



Weakly Compatible Maps in Complex Valued Metric Spaces and an Application to Solve Urysohn Integral Equation

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Abstract. In this paper, we prove common fixed point theorems for a pair of mappings satisfying rational inequality. Also, we prove common fixed point theorems for weakly compatible maps, weakly compatible along with (CLR) and E.A. properties that generalizes the results of Sintunavarat et al. [15]. Further, we apply our results to find the solution of Urysohn integral equations

$$x(t) = \int_a^b K_1(t, s, x(s))ds + g(t),$$

$$x(t) = \int_a^b K_2(t, s, x(s))ds + h(t),$$

where $t \in [a, b] \subseteq \mathbb{R}$, $x, g, h \in X$ and $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

1. Introduction

In 2011, Azam et al. [2] introduced the notion of complex valued metric space which is a generalization of the classical metric spaces. They established some fixed point results for a pair of mapping satisfying a rational inequality.

A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $\text{Re}(z)$ and second coordinate is called $\text{Im}(z)$. A complex-valued metric d is a function from $X \times X$ into \mathbb{C} , where X is a nonempty set and \mathbb{C} is the set of complex numbers.

Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$z_1 \preceq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$, that is, $z_1 \preceq z_2$ if one of the following holds:

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(C1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$;

(C2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$;

(C3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$;

(C4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

In particular, we will write $z_1 \not\prec z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 < z_2$ if only (C4) is satisfied.

Remark 1.1. We note that the following statements hold:

(i) $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \preceq bz \forall z \in \mathbb{C}$.

(ii) $0 \preceq z_1 \not\prec z_2 \Rightarrow |z_1| < |z_2|$,

(iii) $z_1 \preceq z_2$ and $z_2 < z_3 \Rightarrow z_1 < z_3$.

Definition 1.2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$ satisfies the following conditions:

(i) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = 2i|z_1 - z_2|, \quad \text{for all } z_1, z_2 \in X.$$

Then (X, d) is a complex valued metric space.

Definition 1.4. Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 < c$ there is $k \in \mathbb{N}$ such that for all $n > k$,

(i) $d(x_n, x) < c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\{x_n\} \rightarrow x$ as $n \rightarrow \infty$ or $\lim_{n \rightarrow \infty} x_n = x$.

(ii) $d(x_n, x_{n+m}) < c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued metric space.

Lemma 1.5. Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 1.6. Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$, where $m \in \mathbb{N}$.

Further, In 2013, Sintunavarat et al. [15] introduced the notion of a C-Cauchy sequence in C-complete complex valued metric space as follows:

Definition 1.7. Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for any $c \in \mathbb{C}$, with $0 < c$, there exists $k \in \mathbb{N}$ such that, for all $m, n > k$, $d(x_n, x_m) < c$, then $\{x_n\}$ is called a C-Cauchy sequence in X .

(ii) If every C -Cauchy sequence in X is convergent, then (X, d) is said to be a C -complete complex valued metric space.

In 1996, Jungck [8] introduced the notion of weakly compatible maps as follows:

Definition 1.8. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

In 2002, Aamri et al. [1] introduced the notion of E.A. property as follows:

Definition 1.9. Two self-mappings f and g of a metric space (X, d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = t$ for some t in X .

In 2011, Sintunavarat et al. [12] introduced the notion of (CLR) property as follows:

Definition 1.10. Two self-mappings f and g of a metric space (X, d) are said to satisfy (CLR_f) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = f x$ for some x in X .

Some of the interesting works for (CLR) property can be cited in [3, 4, 6, 7, 9–11]. In a similar mode, we use these properties in complex valued metric spaces.

Example 1.11. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = 2i|z_1 - z_2|, \quad \text{for all } z_1, z_2 \in X.$$

Then (X, d) is a complex valued metric space.

Define $S, T : X \rightarrow X$ by

$$S z = z + i \text{ and } T z = 2z, \quad \text{for all } z \in X.$$

Consider a sequence $\{z_n\} = \left\{i - \frac{4}{n}\right\}$, $n \in \mathbb{N}$, in X , then

$$\lim_{n \rightarrow \infty} S z_n = \lim_{n \rightarrow \infty} (z_n + i) = \lim_{n \rightarrow \infty} i - \frac{4}{n} + i = 2i.$$

$$\lim_{n \rightarrow \infty} T z_n = \lim_{n \rightarrow \infty} 2z_n = \lim_{n \rightarrow \infty} 2 \left(i - \frac{4}{n}\right) = 2i, \text{ where } 2i \in X.$$

Thus, S and T satisfies E.A. property.

Also, we have

$$\lim_{n \rightarrow \infty} S z_n = \lim_{n \rightarrow \infty} T z_n = 2i = S(i), \text{ where } i \in X.$$

Thus, S and T satisfies (CLR_S) property.

Lemma 1.12 ([5]). Let X be a nonempty set and $T : X \rightarrow X$ be a function. Then there exists a subset $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \rightarrow X$ is one-to-one.

2. Main Results

Throughout this paper, \mathbb{C}_+ denotes a set $\{c \in \mathbb{C} : 0 \preceq c\}$ and Γ denotes the class of all functions $\delta : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ which satisfies the condition:
for (x_n, y_n) in $\mathbb{C}_+ \times \mathbb{C}_+$,

$$\delta(x_n, y_n) \rightarrow 1 \Rightarrow (x_n, y_n) \rightarrow 0.$$

In 2013, Sintunavarat et al. [15] proved the following fixed point result:

“Let S and T be self mappings of a C -complete complex value metric space (X, d) . If there exists mappings $\alpha, \beta : \mathbb{C}_+ \rightarrow [0, 1)$ such that for all x, y in X :

(a) $\alpha(x) + \beta(x) < 1,$

(b) the mapping $\gamma : \mathbb{C}_+ \rightarrow [0, 1)$ defined by $\gamma(x) = \frac{\alpha(x)}{1 - \beta(x)}$ belongs to $\Gamma,$

(c) $d(Sx, Ty) \lesssim \alpha(d(x, y))d(x, y) + \beta(d(x, y))\frac{d(x, Sx)d(y, Ty)}{1+d(x, y)}.$

Then S and T have a unique common fixed point.”

