Filomat 29:10 (2015), 2367–2379
DOI 10.2298/FIL1510367B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$F-$geodesics on Manifolds

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Dedicated to Academician Professor Mileva Prvanović on her birthday

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Abstract. The notion of $F-$geodesic, which is slightly different from that of $F-$planar curve (see [13], [17], and [18]), generalizes the magnetic curves, and implicitly the geodesics, by using any (1,1)-tensor field on the manifold (in particular the electro-magnetic field or the Lorentz force). We give several examples of $F-$geodesics and the characterizations of the $F-$geodesics w.r.t. Vranceanu connections on foliated manifolds and adapted connections on almost contact manifolds. We generalize the classical projective transformation, holomorphic-projective transformation and $C-$projective transformation, by considering a pair of symmetric connections which have the same $F-$geodesics. We deal with the transformations between such two connections, namely $F-$planar diffeomorphisms ([18]). We obtain a Weyl type tensor field, invariant under any $F-$planar diffeomorphism, on a $1-$codimensional foliation.

1. Introduction

Recently, in mathematics literature, a series of papers on magnetic curves, inspired from theoretical physics (see [1]-[3], [5], [8], [11], [21]) have appeared. The Lorentz force, the electro-magnetic tensor field, as well as some special forces involved in the Euler-Lagrange equations from Lagrangian mechanics, lead us to consider an arbitrary $(1,1)-$tensor field F on a differentiable manifold. By using it, we deal here with a notion which generalizes both the classical equations of geodesics and magnetic curves, namely the $F-$geodesics on manifolds, with the purpose to unify these classes of curves on one side, and to provide a geometrical model for some physical particles, satisfying certain differential equations, on the other side. The notion of $F-$geodesic is slightly different from $F-$planar curve (see [18] and the references therein).

We provide several classes of $F-$geodesics, which highlight trajectories in Lagrangian mechanics, magnetic curves (described by particles moving under the influence of the Lorentz force), and curves on the total space of the tangent bundle (obtained by using different types of lifts).

We obtain some characterizations of $F-$geodesics w.r.t. special connections, namely Vranceanu connections (see [6], [30]) on foliated manifolds and adapted connections (see [15]) on almost contact manifolds (see [7]). We give a necessary and sufficient condition for a pair of symmetric connections to have the same system of $F-$geodesics. Moreover, we use here the notion of $F-$planar diffeomorphism (see [13], [17], and [18]), which extends the classical projective transformation (see [29]), the holomorphic-projective

2010 *Mathematics Subject Classification*. Primary 53B05, 53C22, 53C12, Secondary 53C56, 53C80, 53D10.
Keywords. geodesic, magnetic curve, projective transformation, holomorphically projective transformation, C-projective transformation, foliated manifold, almost contact manifold, Vranceanu connection.
Received: 20 August 2014; Accepted: 03 November 2014
Communicated by Ljubica S. Velimirovic
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($H-$projective) transformation from both the complex (see [31], [26]) and locally product (para-complex) context (see [24], [25]), as well as the $C-$projective transformation from the almost contact case (see [15], [22]).

On a $1-$codimensional foliation, we construct a tensor field of Weyl type which is invariant under any $F-$planar diffeomorphism.

Throughout this note, all geometric objects are assumed to be smooth, the Einstein convention summation is used, and the derivative $\dot{\gamma}(t)$ with respect to t of a curve $\gamma(t)$ on a manifold denotes the speed vector field, while the derivative of a function f is denoted by f'.

2. $F-$geodesics

The main ingredients used in the present note are provided in the following:

Notations 1: By a couple (M, F) (resp. a triple (M, F, ∇)) we mean a manifold M endowed with a (1,1)-tensor field F (resp. a couple as above, with a linear connection ∇).

The following notion is slightly different from the notion of $F-$planar curve (see [13], [17], and [18]), it generalizes the geodesics, and it is followed by some examples.

Definition 2.1. *We say that a smooth curve $\gamma : I \to M$ on a manifold (M, F, ∇) is an $F-$geodesic if $\gamma(u)$ satisfies:*

$$\nabla_{\dot{\gamma}(u)}\dot{\gamma}(u) = F\dot{\gamma}(u). \tag{1}$$

Note that the above notion is completely different from that of $\Phi-$geodesic (see [28]), which means a classical geodesic on a Sasakian manifold, whose velocity vector field is horizontal.

Remark 2.2. *(a) If t is another parameter for the same curve $\gamma(u)$ then the relation* (1) *becomes:*

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = \alpha(t)\dot{\gamma}(t) + \beta(t)F\dot{\gamma}(t), \tag{2}$$

where α and β are some functions on the curve $\gamma(t)$.

(b) A curve $\gamma(t)$ satisfying the relation (2) *describes an $F-$geodesic up to a reparameterization.*

(c) From geometrical point of view, an $F-$geodesic (up to a reparameterization) is defined as a curve $\gamma(t)$ such that the parallel transport along the curve preserves the tangent subspace (of dimension 1 or 2) spanned by $\dot{\gamma}(t)$ and $F\dot{\gamma}(t)$.

(d) $F-$geodesics are a special case of $F-$planar curves. Not every $F-$planar curve is an $F-$geodesic, because generally, a transformation to a canonical parameter in equation (2)*, with a given tensor field F does not necessarily lead to the form* (1)*, but to a form*

$$\nabla_{\dot{\gamma}(u)}\dot{\gamma}(u) = f(t)F\dot{\gamma}(u),$$

with a function f of parameter t.

e) The variational problem of $F-$planar curves was solved in [14] (*see* [16]).

