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The Irreducibility of C^*–algebras Acting on Hilbert C^*–modules

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Abstract. Let $\mathcal{B}$ be a C^*-algebra, E be a Hilbert $\mathcal{B}$ module and $\mathbb{L}(E)$ be the set of adjointable operators on E. Let $\mathcal{A}$ be a non-zero C^*-subalgebra of $\mathbb{L}(E)$. In this paper, some new kinds of irreducibilities of $\mathcal{A}$ acting on E are introduced, which are all the generalizations of those associated to Hilbert spaces. The difference between these irreducibilities are illustrated by a number of counterexamples. It is concluded that for a full Hilbert $\mathcal{B}$-module, these irreducibilities are all equivalent if and only if the underlying C^*-algebra $\mathcal{B}$ is isomorphic to the C^*–algebra of all compact operators on a Hilbert space.

1. Introduction

By a Hilbert module over a C^*–algebra $\mathcal{B}$ [2] we mean a right $\mathcal{B}$–module E equipped with a $\mathcal{B}$–valued inner product $<\cdot,\ \cdot>_{\mathcal{B}}$, which is $\mathcal{B}$–linear in the second variable and conjugate linear in the first variable, such that E is a Banach space with the induced norm given by

$$||x|| = ||<x,x>_{\mathcal{B}}||^{\frac{1}{2}}, x \in E.$$

If the closed linear span of $\{<x,\ y>_{\mathcal{B}} | x,y \in E\}$ coincides with $\mathcal{B}$, then E is called a *full Hilbert $\mathcal{B}$–module*. Note that every C^*–algebra $\mathcal{B}$ can be considered as a full Hilbert C^*–module over itself, where the inner product is given by

$$<x,y>_{\mathcal{B}}= x^*y,\ x,y \in \mathcal{B}. \tag{1}$$

For any submodule E_0 of E, we put $E_0^{\perp} = \{e \in E| <e,x>_{\mathcal{B}}= 0, \forall x \in E_0\}$. We denote by $\mathbb{L}(E)$ the C^*–algebra of all adjointable operators on E. For any $x,y \in E$ there is an operator $\theta_{x,y} \in \mathbb{L}(E)$ defined by

$$\theta_{x,y}(z) = x<y,z>_{\mathcal{B}}\ z \in E.$$

The linear span of such operators will be denoted by $\mathbb{F}(E)$. We set $\mathbb{K}(E)$ to be the norm closure of $\mathbb{F}(E)$, which is a closed two-sided ideal (not necessarily proper) of $\mathbb{L}(E)$. Note that if $E = \mathcal{B}$ with the inner product given by (1), then $\mathbb{K}(E) \cong \mathcal{B}$ through an isomorphism from $\mathbb{K}(E)$ to $\mathcal{B}$ by identifying $\theta_{x,y}$ with xy^* for $x,y \in \mathcal{B}$. As a result, $\mathbb{L}(E) \cong \mathbb{M}(\mathcal{B})$, where $\mathbb{M}(\mathcal{B})$ is the multiplier algebra of $\mathcal{B}$.

In this paper, we study the irreducibility of C^*–algebras acting on Hilbert $\mathcal{B}$–modules. Let E be a Hilbert $\mathcal{B}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. Denote by $C[\mathcal{A},\mathbb{L}(E)]$ the set of all elements of $\mathbb{L}(E)$ that commute with all the elements of $\mathcal{A}$. We consider the following chain of conditions:

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(SD) $\mathcal{A}$ is strictly dense in $\mathbb{L}(E)$;

(TI) The only closed $\mathcal{B}$–submodules of E that are invariant for $\mathcal{A}$ are 0 and E;

(EI) For every $\mathcal{A}$–invariant $\mathcal{B}$–submodule E_0 of E, either $E_0 = 0$ or $E_0^{\perp} = 0$;

(MC) $C[\mathcal{A}, \mathbb{L}(E)] = C[\mathbb{L}(E), \mathbb{L}(E)]$.

In the special case where $\mathcal{B}$ is the complex field, it can be found in [3] that conditions (SD)–(MC) are all equivalent. It is interesting to study whether the above four conditions still remain to be equivalent if the complex field is replaced by a general C^*-algebra $\mathcal{B}$. Clearly, (TI) ⇒ (EI) ⇒ (MC); besides if $\mathcal{B}$ is supposed to be simple, (SD) implies (TI). So the question under discussion is to show whether the reverse implications of each step from (MC) to (SD) are true or not. In this paper we will give a negative answer for each step by constructing an associated counterexample based on some concrete C^*-algebra $\mathcal{B}$.