Now, we prove our results in a more general way as follows:

Theorem 2.1. Let S and T be self mappings of a C -complete complex valued metric space (X, d) . If there exists mappings $\alpha, \beta, \gamma : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ such that for all x, y in X :

$$\alpha(x, y) + \beta(x, y) + \gamma(x, y) < 1, \tag{2.1}$$

the mapping $\delta : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ defined by $\delta(x, y) = \frac{\alpha(x, y)}{1 - \beta(x, y)}$ belongs to $\Gamma,$ (2.2)

$$d(Sx, Ty) \lesssim \alpha(x, y)d(x, y) + \beta(x, y)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma(x, y)\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}. \tag{2.3}$$

Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X . We construct the sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}, \text{ for all } n \geq 0. \tag{2.4}$$

For $n \geq 0$, we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\lesssim \alpha(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \beta(x_{2n}, x_{2n+1})\frac{d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ &\quad + \gamma(x_{2n}, x_{2n+1})\frac{d(x_{2n+1}, Sx_{2n})d(x_{2n}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ &= \alpha(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \beta(x_{2n}, x_{2n+1})\frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\ &\quad + \gamma(x_{2n}, x_{2n+1})\frac{d(x_{2n+1}, x_{2n+1})d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\ &= \alpha(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \beta(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})\frac{d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ &\lesssim \alpha(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \beta(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), \end{aligned}$$

which implies that,

$$d(x_{2n+1}, x_{2n+2}) \lesssim \delta(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}), \quad \text{where } \delta(x, y) = \frac{\alpha(x, y)}{1 - \beta(x, y)}. \tag{2.5}$$

Similarly, for $n \geq 0$, we get

$$\begin{aligned}
 d(x_{2n+2}, x_{2n+3}) &= d(x_{2n+3}, x_{2n+2}) \\
 &= d(Sx_{2n+2}, Tx_{2n+1}) \\
 &\lesssim \alpha(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x_{2n+1}) + \beta(x_{2n+2}, x_{2n+1}) \frac{d(x_{2n+2}, Sx_{2n+2})d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n+2}, x_{2n+1})} \\
 &\quad + \gamma(x_{2n+2}, x_{2n+1}) \frac{d(x_{2n+1}, Sx_{2n+2})d(x_{2n+2}, Tx_{2n+1})}{1 + d(x_{2n+2}, x_{2n+1})} \\
 &= \alpha(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x_{2n+1}) + \beta(x_{2n+2}, x_{2n+1}) \frac{d(x_{2n+2}, x(2n+3))d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} \\
 &\quad + \gamma(x_{2n+2}, x_{2n+1}) \frac{d(x_{2n+1}, x(2n+3))d(x_{2n+2}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} \\
 &= \alpha(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x_{2n+1}) + \beta(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x(2n+3)) \frac{d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} \\
 &\lesssim \alpha(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x_{2n+1}) + \beta(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x(2n+3)),
 \end{aligned}$$

that is,

$$d(x_{2n+3}, x_{2n+2}) \lesssim \delta(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x_{2n+1}). \tag{2.6}$$

From (2.5) and (2.6), we get

$$d(x_n, x_{n+1}) \lesssim \delta(x_{n-1}, x_n)d(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N}.$$

Therefore, we get

$$|d(x_n, x_{n+1})| \leq \delta(x_{n-1}, x_n)|d(x_{n-1}, x_n)| \leq |d(x_{n-1}, x_n)|, \quad \text{for all } n \in \mathbb{N}. \tag{2.7}$$

This implies that the sequence $\{|d(x_{n-1}, x_n)|\}$, $n \in \mathbb{N}$ is monotone non-increasing and bounded below, therefore, $|d(x_{n-1}, x_n)| \rightarrow r$ for some $r \geq 0$.

Next, we claim that $r = 0$. Assume to the contrary that $r > 0$. Proceeding limit as $n \rightarrow \infty$, we have from (2.7) $\delta(x_{n-1}, x_n) \rightarrow 1$. Since $\delta \in \Gamma$, we get $(x_{n-1}, x_n) \rightarrow 0$, that is, $|d(x_{n-1}, x_n)| \rightarrow 0$, which is a contradiction. Therefore, we have $r = 0$, that is,

$$|d(x_{n-1}, x_n)| \rightarrow 0. \tag{2.8}$$

Next, we show that $\{x_n\}$ is a C-Cauchy sequence. According to (2.8), it is sufficient to prove that the subsequence $\{x_{2n}\}$ is a C-Cauchy sequence. Let, if possible, $\{x_{2n}\}$ is not a C-Cauchy sequence. So, there is $c \in \mathbb{C}$ with $0 < c$, for which, for all $k \in \mathbb{N}$, there exists $m(k) > n(k) \geq k$, such that

$$d(x_{2n(k)}, x_{2m(k)}) \gtrsim c. \tag{2.9}$$

Further, corresponding to $n(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k) \geq k$ satisfying (2.9). Then, we have

$$d(x_{2n(k)}, x_{2m(k)}) \gtrsim c \tag{2.10}$$

and

$$d(x_{2n(k)}, x_{2m(k)-2}) < c. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned}
 c &\lesssim d(x_{2n(k)}, x_{2m(k)}) \\
 &\lesssim d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}) \\
 &< c + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}).
 \end{aligned}$$