Recall from the Riemannian context, the existence and uniqueness of the solution of a second order differential equation with initial data, which gives the existence and uniqueness of a geodesic passing through a given point $p \in M$, with a given velocity $X_p \in T_pM$. These properties are extended in [3] to magnetic curves corresponding to an arbitrary magnetic field. The first question arising on a triple (M, F, ∇) is about the existence of the $F-$geodesics. The theory of differential systems with Cauchy condition leads to the following generalization of the mentioned result.

Lemma 2.3. *Let (M, F, ∇) be as in Notations 1. Then, for any $p \in M$ and $X_p \in T_pM$, there exists a unique maximal $F-$geodesic passing through p and having the velocity X_p.*

Examples of $F-$geodesics

(*i*) If F is identically zero, then an $F-$*geodesic becomes a classical geodesic*, and moreover *an* $F-$*geodesic up to a reparameterization becomes a geodesic up to a reparameterization*.

(*ii*) When F is the identity endomorphism up to a multiplicative function, then an $F-$ *geodesic* is a *geodesic up to a reparameterization*.

(*iii*) In the context of Lagrangian mechanics, the Euler-Lagrange equations of systems with frictions, i.e. with non-conservative forces F_i (not of gradient type) take the form

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\Big) - \frac{\partial \mathcal{L}}{\partial q^i} = F_i, \quad i = \overline{1,k}, \tag{3}$$

where $\mathcal{L}(t, q, \dot{q})$ denotes a Lagrangian function, depending on the coordinates (q^i) (and on their derivatives $\dot{q}^i$) of a submanifold M, which are given by $x^i = x^i(q^i, \ldots, q^k)$, $i = \overline{1,n}$, in $\mathbb{R}^n$, with the cartesian coordinates $(x^1, \ldots, x^n)$. For the general theory of a Lagrange space $(M, \mathcal{L})$ we refer to [19].

We assume here that the dissipative forces F_i are expressed by:

$$F_i = -\sum_{j=1}^{k} f_{ij}\dot{q}^j, \quad i = \overline{1,k}, \tag{4}$$

where $(f_{ij})_{i,j=\overline{1,k}}$ are the function coefficients.

To focus on the classical example in mechanics, we take in particular the function $\mathcal{L}(q, \dot{q})$ to be the kinetic energy T for a particle of mass m:

$$T = \frac{m}{2}\sum_{i=1}^{k}(\dot{x}^i)^2 = \frac{1}{2}\sum_{i,j=1}^{k} g_{ij}(q)\dot{q}^i\dot{q}^j, \tag{5}$$

where $(g_{ij})_{i,j=\overline{1,k}}$ is a Riemannian metric on M. Then, in view of Definition 2.1, the trajectory of a particle described by (3) (with (4) and (5)) is an $F-$geodesic.

(*iv*) In the 3-dimensional Riemannian case, let F be the Lorentz force setting as:

$$FX = B \times X, \qquad \forall X \in \Gamma(TM), \tag{6}$$

where B is the magnetic induction.

Then the notion known in literature as a *normal magnetic curve* (see [1]-[3], [8], [11], [21], [27]), can be redefined in view of Definition 2.1, as being an $F-$geodesic $\gamma(s)$, parameterized by its arc length, where F is the Lorentz force.

To extend the above statement to the 3-dimensional pseudo-Riemannian case, we take into account that a lightlike curve (see e.g. [12]) cannot be parameterized by its arc length. Therefore, the statement remains true for any spacelike or timelike arc length parameterized curve $\gamma(s)$, only.

(*v*) In higher dimensions, F may be the *electro-magnetic tensor field*, whose action on particle trajectories was studied e.g. in [23].

(*vi*) We provide now another example of $F-$geodesics, by using the Lorentz force defined on a (pseudo) Riemannian manifold of arbitrary dimension.

To do this, we recall the following notions for which we quote e.g. [2]:

Definition 2.4. On a (pseudo) Riemannian manifold (M, g), a closed 2-form Ω is called a *magnetic field* if it is associated by the following relation to the *Lorentz force* Φ, defined as a skew symmetric (w.r.t. g) endomorphism field on M:

$$g(\Phi(X), Y) = \Omega(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{7}$$

The *Lorentz force* Φ is a divergence free (1,1)-tensor field (i.e. div $\Phi = 0$).

Let ∇ be the Levi-Civita connection of g, and let q be the charge of a particle, describing a smooth trajectory γ on M. Then the curve $\gamma(t)$ whose speed $\dot{\gamma}(t)$ satisfies the *Lorentz equation*

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = q\Phi(\dot{\gamma}(t)), \tag{8}$$

is known in the literature as a *magnetic curve* of the magnetic field Ω.

According to Definition 2.1, the above Lorentz equation expresses the relation satisfied by an $F-$geodesic of M, where F is defined by $FX = q\Phi(X)$, $\forall X \in \Gamma(TM)$.

The action of the Lorentz force on particle trajectories in the sense of the present paper was studied e.g. in [23].

3. Constructions of $F-$geodesics on TM by using lifts

Here, we use the well known method of lifting some geometric objects from the base manifold M to the total space of its tangent bundle TM (for which we mention the classical monograph [32]), aiming to provide some new classes of $F-$geodesics on TM.