It is also very interesting to find out some partial positive answers. We prove that if $\mathcal{B}$ is isomorphic to the C^*–algebra of all compact operators on a Hilbert space, then for any Hilbert $\mathcal{B}$–module E and any C^*–subalgebra $\mathcal{A}$ of $\mathbb{L}(E)$, the implication (MC) ⇒ (SD) is true. Moreover, given any C^*-algebra $\mathcal{B}$, if there exists a full Hilbert $\mathcal{B}$-module E such that the implication (EI) ⇒ (TI) is true for any C^*-subalgebra $\mathcal{A}$ of $\mathbb{L}(E)$, then $\mathcal{B}$ must be isomorphic to the C^*–algebra of all compact operators on a Hilbert space. So, we conclude that for a full Hilbert $\mathcal{B}$-module, conditions (SD)–(MC) are all equivalent if and only if $\mathcal{B}$ is isomorphic to the C^*–algebra of all compact operators on a Hilbert space.

The paper is organized as follows. In Section 2, we introduce and study some new kinds of irreducibilities of C^*-algebras acting on Hilbert C^*–modules. In Section 3, we focus on the study of those irreducibilities with the underlying C^*-algebra $\mathcal{B}$ being isomorphic to the C^*–algebra of all compact operators on a Hilbert space.

2. Irreducibility of Module Operators

From now on, $\mathbb{C}$, $\mathbb{R}$ and $\mathcal{N}$ are the complex field, the real field and the positive integer set respectively; $\mathcal{A}$ and $\mathcal{B}$ are two non-zero C^*–algebras. Let

$$\mathcal{A}_{sa} = \{a \in \mathcal{A} | a^* = a\} \text{ and } \mathcal{A}_+ = \{a \in \mathcal{A}_{sa} | a \geq 0\}.$$

If Γ is a subset of a Banach space, we use the notation $\overline{\Gamma}$ to denote the norm closure of Γ. If furthermore, Γ is a subset of a Hilbert $\mathcal{B}$–module E, we put

$$\Gamma^{\perp} = \{e \in E | < e, \ x >_{\mathcal{B}} = 0, \forall x \in \Gamma\}.$$

For convenience, throughout this paper, we assume that the inner product on every Hilbert space H is linear in the second variable and conjugate linear in the first variable.

Definition 2.1. *Let E be a Hilbert $\mathcal{B}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. We say $\mathcal{A}$ acts* topologically irreducibly *(briefly, TI) on E if E and 0 are the only closed $\mathcal{B}$–submodules of E that are invariant for $\mathcal{A}$. Equivalently,*

$$\overline{\mathcal{A}(x\mathcal{B})} := \overline{span\{a(x \cdot b) | a \in \mathcal{A}, b \in \mathcal{B}\}} = E, \ \forall x \in E \setminus \{0\}.$$

Definition 2.2. *Let E be a Hilbert $\mathcal{B}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. We say $\mathcal{A}$ acts* essentially irreducibly *(briefly, EI) on E if for every $\mathcal{A}$–invariant $\mathcal{B}$–submodule E_0 of E, either $E_0 = 0$ or $E_0^{\perp} = 0$.*

Remark 2.3. *Suppose that $\mathcal{A}$ acts essentially irreducibly on E. Denote by N the set $\{x \in E | \mathcal{A}(x) = 0\}$. Since $\mathcal{A}$ is non-zero, there is a $y \in E \setminus \{0\}$ such that $\mathcal{A}(y) \neq 0$, which implies that $\mathcal{A}(y\mathcal{B}) \neq 0$, hence $[\mathcal{A}(y\mathcal{B})]^{\perp} = 0$, which in turn implies that $N = 0$ since $N \subseteq [\mathcal{A}(y\mathcal{B})]^{\perp}$. We have proved that $\mathcal{A}(x\mathcal{B}) \neq 0$ whenever $x \in E \setminus \{0\}$. In view of this observation, we conclude that $\mathcal{A}$ acts essentially irreducibly on E if and only if*

$$[\mathcal{A}(x)]^{\perp} = 0, \forall x \in E \setminus \{0\}.$$

For any C^*–subalgebra $\mathcal{A}$ of $\mathcal{B}$, we denote by $C[\mathcal{A},\mathcal{B}]$ the set of all elements of $\mathcal{B}$ that commute with all the elements of $\mathcal{A}$, which is also a C^*–subalgebra of $\mathcal{B}$.

Definition 2.4. *Let E be a Hilbert $\mathcal{B}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. We say $\mathcal{A}$ has* minimal commutant *(briefly, MC) if $C[\mathcal{A},\mathbb{L}(E)] = C[\mathbb{L}(E),\mathbb{L}(E)]$.*

Taking $\mathbb{K}(E)$ as an example, we know that for each Hilbert $\mathcal{B}$–module E, there always exists some C^*–subalgebra of $\mathbb{L}(E)$ which satisfies (MC). The following proposition indicates that the same is however not true for (TI). Recall that a C^*–algebra $\mathcal{A}$ is *simple* if it has not any non-trivial closed ideal.