This implies that

$$|c| \leq |d(x_{2n(k)}, x_{2m(k)})| \leq |c| + |d(x_{2m(k)-2}, x_{2m(k)-1})| + |d(x_{2m(k)-1}, x_{2m(k)})|.$$

Letting $k \rightarrow \infty$, we get

$$|d(x_{2n(k)}, x_{2m(k)})| \rightarrow |c|. \tag{2.12}$$

Further, we have

$$\begin{aligned} d(x_{2n(k)}, x_{2m(k)}) &\lesssim d(x_{2n(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2m(k)}) \\ &\lesssim d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2m(k)}), \end{aligned}$$

implies that,

$$|d(x_{2n(k)}, x_{2m(k)})| \leq |d(x_{2n(k)}, x_{2m(k)})| + |d(x_{2m(k)}, x_{2m(k)+1})| + |d(x_{2m(k)+1}, x_{2m(k)})|.$$

Letting $k \rightarrow \infty$, and using (2.8) and (2.12), we get

$$|d(x_{2n(k)}, x_{2m(k)+1})| \rightarrow |c|. \tag{2.13}$$

Now,

$$\begin{aligned} d(x_{2n(k)}, x_{2m(k)+1}) &\lesssim d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)+2}) + d(x_{2m(k)+2}, x_{2m(k)+1}) \\ &= d(x_{2n(k)}, x_{2n(k)+1}) + d(Sx_{2n(k)}, Tx_{2m(k)+1}) + d(x_{2m(k)+2}, x_{2m(k)+1}) \\ &\lesssim d(x_{2n(k)}, x_{2n(k)+1}) + \alpha(x_{2n(k)}, x_{2m(k)+1})d(x_{2n(k)}, x_{2m(k)+1}) \\ &\quad + \beta(x_{2n(k)}, x_{2m(k)+1}) \frac{d(x_{2n(k)}, Sx_{2n(k)})d(x_{2m(k)+1}, Tx_{2m(k)+1})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \\ &\quad + \gamma(x_{2n(k)}, x_{2m(k)+1}) \frac{d(x_{2m(k)+1}, Sx_{2n(k)})d(x_{2n(k)}, Tx_{2m(k)+1})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} + d(x_{2m(k)+2}, x_{2m(k)+1}) \\ &= d(x_{2n(k)}, x_{2n(k)+1}) + \alpha(x_{2n(k)}, x_{2m(k)+1})d(x_{2n(k)}, x_{2m(k)+1}) \\ &\quad + \beta(x_{2n(k)}, x_{2m(k)+1}) \frac{d(x_{2n(k)}, x_{2n(k)+1})d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \\ &\quad + \gamma(x_{2n(k)}, x_{2m(k)+1}) \frac{d(x_{2m(k)+1}, x_{2n(k)+1})d(x_{2n(k)}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} + d(x_{2m(k)+2}, x_{2m(k)+1}), \end{aligned}$$

implies that,

$$\begin{aligned} |d(x_{2n(k)}, x_{2m(k)+1})| &\leq |d(x_{2n(k)}, x_{2n(k)+1})| + \alpha(x_{2n(k)}, x_{2m(k)+1})|d(x_{2n(k)}, x_{2m(k)+1})| \\ &\quad + \beta(x_{2n(k)}, x_{2m(k)+1}) \left| \frac{d(x_{2n(k)}, x_{2n(k)+1})d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \right| \\ &\quad + \gamma(x_{2n(k)}, x_{2m(k)+1}) \left| \frac{d(x_{2m(k)+1}, x_{2n(k)+1})d(x_{2n(k)}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \right| + |d(x_{2m(k)+2}, x_{2m(k)+1})| \end{aligned}$$

$$\begin{aligned}
 &\leq |d(x_{2n(k)}, x_{2n(k)+1})| + \alpha(x_{2n(k)}, x_{2m(k)+1})|d(x_{2n(k)}, x_{2m(k)+1})| \\
 &\quad + \left| \frac{d(x_{2n(k)}, x_{2n(k)+1})d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \right| \\
 &\quad + \left| \frac{d(x_{2m(k)+1}, x_{2n(k)+1})d(x_{2n(k)}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \right| + |d(x_{2m(k)+2}, x_{2m(k)+1})| \\
 &\leq |d(x_{2n(k)}, x_{2n(k)+1})| + \frac{\alpha(x_{2n(k)}, x_{2m(k)+1})}{1 - \beta(x_{2n(k)}, x_{2m(k)+1})}|d(x_{2n(k)}, x_{2m(k)+1})| \\
 &\quad + \left| \frac{d(x_{2n(k)}, x_{2n(k)+1})d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \right| \\
 &\quad + \left| \frac{d(x_{2m(k)+1}, x_{2n(k)+1})d(x_{2n(k)}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \right| + |d(x_{2m(k)+2}, x_{2m(k)+1})| \\
 &\leq |d(x_{2n(k)}, x_{2n(k)+1})| + \delta(x_{2n(k)}, x_{2m(k)+1})|d(x_{2n(k)}, x_{2m(k)+1})| \\
 &\quad + \left| \frac{d(x_{2n(k)}, x_{2n(k)+1})d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \right| \\
 &\quad + \left| \frac{d(x_{2m(k)+1}, x_{2n(k)+1})d(x_{2n(k)}, x_{2m(k)+2})}{1 + d(x_{2n(k)}, x_{2m(k)+1})} \right| + |d(x_{2m(k)+2}, x_{2m(k)+1})|.
 \end{aligned}$$

Letting limit as $k \rightarrow \infty$, we get

$$|c| \leq \lim_{k \rightarrow \infty} \delta(x_{2n(k)}, x_{2m(k)+1})|c| \leq |c|,$$

which implies that, $\lim_{k \rightarrow \infty} \delta(x_{2n(k)}, x_{2m(k)+1}) = 1$.