Proposition 3.1. *Let L (resp. ∇) be a $(1,1)-$tensor field (resp. a linear connection) on a manifold M^n, and let L^H (resp. ∇^H) denote its horizontal lift on TM.*

(i) An integral curve of any vector field X on M is an $L-$geodesic w.r.t. ∇ if and only if the integral curve of X^H is an L^H-geodesic w.r.t. ∇^H.

(ii) The above statement remains true, if "$L-$geodesic" and "L^H-geodesic", are replaced by "$L-$geodesic up to a reparameterization" and "L^H-geodesic up to a reparameterization", respectively.

Proof. Let $\pi : TM \to M$, be the tangent bundle of the manifold (M, ∇), and let $(x^1, \dots, x^n)$ (resp. $(x^1, \dots, x^n,$ $y^1, \dots, y^n))$ be the local coordinates on M (resp. on TM). Recall that the horizontal lift of a vector field $X = X^i \frac{\partial}{\partial x^i} \in \Gamma(TM)$ to the total space TM of the tangent bundle has the expression $X^H = X^i \frac{\delta}{\delta x^i}$, where $\Gamma^h_{ki}(x)$ are the coefficients of the connection ∇ and $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{ki} y^k \frac{\partial}{\partial y^h}$.

The horizontal lift of a vector field $X \in \Gamma(TM)$ has the property:

$$(fX)^H = f^V X^H, \tag{9}$$

for every function f on M, where $f^V = f \circ \pi$.

Let γ be an $L-$geodesic up to a reparameterization (w.r.t. ∇) on M. Then relation (2) is satisfied.

Considering the horizontal lift in (2), then using (9) and the following properties of the horizontal lifts of the (1,1)-tensor field L and of the conection ∇:

$$(LX)^H = L^H X^H, \ \nabla^H_{X^H} Y^H = (\nabla_X Y)^H, \quad \forall X, Y \in \Gamma(TM), \tag{10}$$

we obtain

$$\nabla^H_{\dot{\gamma}(t)^H} \dot{\gamma}(t)^H - \alpha(t)^V \dot{\gamma}(t)^H - \beta(t)^V L^H \dot{\gamma}(t)^H = 0. \tag{11}$$

Since any vector field $X \in \Gamma(TM)$ vanishes if and only if its horizontal lift X^H vanishes, then the equivalence between the relations (2) and (11) follows, and hence *(ii)* is proved.

In the particular case when $\alpha(t) = 0$ and $\beta(t) = 1$, one obtains *(i)*. □

Our aim now is to obtain another class of $F-$geodesics on the total space of the tangent bundle, by using metrics of natural type.

On a Riemannian manifold (M, g), let ∇ be the Levi-Civita connection of g. We denote by $\pi : TM \to M$ the tangent bundle of M, whose total space is endowed with a natural diagonal metric G, i.e. a metric defined by:

$$\begin{cases} G(X^H_y, Y^H_y) = c_1 g_{\pi(y)}(X, Y) + d_1 g_{\pi(y)}(X, y) g_{\pi(y)}(Y, y), \\ G(X^V_y, Y^V_y) = c_2 g_{\pi(y)}(X, Y) + d_2 g_{\pi(y)}(X, y) g_{\pi(y)}(Y, y), \\ G(X^V_y, Y^H_y) = 0, \end{cases} \tag{12}$$

for all $X, Y \in \Gamma(TM)$, $y \in TM$, where $c_1,\ c_2,\ d_1,\ d_2$ are smooth functions depending on the energy density ρ of y, defined as

$$\rho = \frac{1}{2} g_{\pi(y)}(y, y). \tag{13}$$

The metric G is positive definite provided that

$$c_1,\ c_2 > 0,\ c_1 + 2\rho d_1,\ c_2 + 2\rho d_2 > 0.$$

When $c_1 = c_2 = 1$ and $d_1 = d_2 = 0$, the metric G reduces to the Sasaki metric g^S.

The Levi-Civita connection of G, denoted by $\widetilde{\nabla}$ has the following expressions on the horizontal and resp. on the vertical distribution of TTM:

$$\begin{aligned} \widetilde{\nabla}_{X^V} Y^V = &\tfrac{c_2'}{2c_2}(g(X, y)Y^V + g(Y, y)X^V) - \\ &- \tfrac{c_2' - 2d_2}{2(c_2 + 2\rho d_2)} g(X, Y) y^V + \tfrac{c_2 d_2' - 2c_2' d_2}{2c_2(c_2 + 2\rho d_2)} g(X, y) g(Y, y) y^V, \end{aligned} \tag{14}$$

$$\begin{aligned} \widetilde{\nabla}_{X^H} Y^H = &(\nabla_X Y)^H - \tfrac{d_1}{2c_1}(g(X, y)Y^V + g(Y, y)X^V) - \\ &- \tfrac{c_1'}{2(c_2 + 2\rho d_2)} g(X, Y) y^V - \tfrac{c_2 d_1' - 2d_1 d_2}{2c_2(c_2 + 2\rho d_2)} g(X, y) g(Y, y) y^V - \tfrac{1}{2}(R(X, Y)y)^V, \end{aligned} \tag{15}$$

for all $X, Y \in \Gamma(TM)$, $y \in TM$, where R is the curvature tensor field on the base manifold M.

