^{\xi-1}\,e^{-x/\phi}}{\phi^{\xi}\,\Gamma(\xi)}\,\mathbf{1}_{\mathbb{R}_+}(x), \qquad \xi,\,\phi > 0\,.$$

- The classical Weibull (Weibull) distribution with density function

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}\,\mathbf{1}_{\mathbb{R}_+}(x), \qquad k,\,\lambda > 0\,.$$

- The gamma exponentiated exponential (GEE) distribution [28] with density function

$$f(x) = \frac{\lambda\,\alpha^{\delta}\,e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\left(-\log\left(1-e^{-\lambda x}\right)\right)^{\delta-1}}{\Gamma(\delta)}\,\mathbf{1}_{\mathbb{R}_+}(x), \qquad \lambda,\,\alpha,\,\delta > 0\,.$$

- The exponential–Weibull (EW) distribution [7] with density function

$$f(x) = \left(\lambda + \beta\, k\, x^{k-1}\right) \mathrm{e}^{-\lambda\, x - \beta\, x^k}\, 1_{\mathbb{R}_+}(x), \qquad \lambda,\, \beta,\, k > 0\,.$$

- The extended Weibull (ExtW) distribution [26] with density function

$$f(x) = a\,(c + b\,x)\, x^{-2+b}\, \exp\left\{-c/x - a x^b \mathrm{e}^{-c/x}\right\} 1_{\mathbb{R}_+}(x), \qquad a > 0\,,\, b > 0\,,\, c \geq 0\,.$$

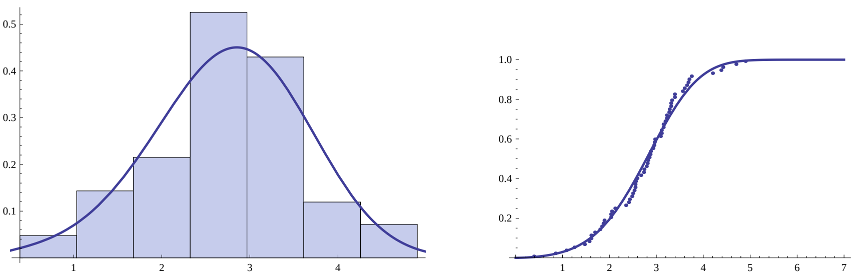

Figure 4: Left panel: The GEEW density estimates superimposed on the histogram for Carbon fibres data. Right panel: The GEEW CDF estimates and empirical CDF.

Table 1: Estimates of the Parameters (Standard Errors) and Goodness-of-Fit Statistics for the Carbon Fibres Data

Distributions	Estimates				A_0^*	W_0^*
Gamma(ξ, ϕ)	7.48803	0.36853			1.32674	0.24815
	(1.2755)	(0.0649)				
Weibull(k, λ)	3.44120	47.0505			0.49168	0.08430
	(0.3309)	(20.119)				
GEE$(\lambda, \alpha, \delta)$	0.26555	10.0365	7.23658		1.43415	0.26682
	(0.2162)	(2.5950)	(7.0529)			
EW(k, λ, β)	3.73666	0.01709	0.01402		0.40365	0.06479
	(0.4458)	(0.0213)	(0.0084)			
ExtW(a, b, c)	16.1979	$1 * 10^{-7}$	8.05671		2.26745	0.41615
	(25.712)	(0.9388)	(1.6531)			
GEEW$(\lambda, \beta, k, \alpha)$	0.15704	0.03692	3.22861	1.77021	**0.3784**	**0.0595**
	(0.3779)	(0.0390)	(0.6368)	(1.3851)		

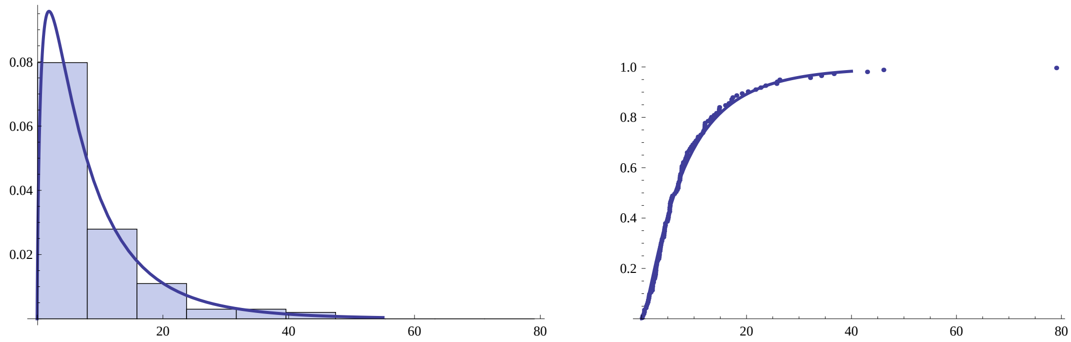

Figure 5: Left panel: The GEEW density estimates superimposed on the histogram for Cancer Patients data. Right panel: The GEEW CDF estimates and empirical CDF.