Since $\delta \in \Gamma$, we get, $(x_{2n(k)}, x_{2m(k)}) \rightarrow 0$, that is, $|d(x_{2n(k)}, x_{2m(k)+1})| \rightarrow 0$, which contradicts $0 < c$. Therefore, we can conclude that $\{x_{2n}\}$ is a C-Cauchy sequence and hence $\{x_n\}$ is a C-Cauchy sequence in X and X is complete, so there exists a point z in X such that $x_n \rightarrow z$ as $n \rightarrow \infty$.

Next, we claim that $Sz = z$. If $Sz \neq z$, then $d(Sz, z) > 0$.

Now,

$$\begin{aligned}
 d(z, Sz) &\lesssim d(z, x_{2n+2}) + d(x_{2n+2}, Sz) \\
 &= d(z, x_{2n+2}) + d(Tx_{2n+1}, Sz) \\
 &= d(z, x_{2n+2}) + d(Sz, Tx_{2n+1}) \\
 &\lesssim d(x_{2n+2}, z) + \alpha(z, x_{2n+1})d(z, x_{2n+1}) + \beta(z, x_{2n+1}) \frac{d(z, Sz)d(x_{2n+1}, Tx_{2n+1})}{1 + d(z, x_{2n+1})} \\
 &\quad + \gamma(z, x_{2n+1}) \frac{d(x_{2n+1}, Sz)d(z, Tx_{2n+1})}{1 + d(z, x_{2n+1})} \\
 &= d(x_{2n+2}, z) + \alpha(z, x_{2n+1})d(z, x_{2n+1}) + \beta(z, x_{2n+1}) \frac{d(z, Sz)d(x_{2n+1}, x_{2n+2})}{1 + d(z, x_{2n+1})} \\
 &\quad + \gamma(z, x_{2n+1}) \frac{d(x_{2n+1}, Sz)d(z, x_{2n+2})}{1 + d(z, x_{2n+1})}.
 \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$d(z, Sz) \lesssim d(z, z) + \alpha(z, z)d(z, z) + \beta(z, z) \frac{d(z, Sz)d(z, z)}{1 + d(z, z)} + \gamma(z, z) \frac{d(z, Sz)d(z, z)}{1 + d(z, z)} = 0,$$

that is, $|d(z, Sz)| = 0$, which is a contradiction. Thus, we get $Sz = z$.

Similarly, we have $Tz = z$. Therefore, $z = Sz = Tz$, that is, z is a common fixed point of S and T .

Finally, we show that z is the unique common fixed point of S and T . Assume that there exists another point w such that $w = Sw = Tw$.

From (2.3), we have

$$\begin{aligned} d(z, w) &= d(Sz, Tw) \\ &\lesssim \alpha(z, w)d(z, w) + \beta(z, w)\frac{d(z, Sz)d(w, Tw)}{1 + d(z, w)} + \gamma(z, w)\frac{d(w, Sz)d(z, Tw)}{1 + d(z, w)} \\ &= \alpha(z, w)d(z, w) + \gamma(z, w)\frac{d(w, Sz)d(z, Tw)}{1 + d(z, w)} \\ &\lesssim [\alpha(z, w) + \gamma(z, w)]d(z, w), \end{aligned}$$

that is,

$$|d(z, w)| \leq [\alpha(z, w) + \gamma(z, w)]|d(z, w)|,$$

which implies that, $\alpha(z, w) + \gamma(z, w) \geq 1$, which is a contradiction and hence $z = w$.

Therefore, z is a unique common fixed point of S and T . \square

Corollary 2.2. Let S and T be self mappings of a C -complete complex valued metric space (X, d) satisfying the following:

$$d(Sx, Ty) \lesssim \lambda d(x, y) + \mu \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \nu \frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}, \quad \text{for all } x, y \text{ in } X, \tag{2.14}$$

where λ, μ, ν are non-negative reals with $\lambda + \mu + \nu < 1$.

Then S and T have a unique common fixed point.

Proof. By putting $\alpha(x, y) = \lambda, \beta(x, y) = \mu, \gamma(x, y) = \nu$ in Theorem 2.1, we get the required result. \square

Corollary 2.3. Let T be a self map of a C -complete complex valued metric space (X, d) . If there exists mappings $\alpha, \beta, \gamma : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ satisfying (2.1), (2.2) and the following:

$$d(Tx, Ty) \lesssim \alpha(x, y)d(x, y) + \beta(x, y)\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \gamma(x, y)\frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)}, \quad \text{for all } x, y \text{ in } X. \tag{2.15}$$

Then T has a unique fixed point in X .

Proof. By putting $S = T$ in Theorem 2.1, we get the required result. \square

Corollary 2.4. Let T be self mapping of a C -complete complex valued metric space (X, d) satisfying the following:

$$d(Tx, Ty) \lesssim \lambda d(x, y) + \mu \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \nu \frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)}, \quad \text{for all } x, y \text{ in } X, \tag{2.16}$$

where λ, μ, ν are non-negative reals with $\lambda + \mu + \nu < 1$.

Then T has a unique fixed point in X .