Proposition 2.5. *Let E be a full Hilbert $\mathcal{B}$–module. The following statements are equivalent:*

(1) *$\mathcal{B}$ is simple;*
(2) *There exists a C^*–subalgebra $\mathcal{A}$ of $\mathbb{L}(E)$ satisfying (TI).*

Proof. (1) $\Rightarrow$ (2). Set $x,\ y \in E\setminus\{0\}$. By [4, Proposition 2.31], there exists a unique $e \in E$ such that $y = e < e, e >_{\mathcal{B}}$. Since $\mathcal{B}$ is simple, $< e, e >_{\mathcal{B}}$ can be infinitely approximated by the elements of the form $\sum_{i=1}^{n} a_i < x, x >_{\mathcal{B}} b_i$, where $a_i,\ b_i \in \mathcal{B},\ i = 1,\ ...,\ n$. Hence y can be approximated by the elements of the form $\sum_{i=1}^{n} \theta_{e,xa_i^*}(xb_i)$. This proves that $y \in \overline{\mathbb{K}(E)(x\mathcal{B})}$. Thus, $\mathbb{K}(E)$ acts topologically irreducibly on E.

(2) $\Rightarrow$ (1). Let I be any non–zero closed two–side ideal of $\mathcal{B}$. We assert that the submodule $\overline{EI}$ is non-zero. In fact, if $EI = 0$, then

$$I = \overline{I\mathcal{B}I} = \overline{I < E, E >_{\mathcal{B}} I} = \overline{< EI, EI >_{\mathcal{B}}} = 0.$$

Now if $\mathcal{A}$ is a C^*–subalgebra of $\mathbb{L}(E)$ which satisfies (TI), then $\overline{EI}$ is non-zero and $\mathcal{A}$–invariant, thus $\overline{EI} = E$. It follows that

$$I \supseteq \overline{I < E, E >_{\mathcal{B}} I} = \overline{< EI, EI >_{\mathcal{B}}} = \overline{< E, E >_{\mathcal{B}}} = \mathcal{B}.$$

This completes the proof of (2) $\Rightarrow$ (1). $\square$

Remark 2.6. *A C^*–algebra $\mathcal{A}$ is* prime *if 0 is a prime ideal of $\mathcal{A}$ ([3]). It also can be proved that for a full Hilbert $\mathcal{B}$–module E, there exists a C^*–subalgebra of $\mathbb{L}(E)$ acting essentially irreducibly on E if and only if $\mathcal{B}$ is prime.*

By Proposition 2.5, throughout the rest of this paper if there is no special instructions, $\mathcal{B}$ is always assumed to be simple. Then notice that every Hilbert $\mathcal{B}$–module E must be full module. It is clear that (TI) can imply (EI). Unfortunately the reverse is false.

Example 2.7. *For positive integer n, we give a unital *–homomorphism φ_n from matrix algebra $M_{2^n}(\mathbb{C})$ to $M_{2^{n+1}}(\mathbb{C})$*

$$\varphi_n : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C}), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Let $\mathcal{B}$ be the direct limit of $(M_{2^n}(\mathbb{C}), \varphi_n)_{n=1}^{\infty}$ which is a UHF algebra ([3]). Denote by φ^n the natural map from $M_{2^n}(\mathbb{C})$ to $\mathcal{B}$ for each n. By [3, Theorem 6.2.5], we assume $\mathcal{B}$ acts non-degenerately on a Hilbert space H and the von Neumann algebra $\mathcal{B}''$ generated by $\mathcal{B}$ admits a faithful tracial state τ. Observe that $\tau \circ \varphi^n$ is the unique tracial state of $M_{2^n}(\mathbb{C})$.

Let q_1 be a one-rank projection of $M_2(\mathbb{C})$ and set $p_1 = \varphi^1(q_1) \in \mathcal{B}$. Clearly $\tau(p_1) = 2^{-1}$. Element $\varphi_1(q_1)$ is a two-rank projection of $M_4(\mathbb{C})$. Let q_2 be a one-rank projection of $M_4(\mathbb{C})$ with $q_2\varphi_1(q_1) = 0$. Set $p_2 = \varphi^2(q_2) \in \mathcal{B}$. Then we have $p_1p_2 = 0$ and $\tau(p_2) = 2^{-2}$. Element $\varphi_2[\varphi_1(q_1) + q_2]$ is a six-rank projection of $M_8(\mathbb{C})$. Let q_3 be a one-rank projection of $M_8(\mathbb{C})$ with $q_3\varphi_2[\varphi_1(q_1) + q_2] = 0$. Set $p_3 = \varphi^3(q_3) \in \mathcal{B}$. We have $p_3(p_2 + p_1) = 0$ and $\tau(p_3) = 2^{-3}$. Repeating these steps, we can construct a sequence of projections $\{p_i\}_{i=1}^{\infty}$ in $\mathcal{B}$ satisfying the following two conditions:

(1) *If $i \neq j$, $p_ip_j = 0$;*
(2) *$\tau(p_i) = 2^{-i}, i \in \mathcal{N}$.*

Set $T = \sum_{i=1}^{\infty} 2^{-i} p_i$ *and* $Q_i = \sum_{j=1}^{i} p_j, i \in \mathcal{N}$.