By using the expression (15), we provide the following:

Proposition 3.2. *Let (M, g) be a Riemannian manifold endowed with a $(1, 1)-$tensor field L.*

(i) An integral curve of any vector field $X \in \Gamma(TM)$ is an $L-$geodesic w.r.t. the Levi-Civita connection ∇ of g if and only if the integral curve of the horizontal lift X^H is an L^H-geodesic w.r.t. the Levi-Civita connection $\widetilde{\nabla}$ of a natural diagonal metric G, given by (12)*, provided that $c_2 = const \in \mathbb{R}$ and $d_2 = 0$.*

(ii) The above assertion remains true, if instead of an "$L-$geodesic" (resp. an "L^H-geodesic") we take an "$L-$geodesic up to a reparameterization" (resp. an "L^H-geodesic up to a reparameterization").

Proof. Let γ be an $L-$geodesic up to a reparameterization (w.r.t. ∇) on M, i.e. γ satisfies

$$\nabla_{\dot\gamma} \dot\gamma = \alpha \dot\gamma + \beta L \dot\gamma, \tag{16}$$

where α and β are some smooth functions on the curve.

For $X = Y = \dot\gamma$, the relation (15) becomes

$$\widetilde{\nabla}_{\dot\gamma^H} \dot\gamma^H = (\nabla_{\dot\gamma} \dot\gamma)^H - \tfrac{d_1}{c_1} g(\dot\gamma, y) \dot\gamma^V - \tfrac{c_1'}{2(c_2 + 2\rho d_2)} g(\dot\gamma, \dot\gamma) y^V - \tfrac{c_2 d_1' - 2d_1 d_2}{2c_2(c_2 + 2\rho d_2)} (g(\dot\gamma, y))^2 y^V.$$

Replacing (16) into the above relation, and taking into account (9) and (10), if follows that

$$\widetilde{\nabla}_{\dot\gamma^H} \dot\gamma^H = \alpha^V \dot\gamma^H + \beta^V L^H \dot\gamma^H,$$

if and only if

$$\frac{d_1}{c_1} g(\dot\gamma, y) \dot\gamma^V + \frac{c_1'}{2(c_2 + 2\rho d_2)} g(\dot\gamma, \dot\gamma) y^V + \frac{c_2 d_1' - 2d_1 d_2}{2c_2(c_2 + 2\rho d_2)} (g(\dot\gamma, y))^2 y^V = 0.$$

From [10, Lemma 3.2] it follows that the coefficients involved in the above relation vanish, and thus item *(ii)* is proved.

If in particular $\alpha = 0$ and $\beta = 1$, it follows that item *(i)* is also true. □

Corollary 3.3. *Let (M, g) be a Riemannian manifold, endowed with a $(1, 1)-$tensor field L. We denote by ∇ and ${}^S\nabla$ the Levi-Civita connection of g and respectively of the Sasaki metric g^S. Then an integral curve of an arbitrary vector field X on M is:*

(i) an L−geodesic w.r.t. ∇ if and only if the integral curve of the horizontal lift X^H to TM is an L^H-geodesic w.r.t. ${}^S\nabla$.

(ii) an L−geodesic up to a reparameterization w.r.t. ∇ if and only if the integral curve of the horizontal lift X^H to TM is an L^H-geodesic up to a reparameterization w.r.t. ${}^S\nabla$.

At the end of this section we focus on a special class of $F-$geodesics, namely the class of geodesics on the total space of the tangent bundle.

Remark: Based on the fact that the horizontal lift ∇^H and the complete lift ∇^C to TM of a linear connection ∇ from M are both vanishing on the vertical distribution, i.e.

$$\nabla^H_{X^V} Y^V = \nabla^C_{X^V} Y^V = 0, \ \forall X, Y \in \Gamma(TM),$$

we note that the integral curves of the vertical lift of any vector field X on M to TM are geodesics w.r.t. both ∇^H and ∇^C.

On TM, instead of ∇^H and ∇^C, as above, we take now the Levi-Civita connection of a natural metric, to obtain the following:

Proposition 3.4. *Let (TM, G) be the total space of the tangent bundle of a Riemannian manifold (M, g), endowed with a natural diagonal metric G, given by (12), whose Levi-Civita connection is denoted by $\widetilde{\nabla}$. Any integral curve of the vertical lift X^V of an arbitrary vector field X is:*

(i) a geodesic w.r.t. $\widetilde{\nabla}$ if and only if $c_2 = const \in \mathbb{R}$ and $d_2 = 0$.

(ii) a geodesic up to a reparameterization, w.r.t. $\widetilde{\nabla}$, if and only if

$$c_2 = -\frac{1}{k_1\rho + k_2}, \qquad d_2 = \frac{k_1}{2(k_1\rho + k_2)^2}, \tag{17}$$

where k_1 and k_2 are two real constants, chosen such that the function c_2 be well defined.

Proof. *(i)* From (14) if follows that the integral curve of X^V is a geodesic if and only if

$$\tfrac{c_2'}{c_2} g(X, y)X^V - \tfrac{c_2' - 2d_2}{2(c_2+2\rho d_2)} g(X, X)y^V + \tfrac{c_2 d_2' - 2c_2' d_2}{2c_2(c_2+2\rho d_2)} (g(X, y))^2 y^V = 0, \tag{18}$$

for any vector field X on M and any vector y tangent to M.

Writing (18) in local coordinates, and then using [10, Lemma 3.2], we obtain that all the involved coefficients vanish, and item *(i)* is proved.