Table 2: Estimates of the Parameters (Standard Errors) and Goodness-of-Fit Statistics for the Bladder Cancer Patients Data

Distributions	Estimates				A_0^*	W_0^*
Gamma(ξ, ϕ)	1.17251	7.98766			0.77625	0.13606
	(0.2451)	(0.8956)				
Weibull(k, λ)	1.04783	10.6510			0.96345	0.15430
	(0.0676)	(2.1645)				
GEE$(\lambda, \alpha, \delta)$	0.12117	1.21795	1.00156		0.71819	0.12840
	(0.1068)	(0.1877)	(0.8659)			
EW(k, λ, β)	1.04783	$1.005 * 10^{-7}$	0.09389		0.96345	0.15430
	(0.3142)	(0.3013)	(0.1179)			
ExtW(a, b, c)	1.96210	$1 * 10^{-21}$	3.74383		13.3317	2.49818
	(0.7081)	(0.1384)	(0.3895)			
GEEW$(\lambda, \beta, k, \alpha)$	$1 * 10^{-10}$	1.30988	0.52009	3.74791	**0.2991**	**0.0453**
	(0.0983)	(1.9112)	(0.3223)	(3.3941)		

The PDF and CDF estimates of the GEEW(θ), extended Weibull (ExtW) [26], exponential–Weibull (EW) [8], gamma exponentiated exponential (GEE) distribution [28], two parameter Weibull and two parameter Gamma distributions are plotted in Figures 4 and 5 for the Carbon fibres and Cancer patients data, respectively. The estimates of the parameters and the values of the Anderson-Darling and Cramér-von Mises goodness–of–fit statistics are given in Tables 1 and 2. It is seen that the proposed GEEW model provides the best fit for the both data sets.

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Appendix A. The unified Fox–Wright generalized hypergeometric function

Here

$$ {}_p\Psi_q^*\Big[\begin{array}{c} (a,A)_p \\ (b,B)_q \end{array} \Big| z \Big] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p (a_j)_{A_j n}}{\prod_{j=1}^q (b_j)_{B_j n}} \frac{z^n}{n!} $$

stands for the *unified variant of the Fox–Wright generalized hypergeometric function* with p upper and q lower parameters; $(a,A)_p$ denotes the parameter p–tuple $(a_1,A_1),\cdots,(a_p,A_p)$ and $a_j \in \mathbb{C}$, $b_i \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $A_i, B_j > 0$ for all $j = \overline{1,p}, i = \overline{1,q}$, while the series converges for suitably bounded values of $|z|$ when

$$ \Delta_{p,q} := 1 - \sum_{j=1}^p A_j + \sum_{j=1}^q B_j > 0\,. $$

In the case $\Delta = 0$, the convergence holds in the open disc $|z| < \beta = \prod_{j=1}^q B_j^{B_j} \cdot \prod_{j=1}^p A_j^{-A_j}$.

The function ${}_1\Psi_0^*$ we call *confluent*. The convergence condition becomes $\Delta_{1,0} = 1 - A_1 > 0$.

Let us point out that the original definition of the Fox–Wright function ${}_p\Psi_q[z]$ (consult formula collection [15] and the monographs [19], [22]) contains Gamma functions instead of the here used generalized Pochhammer symbols. However, these two functions differ only up to constant multiplying factor, that is

$$ {}_p\Psi_q\Big[\begin{matrix} (a,A)_p \\ (b,B)_q \end{matrix} \Big| z \Big] = \frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)} \, {}_p\Psi_q^*\Big[\begin{matrix} (a,A)_p \\ (b,B)_q \end{matrix} \Big| z \Big]. $$

The unification's motivation is clear - for $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$, ${}_p\Psi_q^*[z]$ one reduces exactly to the generalized hypergeometric function ${}_pF_q[z]$.

Appendix B. Meijer G–function

The symbol $G_{p,q}^{m,n}(\cdot|\cdot)$ denotes Meijer's G–function [23] defined in terms of the Mellin–Barnes integral as

$$ G_{p,q}^{m,n}\Big(z \Big| \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \Big) = \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{C}} \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} \, z^s \, ds, $$

where $0 \le m \le q$, $0 \le n \le p$ and the poles a_j, b_j are such that no pole of $\Gamma(b_j - s)$, $j = \overline{1,m}$ coincides with any pole of $\Gamma(1 - a_j + s)$, $j = \overline{1,n}$; i.e. $a_k - b_j \notin \mathbb{N}$, while $z \neq 0$. $\mathfrak{C}$ is a suitable integration contour which startes at $-\mathrm{i}\infty$ and goes to $\mathrm{i}\infty$ separating the poles of $\Gamma(b_j - s)$, $j = \overline{1,m}$ which lie to the right of the contour, from all poles of $\Gamma(1 - a_j + s)$, $j = \overline{1,n}$, which lie to the left of $\mathfrak{C}$. The integral converges if $\delta = m + n - \frac{1}{2}(p+q) > 0$ and $|\arg(z)| < \delta\pi$, see [21, p. 143] and [23].

The G function's *Mathematica* code reads

```
MeijerG[{{a_1, ..., a_n}, {a_{n+1}, ..., a_p}}, {{b_1, ..., b_m}, {b_{m+1}, ..., b_q}}, z].
```