Next we assert that if $a \in \mathcal{B}$ *with* $aT = 0$, *then* a *must be* 0. *By Vigier's theorem ([3, Theorem 4.1.1]),* $\{Q_i\}_{i\in\mathcal{N}}$ *converges weakly to some* Q *in* $\mathcal{B}''_+$ *with* $\|Q\| \leq 1$. *Obviously,* $Q \geq Q_i$ *for each* $i \in \mathcal{N}$, *so* $\tau(Q) \geq 1 - 2^{-i}$ *for all* $i \in \mathcal{N}$. *Thus* $\tau(Q) = 1$. *Since* τ *is faithful on* $\mathcal{B}''$, Q *must be the identical operator on* H. *Let* a *be an element of* $\mathcal{B}$, *then*

$$\begin{aligned} aT = 0 \quad &\Leftrightarrow \quad aTa^* = 0 \\ &\Leftrightarrow \quad a(2^{-i}p_i)a^* = 0, i \in \mathcal{N} \\ &\Leftrightarrow \quad ap_ia^* = 0, i \in \mathcal{N} \\ &\Leftrightarrow \quad aQ_ia^* = 0, i \in \mathcal{N} \\ &\Leftrightarrow \quad aa^* = 0 \\ &\Leftrightarrow \quad a = 0. \end{aligned}$$

Now consider $\mathcal{B}$ *as a full Hilbert* C^**–module* E *over itself with the inner product given by (1). Since* $\mathcal{B}$ *has a unit,* $\mathbb{L}(E) = \mathbb{K}(E) \cong \mathcal{B}$. *Denote by* l *the isomorphism from* $\mathcal{B}$ *to* $\mathbb{L}(E)$. *Let* $\mathcal{A}$ *be* $\overline{T\mathcal{B}T}$, *the hereditary* C^**–subalgebra of* $\mathcal{B}$ *generated by* T. *Notice that* $\mathcal{A}$ *is simple. For any* $x \in \mathcal{B} \setminus \{0\}$, *we have* $Txx^*T \in \mathcal{A} \setminus \{0\}$. *Then*

$$\overline{l(\mathcal{A})(x\mathcal{B})} = \overline{\mathcal{A}x\mathcal{B}} \supseteq \overline{\mathcal{A}Txx^*T\mathcal{B}} \supseteq \overline{\mathcal{A}(Txx^*T)\mathcal{A}} = \mathcal{A} \supseteq \{T\},$$

which means

$$[l(\mathcal{A})(x)]^{\perp} = [l(\mathcal{A})(x\mathcal{B})]^{\perp} \subseteq \{T\}^{\perp} = 0.$$

However, $l(\mathcal{A})$ *does not act topologically irreducibly on* E. *Actually if* $l(\mathcal{A})$ *acts topologically irreducibly on* E, *we can get*

$$\overline{\mathcal{A}\mathcal{B}} = \overline{l(\mathcal{A})E} = E = \mathcal{B},$$

which follows T *is a strictly positive element of* $\mathcal{B}$, *namely* $\overline{T\mathcal{B}T} = \mathcal{B}$. *Since* $\mathcal{B}$ *has a unit, any strictly positive element of* $\mathcal{B}$ *must be invertible. On the other hand, it is clear that* 2^{-i} *is in the spectral set of* T *for any* $i \in \mathcal{N}$, *so* T *can not be invertible.*

The following results show that (EI) can imply (MC).

Proposition 2.8. *Let* E *be a Hilbert* $\mathcal{B}$*–module. If* $\mathcal{A}$ *is a* C^**–subalgebra of* $\mathbb{L}(E)$ *acting essentially irreducibly on* E, *then* $C[\mathcal{A}, \mathbb{L}(E)] = C[\mathbb{L}(E), \mathbb{L}(E)] = \mathbb{C} \cdot id_E$ *where* id_E *is the identical operator on* E.

Proof. It is enough to prove that each element of $(C[\mathcal{A}, \mathbb{L}(E)])_{sa}$ has only one spectral point.

Let a be a non–zero element of $(C[\mathcal{A}, \mathbb{L}(E)])_{sa}$. Denote by $\sigma(a)$ the spectral set of a. If a has more than two spectral points, by Urysohn's lemma, there are two non–zero bounded continuous real-valued functions f_1, f_2 on $\sigma(a)$ with $f_1 \cdot f_2 = 0$. Set $a_1 = f_1(a)$ and $a_2 = f_2(a)$ and they are still in $(C[\mathcal{A}, \mathbb{L}(E)])_{sa}$. It is obvious that $\overline{a_1(E)}$ and $\overline{a_2(E)}$ are two non–zero closed submodules of E with $< \overline{a_1(E)}, \overline{a_2(E)} >_{\mathcal{B}} = 0$, which contradicts the assumption. □

In the next example, $\mathcal{B}$ is not simple and we will show that (MC) can not imply (EI) when $\mathbb{L}(E)$ satisfies (EI).