Proof. By putting $\alpha(x, y) = \lambda, \beta(x, y) = \mu, \gamma(x, y) = \nu$ in Corollary 2.3, we get the required result. \square

Theorem 2.5. Let T be a self map of a C -complete complex valued metric space (X, d) . If there exists mappings $\alpha, \beta, \gamma : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ satisfying (2.1), (2.2) and the following:

$$d(T^n x, T^n y) \lesssim \alpha(x, y)d(x, y) + \beta(x, y)\frac{d(x, T^n x)d(y, T^n y)}{1 + d(x, y)} + \gamma(x, y)\frac{d(y, T^n x)d(x, T^n y)}{1 + d(x, y)}, \tag{2.17}$$

for all x, y in X and some $n \in \mathbb{N}$.

Then T has a unique fixed point in X .

Proof. From Corollary 2.3, T^n has a fixed point z . But T^n has a fixed point Tz , since $T^n(Tz) = T(T^n z) = Tz$. Therefore, $Tz = z$ by the uniqueness of a fixed point T^n . Therefore, z is also a fixed point of T . Since the fixed point of T is also a fixed point of T^n , the fixed point of T is also unique. \square

Corollary 2.6. *Let T be a self mapping of a C -complete complex valued metric space (X, d) satisfying the following:*

$$d(T^n x, T^n y) \lesssim \lambda d(x, y) + \mu \frac{d(x, T^n x)d(y, T^n y)}{1 + d(x, y)} + \nu \frac{d(y, T^n x)d(x, T^n y)}{1 + d(x, y)}, \text{ for all } x, y \text{ in } X \text{ and some } n \in \mathbb{N}, \tag{2.18}$$

where λ, μ, ν are non-negative reals with $\lambda + \mu + \nu < 1$.

Then T has a unique fixed point in X .

Proof. By putting $\alpha(x, y) = \lambda, \beta(x, y) = \mu, \gamma(x, y) = \nu$ in Theorem 2.5, we get the required result. \square

3. Weakly Compatible Maps

Theorem 3.1. *Let S and T be self mappings of a complex value metric space (X, d) such that $T(X) \subseteq S(X)$ and $S(X)$ is C -complete. If there exists mappings $\alpha, \beta, \gamma : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ satisfying (2.1), (2.2) and the following:*

$$d(Tx, Ty) \lesssim \alpha(Sx, Sy)d(Sx, Sy) + \beta(Sx, Sy) \frac{d(Sx, Tx)d(Sy, Ty)}{1 + d(Sx, Sy)} + \gamma(Sx, Sy) \frac{d(Sx, Ty)d(Sy, Tx)}{1 + d(Sx, Sy)}, \tag{3.1}$$

for all x, y in X . Then S and T have a unique point of coincidence in X .

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof. Consider the mapping $S : X \rightarrow X$. By Lemma 1.12, there exists $E \subseteq X$ such that $S(E) = S(X)$ and $S : E \rightarrow X$ is one-to-one.

Next, we define a mapping $\Theta : S(E) \rightarrow S(E)$ by $\Theta(Sx) = Tx$ for all $Sx \in S(E)$. Therefore, Θ is well defined, since S is one-to-one on E . Since $\Theta \circ S = T$, using (3.1), we get

$$d(\Theta(Sx), \Theta(Sy)) \lesssim \alpha(Sx, Sy)d(Sx, Sy) + \beta(Sx, Sy) \frac{d(Sx, \Theta(Sx))d(Sy, \Theta(Sy))}{1 + d(Sx, Sy)} + \gamma(Sx, Sy) \frac{d(Sx, \Theta(Sy))d(Sy, \Theta(Sx))}{1 + d(Sx, Sy)}, \tag{3.2}$$

for all Sx, Sy in $S(E)$. Since $S(E) = S(X)$ is C -complete and 3.2 holds, we can apply Corollary 2.3 with a mapping Θ . Therefore, there exists a unique fixed point z in $S(X)$ such that $\Theta z = z$. Now, since $z \in S(X)$, there exists a point v in X such that $z = Sv$. So, $\Theta(Sv) = Sv$, that is, $Tv = Sv$. Therefore, T and S have a unique point of coincidence.

Next, we show that S and T have a common fixed point. Now, we have $z = Tv = Sv$. Since S and T are weakly compatible, we get $Sz = STv = TSv = Tz$. This implies that z is a point of coincidence of S and T .

Finally, we prove the uniqueness of a common fixed point of S and T . Assume that w is another common fixed point of S and T . So, $w = Sw = Tw$ and then w is also a point of coincidence of S and T . However, we know that z is a unique point of coincidence of S and T . Therefore, we get $w = z$, that is, z is a unique common fixed point of S and T . \square

4. Weakly Compatible and (CLR) Property

Theorem 4.1. *Let S and T be self mappings of a complex valued metric space (X, d) such that*

$$S \text{ and } T \text{ are weakly compatible,} \tag{4.1}$$

$$S \text{ and } T \text{ satisfy } (CLR_S) \text{ property,} \tag{4.2}$$

$$d(Tx, Ty) \lesssim \alpha(Sx, Sy)d(Sx, Sy) + \beta(Sx, Sy) \frac{d(Sx, Tx)d(Sy, Ty)}{1 + d(Sx, Sy)} + \gamma(Sx, Sy) \frac{d(Sx, Ty)d(Sy, Tx)}{1 + d(Sx, Sy)}, \tag{4.3}$$

where $\alpha, \beta, \gamma : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ be the mappings satisfying (2.1).

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy the (CLR_S) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = Sx, \quad \text{for some } x \text{ in } X.$$

We claim that $Sx = Tx$.