(ii) The integral curve of the vertical lift X^V of an arbitrary vector field X on M is a geodesic up to a reparameterization, w.r.t. $\widetilde{\nabla}$, if and only if there exists a function α on $\gamma(t)$ such that

$$\alpha(t)X^V = \tfrac{c_2'}{c_2} g(X, y)X^V - \tfrac{c_2' - 2d_2}{2(c_2+2\rho d_2)} g(X, X)y^V + \tfrac{c_2 d_2' - 2c_2' d_2}{2c_2(c_2+2\rho d_2)} (g(X, y))^2 y^V = 0, \tag{19}$$

for any vector field X on M and any point of the curve $\gamma(t) = (x(t), y(t)) \in TM$.

The relation (19) holds good if and only if the last two involved coefficients vanish, i.e. the functions c_2 and d_2 are expressed by the relations (17). □

Corollary 3.5. *On a Riemannian manifold (M, g), let X be an arbitrary vector field, and denote by (TM, g^S) the total space of the tangent bundle, endowed with the Sasaki metric. The integral curve of the vertical lift X^V of X is a geodesic w.r.t. the Levi-Civita connection of g^S.*

4. F−geodesics w.r.t. special connections

Vranceanu connections

On a Riemannian manifold (M^{n+1}, g), let $\mathcal{F}$ be a 1-codimensional foliation. Then there exists a distribution $\mathcal{D}$ of dimension n, tangent to $\mathcal{F}$ and a vector field ξ such that the orthogonal distribution $\mathcal{D}^\perp$ can be written as $\mathcal{D}^\perp = span\{\xi\}$. The latest distribution is called the transversal distribution, to the foliation $\mathcal{F}$. If ∇ is the Levi-Civita connection of g, then the Vranceanu connection ∇^* on $(M, g, \mathcal{F})$ is defined (see [6]) by

$$\nabla^*_X Y = P\nabla_{PX}PY + Q\nabla_{QX}QY + P[QX, PY] + Q[PX, QY],\ \forall X, Y \in \Gamma(TM), \tag{20}$$

where P and Q are the projection morphisms from TM on $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively.

Proposition 4.1. *Under the above notations, we assume that a vector field V tangent to the foliation is parallel w.r.t. ξ. Then V is self parallel w.r.t. the Levi Civita connection of the foliation $\mathcal{F}$ if and only if any integral curve of $V + \xi$ is an F−geodesic on M w.r.t. Vranceanu connection, where F is defined by*

$$FX = Q\nabla_{QX}\xi + (Q - P)\nabla_{PX}\xi,\ \forall X \in \Gamma(TM). \tag{21}$$

Proof. From the above hypothesis we have that $\dot\gamma$ can be written as

$$\dot\gamma = V + \xi,$$

and then, relation (20) yields:

$$\nabla^*_{\dot\gamma}\dot\gamma = P\nabla_V V + Q\nabla_\xi \xi + (P - Q)[\xi, V]. \tag{22}$$

Since $V = P\dot\gamma$ and $\nabla_\xi V = 0$, we obtain

$$\nabla^*_{\dot\gamma}\dot\gamma = P\nabla_V V + Q\nabla_\xi \xi + (Q - P)\nabla_{P\dot\gamma}\xi,$$

which can be written as:

$$\nabla^*_{\dot\gamma}\dot\gamma = P\nabla_V V + F\dot\gamma.$$

Note that the restriction $P\nabla_{PX}PY$, for every $X, Y \in \Gamma(TM)$, defines the Levi-Civita connection on the foliation $\mathcal{F}$, and thus the proof is complete. □

Adapted connections

Let (M, φ, ξ, η) be an almost contact manifold, that is φ, ξ, η are respectively a $(1, 1)$−tensor field, a structure vector field and its dual 1−form, such that

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi,\ \eta(\xi) = 1. \tag{23}$$

From (23) one may easily deduce that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$ (see [7]). Therefore, TM splits into the direct sum:

$$TM = \ker \eta \oplus span\{\xi\}. \tag{24}$$

The class of almost contact manifolds (see e.g. [7], [9]), as well as the almost para-contact ones are used in physics (see [4], [20]).

Let $\dot\nabla$ be an arbitrary connection on M. From [15] and [22], a connection ∇ is adapted to the almost contact structure (i.e. ∇ satisfies certain compatibility relations with (φ, ξ, η)) if and only if

$$\nabla_X Y = \dot\nabla_X Y + \varphi^2 H(X, \varphi^2 Y) - \varphi H(X, \varphi Y), \forall X, Y \in \Gamma(TM). \tag{25}$$

where H is an arbitrary $(1, 2)$−tensor field.