Example 2.9. *Let* H *be an infinite-dimensional Hilbert space. Set* $\mathcal{B}$ *to be the unitization of* $\mathbb{K}(H \oplus H)$, *namely* $M_2(\mathbb{K}(H))^{\sim}$. *Let* $\mathcal{A}$ *to be the set* $\{T \oplus T | T \in \mathbb{K}(H)\}$. *Clearly,* $\mathcal{A}$ *is a* C^**–subalgebra of* $\mathcal{B}$ *and*

$$C[\mathcal{A}, \mathcal{B}] = \mathbb{C} \cdot 1 = C[\mathcal{B}, \mathcal{B}].$$

Now consider $\mathcal{B}$ *as a full Hilbert* C^**–module* E *over itself. Since* $\mathcal{B}$ *has a unit,* $\mathbb{K}(E) = \mathbb{L}(E) \cong \mathcal{B}$. *We still denote by* l *the isomorphism from* $\mathcal{B}$ *to* $\mathbb{L}(E)$. *Hence* $l(\mathcal{A})$ *has minimal commutant. Put* $a = 0 \oplus T$ *and* $b = T \oplus 0$, *where* $T \in \mathbb{K}(H)$. *It is easy to see* $< l(\mathcal{A})(a), b >_{\mathcal{B}} = 0$, *which means* $l(\mathcal{A})$ *does not satisfy (EI). However notice that* $\mathbb{L}(E)$ *does act essentially irreducibly on* E.

Conjecture 2.10. *When $\mathcal{B}$ is simple, (MC) can not imply (EI).*

The *strict topology* on $\mathbb{L}(E)$ ([2]) is defined to be the topology given by the seminorms

$$t \mapsto \|t(x)\|, \quad t \mapsto \|t^*(x)\|, \quad (t \in \mathbb{L}(E),\ x \in E).$$

When $\mathcal{B}$ is $\mathbb{C}$, the strict topology is just the strong * topology.

If $\mathcal{A}$ is a strictly dense C^*–subalgebra of $\mathbb{L}(E)$, then for any $x \in E \setminus \{0\}$, we have $\overline{\mathcal{A}(x)} = \overline{\mathbb{L}(E)(x)}$. Since we assume $\mathcal{B}$ is simple, $\overline{\mathcal{A}(x\mathcal{B})} = \overline{\mathbb{L}(E)(x\mathcal{B})} = E$. (SD) must imply (TI).

[1, Theorem 1.13] showed that the Cuntz algebra O_n [1] is a simple C^*–algebra when n is a finite positive integer. In fact the proof of that theorem can show a stronger result. For the reader's convenience, we give a short proof of the next lemma and we will follow standard terminology and notation in [1].

Lemma 2.11. *Let n be a finite positive integer. For each $X \in O_n \setminus \{0\}$, it holds that*

$$\mathcal{F}^n X O_n := span\{A \cdot X \cdot B | A \in \mathcal{F}^n,\ B \in O_n\} = O_n,$$

where $\mathcal{F}^n$ is a UHF C^*–algebra contained in O_n which has the unit of O_n.

Proof. Replace the element X^*X with XX^* in the proof of [1, Theorem 1.13]. Define ε, S_1, Y, Q, λ_{i_0} and U in the same way they were defined in [1, Theorem 1.13]. Remember UU^* is a non-zero projection of $\mathcal{F}^n$. Set $A = UQ \in \mathcal{F}^n$ instead of S_1^*UQ. Then we have $AYA^* = \lambda_{i_0}UU^*$ and

$$\|AXX^*A^* - UU^*\| \leq \|AXX^*A^* - AYA^*\| + \|AYA^* - UU^*\| \leq 2\varepsilon < 1.$$

This shows that AXX^*A^* is invertible in the C^*–algebra $(UU^*)O_n(UU^*)$, so there is $B \in (UU^*)O_n(UU^*)$ such that $AXX^*A^*B = UU^* \in \mathcal{F}^n$. Note that $\mathcal{F}^n$ is algebraical simple, namely $\mathcal{F}^n$ without any non-trivial ideal. Thus $\mathcal{F}^nUU^*\mathcal{F}^n$, the ideal generated by UU^* of $\mathcal{F}^n$ must be $\mathcal{F}^n$. Hence the unit of O_n is in $\mathcal{F}^nXO_n$. Consequently, $\mathcal{F}^nXO_n$ contains all the elements of O_n. $\square$