From (4.3), we have

$$d(Tx_n, Tx) \lesssim \alpha(Sx_n, Sx)d(Sx_n, Sx) + \beta(Sx_n, Sx) \frac{d(Sx_n, Tx_n)d(Sx, Tx)}{1 + d(Sx_n, Sx)} + \gamma(Sx_n, Sx) \frac{d(Sx_n, Tx)d(Sx, Tx_n)}{1 + d(Sx_n, Sx)}.$$

Letting $n \rightarrow \infty$, we get

$$d(Sx, Tx) \lesssim 0, \quad \text{which implies that, } |d(Sx, Tx)| \leq 0, \quad \text{that is, } Sx = Tx.$$

Let $z = Sx = Tx$. Since S and T are weakly compatible mappings, therefore, $STx = TSx$, implies that, $Sz = STx = TSx = Tz$.

Now, we claim that, $Tz = z$. Let, if possible, $Tz \neq z$.

From (4.3), we have

$$\begin{aligned} d(Tz, z) &= d(Tz, Tx) \\ &\lesssim \alpha(Sz, Sx)d(Sz, Sx) + \beta(Sz, Sx) \frac{d(Sz, Tz)d(Sx, Tx)}{1 + d(Sz, Sx)} + \gamma(Sz, Sx) \frac{d(Sz, Tx)d(Sx, Tz)}{1 + d(Sz, Sx)} \\ &= \alpha(Tz, z)d(Tz, z) + 0 + \gamma(Tz, z) \frac{d(Tz, z)d(z, Tz)}{1 + d(Tz, z)} \\ &\lesssim \alpha(Tz, z)d(Tz, z) + \gamma(Tz, z)d(Tz, z), \end{aligned}$$

which implies that, $|d(Tz, z)| \leq [\alpha(Tz, z) + \gamma(Tz, z)]|d(Tz, z)|$, a contradiction.

Therefore, $Tz = z = Sz$. So, z is the common fixed point of S and T .

For the uniqueness, let w be another common fixed point of S and T such that $w \neq z$.

From (4.3), we have

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \\ &\lesssim \alpha(Sz, Sw)d(Sz, Sw) + \beta(Sz, Sw) \frac{d(Sz, Tz)d(Sw, Tw)}{1 + d(Sz, Sw)} + \gamma(Sz, Sw) \frac{d(Sz, Tw)d(Sw, Tz)}{1 + d(Sz, Sw)} \\ &= \alpha(z, w)d(z, w) + 0 + \gamma(z, w) \frac{d(z, w)d(w, z)}{1 + d(z, w)} \\ &\lesssim \alpha(z, w)d(z, w) + \gamma(z, w)d(z, w), \end{aligned}$$

which implies that,

$$|d(z, w)| \leq [\alpha(z, w) + \gamma(z, w)]|d(z, w)|, \quad \text{a contradiction.}$$

Hence $w = z$.

Therefore, S and T have a unique common fixed point. \square

5. Weakly Compatible and E.A. Property

Theorem 5.1. Let S and T be self mappings of a complex valued metric space (X, d) satisfying (4.1), (4.3) and the following:

$$S \text{ and } T \text{ satisfy the E.A. property,} \tag{5.1}$$

$$T(X) \subseteq S(X). \tag{5.2}$$

If the range of S or T is a C -complete subspace of X , then S and T have a unique common fixed point in X .

Proof. Since S and T satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = z, \quad \text{for some } z \text{ in } X. \tag{5.3}$$

Since $T(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Tx_n = Sy_n$. Hence $\lim_{n \rightarrow \infty} Sy_n = z$.

Now, we shall show that $\lim_{n \rightarrow \infty} Ty_n = z$. Let $\lim_{n \rightarrow \infty} Ty_n = t$.

From (4.3), we have

$$\begin{aligned} d(Tx_n, Ty_n) &\lesssim \alpha(Sx_n, Sy_n)d(Sx_n, Sy_n) + \beta(Sx_n, Sy_n) \frac{d(Sx_n, Tx_n)d(Sy_n, Ty_n)}{1 + d(Sx_n, Sy_n)} \\ &\quad + \gamma(Sx_n, Sy_n) \frac{d(Sx_n, Ty_n)d(Sy_n, Tx_n)}{1 + d(Sx_n, Sy_n)}. \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$d(z, t) \lesssim \alpha(z, z)d(z, z) + \beta(z, z) \frac{d(z, z)d(z, t)}{1 + d(z, z)} + \gamma(z, z) \frac{d(z, t)d(z, z)}{1 + d(z, z)} = 0,$$

that is,

$$|d(z, t)| \leq 0, \quad \text{which implies that, } t = z.$$

Hence $\lim_{n \rightarrow \infty} Ty_n = z$.

Now, suppose that $S(X)$ is C-complete subspace of X . Then, there exists, u in X such that $z = Su$.

Subsequently, we have

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = \lim_{n \rightarrow \infty} Ty_n = z = Su.$$

Now, we show that $Su = Tu$.

From (4.3), we have

$$d(Tx_n, Tu) \lesssim \alpha(Sx_n, Su)d(Sx_n, Su) + \beta(Sx_n, Su) \frac{d(Sx_n, Tx_n)d(Su, Tu)}{1 + d(Sx_n, Su)} + \gamma(Sx_n, Su) \frac{d(Sx_n, Tu)d(Su, Tx_n)}{1 + d(Sx_n, Su)}.$$

Letting $n \rightarrow \infty$, we get

$$d(Su, Tu) \lesssim \alpha(Su, Su)d(Su, Su) + \beta(Su, Su) \frac{d(Su, Tu)d(Su, Tu)}{1 + d(Su, Su)} + \gamma(Su, Su) \frac{d(Su, Tu)d(Su, Su)}{1 + d(Su, Su)} = 0,$$

that is, $|d(Su, Tu)| \leq 0$, which implies that, $Su = Tu = z$.