By using (1) and (25), after a straightforward computation we obtain:

Proposition 4.2. *Let $\gamma(t)$ be a smooth curve on an almost contact manifold (M, φ, ξ, η), whose velocity $\dot\gamma(t)$ has constant projection on ξ (when it is decomposed by using formula* (24)*). Then γ is a geodesic w.r.t. an arbitrary linear connection $\dot\nabla$ if and only if it is a 2φ−geodesic w.r.t. the adapted connection* (25)*, where H is given by*

$$H(X, Y) = \eta(X)\varphi Y,\ \forall X, Y \in \Gamma(TM).$$

5. $F-$planar diffeomorphisms

On a couple (M, F), a natural question would be how are related two linear connections having the same $F-$geodesics. For this purpose we introduce here the following:

Definition 5.1. *Let (M, F) be a manifold endowed with a $(1,1)-$tensor field. Two linear connections $\overline{\nabla}$ and ∇ are called $F-$projectively related to each other, if they have the same system of $F-$geodesics up to a reparameterization.*

Notations 2: (i) If ∇ and $\overline{\nabla}$ are two symmetric linear connections on a manifold M, then the deformation tensor field S will denote the symmetric $(1,2)-$tensor field, given by:

$$S(X, Y) = \overline{\nabla}_X Y - \nabla_X Y, \ \forall X, Y \in \Gamma(TM). \tag{26}$$

Obviously, for any common $F-$geodesic $\gamma(t)$ of $\overline{\nabla}$ and ∇, one has

$$S(\dot{\gamma}(t), \dot{\gamma}(t)) = \overline{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) - \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = a(t)\dot{\gamma}(t) + b(t)F\dot{\gamma}(t), \tag{27}$$

where a and b are some smooth functions on the curve $\gamma(t)$.

(ii) We say that the deformation tensor field S satisfies the coefficients linearity (CL) condition, if for any common $F-$geodesic in the relation (27), the coefficients a and b depend linearly on the speed of the curve. Precisely, S satisfies the (CL) condition, if there exist two $1-$forms $\nu, \mu \in \Gamma(T^*M)$, such that

$$a(t) = \nu(\dot{\gamma}(t)), \ b(t) = \mu(\dot{\gamma}(t)), \tag{28}$$

for each common $F-$geodesic of $\overline{\nabla}$ and ∇.

We recall here the following notion ([13], [17]):

Definition 5.2. *We say that two symmetric linear connections ∇ and $\overline{\nabla}$ on M are related by an $F-$planar diffeomorphism if:*

$$\overline{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X + \theta(X)FY + \theta(Y)FX, \ \forall X, Y \in \Gamma(TM), \tag{29}$$

for some $1-$forms ω and θ on M.

We recall that a linear connection ∇ is called an $F-$connection if F is parallel w.r.t. ∇, i.e.

$$\nabla F = 0. \tag{30}$$

Remarks: (a) Obviously, for a symmetric linear connection ∇, it follows that $\overline{\nabla}$ given by (29) is also symmetric and the relation (29) establishes an equivalence relation on the set of all symmetric connections.

(b) In the case when one of the symmetric linear connections ∇ or $\overline{\nabla}$ is an $F-$connection it follows that the other one is an $F-$connection, too, if and only if

$$\omega(FY)X + \theta(FY)F^2X = (\omega(Y) - \theta(FY))FX, \ \forall X, Y \in \Gamma(TM). \tag{31}$$

The following theorem is similar to [18, Theorem 9.2], obtained under a different hypothesis:

Theorem 5.3. *On a manifold M, endowed with a $(1,1)-$tensor field F, any two symmetric linear connections which are $F-$projectively related to each other, and whose deformation tensor field S satisfies the (CL) condition, are related by an $F-$planar diffeomorphism.*

Proof. If we assume that $\overline{\nabla}$ and ∇ are $F-$projectively related to each other, then from (2) one has (27). By taking into account Lemma 2.3 it follows that

$$S(X,X) = a(t)X + b(t)FX, \forall X \in \Gamma(TM). \tag{32}$$

Since S satisfies the (CL) condition, one has:

$$S(X,X) = \nu(X)X + \mu(X)FX, \quad \forall X \in \Gamma(TM). \tag{33}$$

From the symmetry of S, by taking $\omega = 2\nu$ and $\theta = 2\mu$, one has:

$$S(X,Y) = \omega(X)Y + \omega(Y)X + \theta(X)FY + \theta(Y)FX, \quad \forall X,Y \in \Gamma(TM), \tag{34}$$

which gives the relation (29). The converse is obvious and the proof is complete. □

In order to show the consistency of (CL) condition, and to highlight some cases in which the hypothesis of Theorem 5.3 is satisfied, we provide the following:

Examples: On a couple (M,F), let S be the deformation tensor field of two symmetric connections $\overline{\nabla}$ and ∇.

(i) When F is zero, then $\overline{\nabla}$ and ∇ are $F-$projectively related if and only if they are projectively related (i.e. their system of geodesics coincide), which is equivalent (see [29]) to the fact that $\overline{\nabla}$ is obtained by a projective transformation of ∇, i.e. $\overline{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X$, $\forall X,Y \in \Gamma(TM)$. In this case S satisfies the (CL) condition with $a(t) = 2\omega(\dot{\gamma}(t))$ and $b(t) = 0$.

(ii) When F is a complex (resp. product) structure, parallel w.r.t. both connections, then $\overline{\nabla}$ and ∇ are F-projectively related if and only if they are holomorphic projectively related, which is equivalent from [31, pp. 255–266] (resp. [24] and [25]) to the fact that $\overline{\nabla}$ is a holomorphic-projective transformation of ∇, i. e.

$$\overline{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X + \varepsilon\omega(FX)FY + \varepsilon\omega(FY)FX, \quad \forall X,Y \in \Gamma(TM),$$

where $\varepsilon = -1$ (resp. 1).

In this case S satisfies (CL) condition with

$$a(t) = 2\omega(\dot{\gamma}(t)) \text{ and } b(t) = 2\varepsilon\omega(\mathrm{F}(\dot{\gamma}(\mathrm{t}))),$$

where $\varepsilon = -1$ (resp. 1).