Example 2.12. *Let $\mathcal{B}$ be O_n where n is a finite positive integer and consider it as a full Hilbert C^*–module E over itself. Since $\mathcal{B}$ has a unit, $\mathbb{K}(E) = \mathbb{L}(E) \cong O_n$ and the strict topology on $\mathbb{L}(E)$ coincide with its norm topology. Denote by l the isomorphism from $\mathcal{B}$ to $\mathbb{L}(E)$. Let $\mathcal{A}$ be $\mathcal{F}^n$. The preceding lemma shows that $l(\mathcal{A})$ acts topologically irreducibly on E and clearly $l(\mathcal{A})$ is not strictly dense in $\mathbb{L}(E)$.*

It is well known [3] that if $\mathcal{A}$ is a C^*–algebra acting irreducibly on a Hilbert space H which has non-zero intersection with $\mathbb{K}(H)$, the set of all compact operators on H, then $\mathcal{A}$ must contain $\mathbb{K}(H)$. This result can be generalized to Hilbert module case. First we need do some preparations. By the polarisation identity

$$\theta_{x,y} = \frac{1}{4}\sum_{j=0}^{3} i^j(\theta_{x+i^j y, x+i^j y}), \quad x, y \in E \text{ (where } i^2 = -1),$$

the following lemma is clear.

Lemma 2.13. *Suppose that E is a Hilbert $\mathcal{B}$–module. If $T \in \mathbb{K}(E)_{sa}$, then for any $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_m \in E$ such that*

$$\left\| T - [\sum_{i=1}^{n} \theta_{x_i,x_i} - \sum_{j=1}^{m} \theta_{y_j,y_j}] \right\| < \varepsilon.$$

Theorem 2.14. *Let E be a Hilbert $\mathcal{B}$–module. If $\mathcal{A}$ is a strictly dense C^*–subalgebra of $\mathbb{L}(E)$ which has non-zero intersection with $\mathbb{K}(E)$, then $\mathcal{A}$ must contain $\mathbb{K}(E)$.*

Proof. Set $I = \mathcal{A} \cap \mathbb{K}(E)$ and let $\{u_\lambda\}$ be an approximate unit of I. Since I is an ideal of $\mathcal{A}$, by the assumption $\mathcal{A}$ acts topologically irreducibly on E, $\overline{I(E)}$ must be a non-zero closed $\mathcal{A}$–invariant submodule of E. Thus, $\overline{I(E)} = E$, which implies $\{u_\lambda\}$ is also an approximate unit of $\mathbb{K}(E)$.

Set $T \in \mathbb{K}(E)_+$ and assume $\|T\| = 1$. For $\varepsilon > 0$, we have shown that there exists $u \in I$ with $\|u\| \leq 1$ such that $\|T - T \cdot u\| < \frac{1}{5}\varepsilon$. By Lemma 2.13, we have $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m \in E$ with

$$\Big\|u - [\sum_{i=1}^{n} \theta_{x_i,x_i} - \sum_{j=1}^{m} \theta_{y_j,y_j}]\Big\| < \frac{1}{5}\varepsilon.$$

Set $v = \sum_{i=1}^{n} \theta_{x_i,x_i} - \sum_{j=1}^{m} \theta_{y_j,y_j}$. By [2, Proposition 1.4], there is $a \in \mathcal{A}$ with $\|a\| \leq 1$ such that

$$\|(T-a)(x_i)\| < \frac{\varepsilon}{5n\|x_i\|}, \quad i = 1, ..., n; \ \|(T-a)(y_j)\| < \frac{\varepsilon}{5m\|y_j\|}, \quad j = 1, ..., m.$$

Hence

$$\begin{aligned} \|(T-a)\cdot\theta_{x_i,x_i}\| &= \|\theta_{(T-a)(x_i),x_i}\| \\ &\leq \|(T-a)(x_i)\| \cdot \|x_i\| \\ &\leq \frac{\varepsilon}{5n}, \quad i = 1, ..., n. \end{aligned}$$

Similarly, we also have $\|(T-a)\cdot\theta_{y_j,y_j}\| \leq \frac{\varepsilon}{5m}, \quad j = 1, ..., m$. By those two inequalities, we get $\|T\cdot v - a\cdot v\| < \frac{2}{5}\varepsilon$. Finally,

$$\begin{aligned} \|T - a\cdot u\| &\leq \|T - T\cdot u\| + \|T\cdot u - T\cdot v\| + \|T\cdot v - a\cdot v\| + \|a\cdot v - a\cdot u\| \\ &\leq \frac{1}{5}\varepsilon + \frac{1}{5}\|T\|\cdot\varepsilon + \frac{2}{5}\varepsilon + \frac{1}{5}\|a\|\cdot\varepsilon \\ &\leq \varepsilon. \end{aligned}$$

Notice I is closed and $a \cdot u \in I$, so $T \in I$. Consequently $\mathbb{K}(E) \subseteq I$. □