Since S and T are weakly compatible, therefore, $STu = TSu$, implies that, $STu = SSu = TTu = TSu$.

Now, we claim that Tu is the common fixed point of S and T . Let, if possible, $Tu \neq TTu$.

From (4.3), we have

$$\begin{aligned} d(Tu, TTu) &\lesssim \alpha(Su, STu)d(Su, STu) + \beta(Su, STu) \frac{d(Su, Tu)d(STu, TTu)}{1 + d(Su, STu)} \\ &\quad + \gamma(Su, STu) \frac{d(Su, TTu)d(STu, Tu)}{1 + d(Su, STu)} \\ &= \alpha(Tu, TTu)d(Tu, TTu) + 0 + \gamma(Tu, TTu) \frac{d(Tu, TTu)d(TTu, Tu)}{1 + d(Tu, TTu)} \\ &\lesssim [\alpha(Tu, TTu) + \gamma(Tu, TTu)]d(Su, STu), \end{aligned}$$

that is,

$$|d(Tu, TTu)| \leq [\alpha(Tu, TTu) + \gamma(Tu, TTu)]|d(Tu, TTu)|, \text{ a contradiction.}$$

Hence $Tu = TTu = STu$. Therefore, Tu is the common fixed point of S and T .

For the uniqueness, let w and z be two common fixed points of S and T such that $w \neq z$.

From (4.3), we have

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \\ &\lesssim \alpha(Sz, Sw)d(Sz, Sw) + \beta(Sz, Sw)\frac{d(Sz, Tz)d(Sw, Tw)}{1 + d(Sz, Sw)} + \gamma(Sz, Sw)\frac{d(Sz, Tw)d(Sw, Tz)}{1 + d(Sz, Sw)} \\ &= \alpha(z, w)d(z, w) + 0 + \gamma(z, w)\frac{d(z, w)d(w, z)}{1 + d(z, w)} \\ &\lesssim \alpha(z, w)d(z, w) + \gamma(z, w)d(z, w), \end{aligned}$$

which implies that,

$$|d(z, w)| \leq [\alpha(z, w) + \gamma(z, w)]|d(z, w)|, \text{ a contradiction.}$$

Hence $w = z$.

Therefore, S and T have a unique common fixed point. \square

6. Urysohn Integral Equation

As an application, we apply Theorem 2.1 for the existence of a common solution of the system of the Urysohn integral equations.

Theorem 6.1. Let $X = C([a, b], \mathbb{R}^n)$, $a > 0$ and $d : X \times X \rightarrow \mathbb{C}$ be defined by

$$d(x, y) = \max_{t \in [a, b]} \|x(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{(t \tan^{-1} a)}.$$

Consider the Urysohn integral equations:

$$x(t) = \int_a^b K_1(t, s, x(s))ds + g(t), \tag{6.1}$$

$$x(t) = \int_a^b K_2(t, s, x(s))ds + h(t), \tag{6.2}$$

where $t \in [a, b] \subseteq \mathbb{R}$, $x, g, h \in X$ and $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Suppose that K_1, K_2 are such that $F_x, G_x \in X$ for all $x \in X$, where

$$\begin{aligned} F_x(t) &= \int_a^b K_1(t, s, x(s))ds, \\ G_x(t) &= \int_a^b K_2(t, s, x(s))ds, \text{ for all } t \in [a, b]. \end{aligned}$$

If there exists mappings $\alpha, \beta, \gamma : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ be the mappings satisfying (2.1), (2.2) and the following:

$$\begin{aligned} &\|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1 + a^2} e^{(t \tan^{-1} a)} \\ &\lesssim \alpha(\max_{t \in [a, b]} A(x, y)(t), y)A(x, y)(t) + \beta(x, \max_{t \in [a, b]} A(x, y)(t))B(x, y)(t) + \gamma(\max_{t \in [a, b]} A(x, y)(t), y)C(x, y)(t), \end{aligned} \tag{6.3}$$

where

$$\begin{aligned}
 A(x, y)(t) &= \|x(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a} \\
 B(x, y)(t) &= \frac{\|F_x(t) + g(t) - x(t)\|_\infty \|G_y(t) + h(t) - y(t)\|_\infty}{1 + d(x, y)} \sqrt{1 + a^2} e^{i \tan^{-1} a}, \\
 C(x, y)(t) &= \frac{\|F_x(t) + g(t) - y(t)\|_\infty \|G_y(t) + h(t) - x(t)\|_\infty}{1 + d(x, y)} \sqrt{1 + a^2} e^{i \tan^{-1} a},
 \end{aligned}$$

then the system of integral equations (6.1) and (6.2) has a unique solution.

Proof. Define two mappings $S, T : X \rightarrow X$ by $Sx = F_x + g$ and $Tx = G_x + h$. Then, we have

$$\begin{aligned}
 d(Sx, Ty) &= \max_{t \in [a, b]} \|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}, \\
 d(x, Sx) &= \max_{t \in [a, b]} \|F_x(t) + g(t) - x(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}, \\
 d(y, Ty) &= \max_{t \in [a, b]} \|G_y(t) + h(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}, \\
 d(x, Ty) &= \max_{t \in [a, b]} \|G_y(t) + h(t) - x(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}, \text{ and} \\
 d(y, Sx) &= \max_{t \in [a, b]} \|F_x(t) + g(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}.
 \end{aligned}$$

Now, we can easily show that for all $x, y \in X$

$$d(Sx, Ty) \lesssim \alpha(x, y)d(x, y) + \beta(x, y) \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma(x, y) \frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}.$$

Now, we can apply Theorem 2.1. Therefore, we get that Urysohn integral equations (6.1) and (6.2) have a unique solution. \square

7. Deduced Results

7.1. Sintunavarat, Cho and Kumam's results

Now, we deduce the main results of [15] as follows.