Note that the above examples are particular cases of Theorem 5.3.

Lemma 5.4. *On a manifold M endowed with a* $(1,1)-$*tensor field F, let* $\overline{\nabla}$ *and* ∇ *be two symmetric* $F-$*connections, which are* $F-$*projectively related to each other by the relation* (29). *Then their Riemannian curvature tensor fields* $\overline{R}$ *and R are related by*

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z &+ \Omega(X,Z)Y - \Omega(Y,Z)X + [\Omega(X,Y) - \Omega(Y,X)]Z + \\ &+[\Theta(X,Y) - \Theta(Y,X)]FZ + \Theta(X,Z)FY - \Theta(Y,Z)FX + \\ &+\theta(Z)[\theta(X)F^2Y - \theta(Y)F^2X], \quad \forall X,Y,Z \in \Gamma(TM),\end{aligned} \tag{35}$$

where:

$$\begin{aligned}\Omega(X,Y) &= (\nabla_X\omega)Y - \omega(X)\omega(Y) - \theta(X)\omega(FY) - \theta(Y)\omega(FX), \\ \Theta(X,Y) &= (\nabla_X\theta)Y - \theta(X)\theta(FY) - \theta(Y)\theta(FX),\end{aligned} \tag{36}$$

for every $X,Y \in \Gamma(TM)$.

In order to prove the lemma, in the expression of the curvature tensor field $\overline{R}$ associated to $\overline{\nabla}$:

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \ \forall X,Y,Z \in \Gamma(TM), \tag{37}$$

we replace $\overline{\nabla}$ from (29), and by using (30), we obtain (35) after a quite long straightforward computation.

6. Foliations

On a Riemannian manifold (M^{n+1}, g), let $\mathcal{F}$ be a 1-codimensional foliation. Then there exists a distribution $\mathcal{D}$ of dimension n, tangent to $\mathcal{F}$, a unit vector field ξ, orthogonal to the foliation and a projector operator P on $\mathcal{D}$. Let φ and η be a (1,1)-tensor field of maximal rank (i.e. n) on the foliation $\mathcal{F}$, and respectively the dual 1−form of ξ, i.e.

$$\eta(X) = g(X, \xi), \ \forall X \in \Gamma(TM). \tag{38}$$

We can extend φ to the whole manifold M by F defined such that

$$FX = \begin{cases} \varphi X, & \text{for } X \in \Gamma(\mathcal{D}), \\ 0, & \text{for } X = \xi. \end{cases} \tag{39}$$

We denote by C the class of all symmetric linear connections ∇ on M, such that

$$\nabla_X \xi = FX, \ \forall X \in \Gamma(\mathcal{D}) \tag{40}$$

and

$$(\nabla_X F)Y = \begin{cases} 0, & \text{for } Y \in \Gamma(\mathcal{D}), \\ PX, & \text{for } Y = \xi, \end{cases} \tag{41}$$

for all $X \in \Gamma(TM)$.

Lemma 6.1. *(a) Any connection in the class C becomes an F−connection when restricted to the foliation.*
(b) The structures F and P are related by

$$F^2 + P = 0. \tag{42}$$

(c) The commutation relation follows:

$$FP = PF = F. \tag{43}$$

Proof. The statement (a) can be obtained from (41). The relations (40) and (41) yield (b), which implies the first equality in (43), while the last equality holds good since P and F have the same kernel and P is a projection on the image of F. □

Theorem 6.2. *Let $\mathcal{F}$ be a 1-codimensional foliation of a Riemannian manifold (M^{n+1}, g), endowed with the F−structure given by* (39). *Any two connections $\overline{\nabla}$ and ∇ in C are F−projectively related to each other if and only if there exist two* 1*−forms α and β such that*

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X)PY + \alpha(Y)PX + \beta(X)FY + \beta(Y)FX, \ \forall X, Y \in \Gamma(TM). \tag{44}$$

Proof. Let any vector field $X \in \Gamma(TM)$ be decomposed as:

$$X = PX + \eta(X)\xi. \tag{45}$$

Restricted to the foliation, both connections $\overline{\nabla}$ and ∇ become F−connections and we show now that their deformation tensor field S, given by (26), satisfies (CL) condition. If V denotes the speed vector field of an arbitrary F-geodesic (common to $\overline{\nabla}$ and ∇), tangent to the foliation, then V satisfies the relation

$$S(V, V) = aV + bFV, \tag{46}$$

where we have to prove that the coefficients a and b depend linearly on V. In any local frame $\{e_i\}_{i=\overline{1,n}}$ tangent to the foliation, let V, F and S be expressed respectively by V^i, F^i_j and S^h_{ij}, $i, j, h = \overline{1, n}$. Written in local coordinate frame, the relation (46) becomes:

$$S^h_{ij}V^iV^j = aV^h + bF^h_rV^r, \tag{47}$$

where a and b depend on the point and on V^h. Taking an arbitrary point p we denote the vector field V in p by the vector v, i.e. $V_p = v$, whose components are v^h, and we obtain in p:

$$S^h_{ij}v^iv^j = av^h + bF^h_rv^r. \tag{48}$$

Multiplying (48) by v^k, in p it becomes:

$$S^h_{ij}v^iv^jv^k = av^hv^k + bF^h_rv^rv^k. \tag{49}$$

By changing k with h, one has in p:

$$S^k_{ij}v^iv^jv^h = av^kv^h + bF^k_rv^rv^h. \tag{50}$$

By substracting the relations (49) and (50), one obtains in p:

$$(S^h_{ij}\delta^k_l - S^k_{ij}\delta^h_l)v^iv^jv^l = b(F^h_r\delta^k_l - F^k_r\delta^h_l)v^rv^l. \tag{51}$$