Remark 2.15. *Notice that Theorem 2.14 will fail if we remove the condition $\mathcal{B}$ is simple. Here is an example. Let (H, π) be a non-zero irreducible representation of O_n. The C^*–algebra $\pi(O_n)$ is strictly dense in $\mathbb{L}(H)$ and $\pi(O_n) \cap \mathbb{K}(H) = 0$. Let $\mathcal{B} = \mathbb{C} \oplus \mathbb{C}$. Set*

$$E = H \oplus \mathbb{C} = \{(h, \lambda) | h \in H, \lambda \in \mathbb{C}\},$$

which is a Hilbert $\mathcal{B}$–module with the $\mathcal{B}$–valued inner product

$$< (h_1, \lambda_1), \ (h_2, \lambda_2) >_{\mathcal{B}} := < h_1, \ h_2 >_H \oplus \overline{\lambda_1}\lambda_2, \ (h_1, \lambda_1), (h_2, \lambda_2) \in E$$

and the operation

$$(h, \lambda) \cdot (\alpha_1, \alpha_2) := (h\alpha_1, \lambda\alpha_2), (h, \lambda) \in E, (\alpha_1, \alpha_2) \in \mathcal{B}.$$

It is easy to see $\mathbb{K}(E) = \mathbb{K}(H) \oplus \mathbb{C}$ (notice that $\theta_{(h_1,\lambda_1),(h_2,\lambda_2)} = \theta_{h_1,h_2} \oplus \theta_{\lambda_1,\lambda_2}$) and $\mathbb{L}(E) = \mathbb{L}(H) \oplus \mathbb{C}$. Now set $\mathcal{A} = \pi(O_n) \oplus \mathbb{C} \subseteq \mathbb{L}(E)$. It is clear that $\mathcal{A}$ is strictly dense in $\mathbb{L}(E)$ and $\mathcal{A} \cap \mathbb{K}(E) \neq 0$. However $\mathcal{A}$ does not contain $\mathbb{K}(E)$.

Remark 2.16. *Example 2.12 also shows only (SD) can make Theorem 2.14 work.*

3. Hilbert $\mathbb{K}$–modules

Throughout this section, we put $\mathbb{K}$ to be the set of all compact operators on an infinite–dimensional (not necessarily separable) Hilbert space.

Except the strict topology, there exists another useful topology on $\mathbb{L}(E)$, which we call the *ultra–strict topology*. The ultra–strict topology on $\mathbb{L}(E)$ is defined to be the topology given by the seminorms

$$t \mapsto \|tk\|, \quad t \mapsto \|t^*k\|, \quad (t \in \mathbb{L}(E),\ k \in \mathbb{K}(E)).$$

In [2], Lance showed there is some difference between the strict topology and the ultra–strict topology. Fortunately, by [2, Proposition 8.1] the next two conditions are equivalent for any C^*–subalgebra $\mathcal{A}$ of $\mathbb{L}(E)$:

(1) $\mathcal{A}$ is strictly dense in $\mathbb{L}(E)$;

(2) $\mathcal{A}$ is ultra–strictly dense in $\mathbb{L}(E)$.

Let $\mathcal{A}_1, \mathcal{A}_2$ be two C^*–algebras. We write $\mathcal{A}_1 \sim_M \mathcal{A}_2$, and call $\mathcal{A}_1$ and $\mathcal{A}_2$ are *Morita equivalent* [4] if there exists a full Hilbert $\mathcal{A}_2$–module E with $\mathcal{A}_1 \cong \mathbb{K}(E)$. Morita equivalence is an equivalence relation and clearly $\mathbb{K} \sim_M \mathbb{C}$.

Proposition 3.1. *Let E be a Hilbert $\mathbb{K}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. If $\mathcal{A}$ has minimal commutant, then $\mathcal{A}$ must be strictly dense in $\mathbb{L}(E)$.*

Proof. First notice that $\mathbb{K}(E)$ and $\mathbb{C}$ are Morita equivalent, so there exists a Hilbert space H such that $\mathbb{K}(E) \cong \mathbb{K}(H)$. Denote by ϕ the isomorphism from $\mathbb{K}(E)$ to $\mathbb{K}(H)$. By this isomorphism we also have $\mathbb{L}(E) \cong \mathbb{L}(H)$. For convenience, we still put ϕ to be the isomorphism. $\mathcal{A}$ has minimal commutant, so does $\phi(\mathcal{A})$. Then $\phi(\mathcal{A})$ is ultra–strongly * dense in $\mathbb{L}(H)$. Namely for any $T \in \mathbb{L}(H)$, there exists a net $\{a_\lambda\} \subseteq \phi(\mathcal{A})$ such that

$$\|(a_\lambda - T)k\| \to 0, \|(a_\lambda^* - T^*)k\| \to 0, \forall k \in \mathbb{K}(H).$$

By ϕ^{-1}, we get that for any $S \in \mathbb{L}(E)$, there exists a net $\{b_\lambda\} \subseteq \mathcal{A}$ such that

$$\|(b_\lambda - S)k\| \to 0, \|(b_\lambda^* - S^*)k\| \to 0, \forall k \in \mathbb{K}(E).$$

Equivalently $\mathcal{A}$ is ultra–strictly dense in $\mathbb{L}(E)$. $\square$

The following two conclusions are easy and useful.