Theorem 7.1 ([15, Theorem 3.2]). Let S and T be self mappings of a C -complete complex value metric space (X, d) . If there exists mappings $\alpha, \beta : \mathbb{C}_+ \rightarrow [0, 1)$ such that for all x, y in X :

$$\alpha(x) + \beta(x) < 1, \tag{7.1}$$

$$\text{the mapping } \gamma : \mathbb{C}_+ \rightarrow [0, 1) \text{ defined by } \gamma(x) = \frac{\alpha(x)}{1 - \beta(x)} \text{ belongs to } \Gamma, \tag{7.2}$$

$$d(Sx, Ty) \lesssim \alpha(d(x, y))d(x, y) + \beta(d(x, y)) \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}. \tag{7.3}$$

Then S and T have a unique common fixed point in X .

Proof. Define $\alpha, \beta, \gamma : \mathbb{C}_+ \times \mathbb{C}_+ \rightarrow [0, 1)$ by

$$\alpha(x, y) = \alpha(d(x, y)), \beta(x, y) = \beta(d(x, y)), \gamma(x, y) = 0, \quad \text{for all } x, y \text{ in } X.$$

Now, using Theorem 2.1, we get the required result. \square

Corollary 7.2 ([15, Corollary 3.3]). Let S and T be self mappings of a C -complete complex valued metric space (X, d) satisfying the following:

$$d(Sx, Ty) \lesssim \lambda d(x, y) + \mu \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}, \quad (7.4)$$

for all x, y in X , where λ, μ are non-negative reals with $\lambda + \mu < 1$.

Then S and T have a unique common fixed point.

Proof. By putting $\alpha(x) = \lambda, \beta(x) = \mu$ in Theorem 7.1, we get the required result. \square

Corollary 7.3 ([15, Corollary 3.4]). Let T be a self map of a C -complete complex valued metric space (X, d) . If there exists mappings $\alpha, \beta : \mathbb{C}_+ \rightarrow [0, 1)$ satisfying (7.5), (7.5) and the following:

$$d(Tx, Ty) \lesssim \alpha(d(x, y))d(x, y) + \beta(d(x, y)) \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \quad \text{for all } x, y \text{ in } X. \quad (7.5)$$

Then T has a unique fixed point in X .

Proof. By putting $S = T$ in Theorem 7.1, we get the required result. \square

Corollary 7.4 ([15, Corollary 3.5]). Let T be self mapping of a C -complete complex valued metric space (X, d) satisfying the following:

$$d(Tx, Ty) \lesssim \lambda d(x, y) + \mu \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \quad (7.6)$$

for all x, y in X , where λ, μ are non-negative reals with $\lambda + \mu < 1$.

Then T has a unique fixed point in X .

Proof. By putting $\alpha(x) = \lambda, \beta(x) = \mu$ in Corollary 7.3, we get the required result. \square

Theorem 7.5 ([15, Theorem 3.6]). Let T be a self map of a C -complete complex valued metric space (X, d) . If there exists mappings $\alpha, \beta : \mathbb{C}_+ \rightarrow [0, 1)$ satisfying , and the following:

$$d(T^n x, T^n y) \lesssim \alpha(d(x, y))d(x, y) + \beta(d(x, y)) \frac{d(x, T^n x)d(y, T^n y)}{1 + d(x, y)}, \quad (7.7)$$

for all x, y in X and some $n \in \mathbb{N}$.

Then T has a unique fixed point in X .

Proof. From Corollary 7.3, T^n has a fixed point z . Since $T^n(Tz) = T(T^n z) = Tz$, we get Tz is a fixed point of T^n . Therefore, $Tz = z$ by the uniqueness of a fixed point T^n . Therefore, z is also a fixed point of T . Since the fixed point of T is also a fixed point of T^n , we get that fixed point of T is also unique. \square

Corollary 7.6 ([15, Corollary 3.7]). Let T be a self mapping of a C -complete complex valued metric space (X, d) satisfying the following:

$$d(T^n x, T^n y) \lesssim \lambda d(x, y) + \mu \frac{d(x, T^n x)d(y, T^n y)}{1 + d(x, y)}, \quad \text{for all } x, y \text{ in } X \text{ and some } n \in \mathbb{N}, \quad (7.8)$$

where λ, μ are non-negative reals with $\lambda + \mu < 1$.

Then T has a unique fixed point in X .

Proof. By putting $\alpha(x) = \lambda, \beta(x) = \mu$, in Theorem 7.5, we get the required result. \square

Theorem 7.7 ([15, Theorem 4.2]). Let S and T be self mappings of a complex value metric space (X, d) such that $T(X) \subseteq S(X)$ and $S(X)$ is C -complete. If there exists mappings $\alpha, \beta : \mathbb{C}_+ \rightarrow [0, 1)$ satisfying (7.5), (7.5) and the following:

$$d(Tx, Ty) \lesssim \alpha(d(Sx, Sy))d(Sx, Sy) + \beta(d(Sx, Sy)) \frac{d(Sx, Tx)d(Sy, Ty)}{1 + d(Sx, Sy)}, \text{ for all } x, y \text{ in } X. \quad (7.9)$$

Then S and T have a unique point of coincidence in X .

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof. By putting $\alpha(Sx, Sy) = \alpha(d(Sx, Sy))$, $\beta(Sx, Sy) = \beta(d(Sx, Sy))$ and $\gamma(Sx, Sy) = 0$ in Theorem 3.1, we get the required result. \square

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