Note that the bracket in the right-hand side is independent of v. Since p and v are arbitrary from Lemma 2.3, it follows from (51) that b should depend linearly on v in any point p and therefore there exists a 1-form μ such that

$$b = \mu(V), \tag{52}$$

which locally can be written as:

$$b = \mu_iV^i. \tag{53}$$

If we replace the relation (53) into (47), it follows that

$$(S^h_{ij} - \mu_iF^h_j)V^iV^j = a\delta^h_iV^i,$$

which shows that a should also depend linearly on V, and therefore (CL) condition is satisfied on the foliation.

The $F-$projectively relation between ∇ and $\overline{\nabla}$ restricted to the foliation is equivalent (from Theorem 5.3) to the existence of two 1-forms ω and θ on the foliation, such that:

$$\begin{aligned} S(PX, PY) = \overline{\nabla}_{PX}PY - \nabla_{PX}PY = \omega(PX)PY + \omega(PY)PX+ \\ +\theta(PX)FPY + \theta(PY)FPX, \ \forall X, Y \in \Gamma(TM). \end{aligned} \tag{54}$$

From now on, we assume the two linear connections belonging to $\mathcal{C}$. Then $S(X, Y) = 0$, if at least one of its variables (X or Y) belongs to $span\{\xi\}$, and therefore

$$S(X, Y) = S(PX, PY), \ \forall X, Y \in \Gamma(TM).$$

Moreover, by using (43), one can replace $FPX = FX$ and $FPY = FY$, $\forall X, Y \in \Gamma(TM)$ into (54).

Now, if the 1-form ω (resp. θ) is extended from the foliation $\mathcal{F}$ to the $1-$ form α (resp. β) on the whole manifold M, by

$$\alpha(X) = \omega(PX) \text{ (resp. } \beta(X) = \theta(PX)), \ \forall X \in \Gamma(TM),$$

then the proof is complete. □

Remark: In the above theorem, if U and F^2U are collinear, but not collinear with FU, then from Lemma 6.1 (a), we may apply (31), which yields $\beta(FU) = \alpha(U)$, for any vector field U tangent to the foliation $\mathcal{F}$, (since restricted to the foliation, α and β coincide respectively with ω and θ).

Theorem 6.3. *On a Riemannian manifold* (M^{n+1}, g)*, carrying a 1-codimensional foliation* $\mathcal{F}$ *and an* $F-$*structure defined by* (39)*, the following tensor field* W *is invariant for any* $F-$*geodesic transformation given by* (44)*:*

$$\begin{aligned} W(X,Y)Z = PR(X,Y)Z + [L(X,Y) - L(Y,X)]PZ + \\ +[L(X,Z) + \eta(X)\eta(Z)]PY - [L(Y,Z) + \eta(Y)\eta(Z))]PX - \\ -[L(X,FY) - L(Y,FX) + d\eta(X,Y)]FZ - \\ -[L(X,FZ) + d\eta(X,Z)/2]FY + [L(Y,FZ) + d\eta(Y,Z)/2]FX, \quad \forall X,Y,Z \in \Gamma(TM), \end{aligned} \tag{55}$$

where P *and* η *are given above, and*

$$\begin{aligned} L(X,Y) = \tfrac{1}{n+2}Ric(X,PY) + \tfrac{1}{n^2-4}[Ric(PX,Y) + Ric(PY,X) - \\ -Ric(FX,FY) - Ric(FY,FX)] + \tfrac{1}{n+2}d\eta(X,FY). \end{aligned} \tag{56}$$

Proof. From Lemma 5.4, by taking into account the relations (38)-(43), we obtain:

$$\begin{aligned} \overline{R}(X,Y)Z = R(X,Y)Z + [A(X,Y) - A(Y,X)]PZ + A(X,Z)PY - \\ -A(Y,Z)PX - [A(X,FY) - A(Y,FX)]FZ - \\ -A(X,FZ)FY + A(Y,FZ)FX - \alpha(Z)d\eta(X,Y)\xi + \\ +\tfrac{1}{2}[\beta(Y)d\eta(X,FZ) - \beta(X)d\eta(X,FZ) - \\ -\alpha(Y)d\eta(X,Z) + \alpha(X)d\eta(Y,Z)]\xi, \quad \forall X,Y,Z \in \Gamma(TM), \end{aligned} \tag{57}$$

where

$$A(X,Y) = (\nabla_X\alpha)Y - \alpha(X)\alpha(Y) + \beta(X)\beta(Y) + \beta(X)\eta(Y) + \eta(X)\beta(Y). \tag{58}$$

By a long calculus similar to the one given in [15] and [22] we obtain:

$$A(X,Y) = L(X,Y) - \overline{L}(X,Y),$$

where $\overline{L}$ is defined in the same way as L, by using $\overline{\nabla}$ instead of ∇. With the substitution of (58) into (57), the invariance of W follows. □

7. Acknowledgement:

Both authors kindly thank the referee for reading the manuscript very carefully, for giving valuable suggestions and for bringing into attention the references [13], [14], [16] and [23].

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