Lemma 3.2. *Let E be a Hilbert $\mathcal{B}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. The next two conditions are equivalent:*

(1) $\mathcal{A}$ acts topologically irreducibly on E;

(2) $\overline{\mathcal{A}T\mathbb{K}(E)} := \overline{span\{aTk | a \in \mathcal{A}, k \in \mathbb{K}(E)\}} = \mathbb{K}(E),\ \forall T \in \mathbb{L}(E) \setminus \{0\}$.

Proof. If Γ_1 and Γ_2 are two sets of E, we denote by $\theta_{\Gamma_1,\Gamma_2}$ the set $span\{\theta_{x_1,x_2} | x_1 \in \Gamma_1, x_2 \in \Gamma_2\}$. Particularly $\theta_{E,E} = \mathbb{F}(E)$.

(1) $\Rightarrow$ (2). Set $T \in \mathbb{L}(E) \setminus \{0\}$. There is a $x \in E$ such that $T(x) \neq 0$. Since $\mathcal{A}$ acts topologically irreducibly on E, we have $\overline{\mathcal{A}[T(x)\mathcal{B}]} = E$, which follows

$$\overline{\mathcal{A}T\theta_{x,E}} = \overline{\mathcal{A}T\theta_{x,E\mathcal{B}}} = \overline{\mathcal{A}T\theta_{(x)\mathcal{B},E}} = \overline{\theta_{\mathcal{A}[T(x)\mathcal{B}],E}} = \overline{\theta_{E,E}} = \mathbb{K}(E).$$

Thus $\overline{\mathcal{A}T\mathbb{K}(E)} = \mathbb{K}(E)$.

(2) $\Rightarrow$ (1). Let x and y be two non-zero elements of E. By [4, Proposition 2.31], there exists a unique $e \in E$ such that $y = \theta_{e,e}(e)$. Since $\overline{\mathcal{A}\theta_{x,x}\mathbb{F}(E)} = \overline{\mathcal{A}\theta_{x,x}\mathbb{K}(E)} = \mathbb{K}(E)$, $\theta_{e,e}$ can be infinitely approximated by the elements of the form $\sum_{i=1}^n a_i\theta_{x,x}\theta_{z_i,z_i'}$, where $a_i \in \mathcal{A}, and\ z_i, z_i' \in E,\ i = 1,\ ...,\ n$. Hence y can be approximated by the elements of the form $\sum_{i=1}^n a_i(x < x, z_i >< z_i', e >)$. This proves that $y \in \overline{\mathcal{A}(x\mathcal{B})}$. $\square$

Lemma 3.3. *Let E be a Hilbert $\mathcal{B}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. The next two conditions are equivalent:*

(1) $\mathcal{A}$ acts essentially irreducibly on E;
(2) $\{A \in \mathbb{L}(E) | A^*B = 0, \forall B \in \mathcal{A}T\} = 0,\ \forall T \in \mathbb{L}(E) \setminus \{0\}$.

Proof. (1) ⇒ (2). For convenience, put $(\mathcal{A}T)^\perp = \{A \in \mathbb{L}(E) | A^*B = 0, \forall B \in \mathcal{A}T\}$. Set $T \in \mathbb{L}(E) \setminus \{0\}$. There is a $x \in E$ such that $T(x) \neq 0$. If $T' \in (\mathcal{A}T)^\perp$, the rang of T' must be contained in $[\mathcal{A}T(x)]^\perp$. Since $\mathcal{A}$ acts essentially irreducibly on E, T' has to be 0.

(2) ⇒ (1). It is easy to see that if $x \in E \setminus \{0\}$ and $y \in [\mathcal{A}(x)]^\perp$, then $\theta_{y,y}\mathcal{A}\theta_{x,x} = 0$. By assumption, $\theta_{y,y} = 0$, so $y = 0$. □

Proposition 3.4. *Let E be a Hilbert $\mathcal{B}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. If the following conditions are equivalent, then $\mathcal{B} \sim_M \mathbb{C}$.*

(1) $\mathcal{A}$ acts topologically irreducibly on E;
(2) $\mathcal{A}$ acts essentially irreducibly on E.

Proof. It suffices to show $\mathbb{K}(E) \sim_M \mathbb{C}$. By the Rieffel Correspondence ([4]), $\mathbb{K}(E)$ is simple. Let (H, π) be a non-zero irreducible representation of $\mathbb{K}(E)$. Since $\mathbb{L}(E)$ is the multiplier algebra of $\mathbb{K}(E)$, π can be extended to a faithful irreducible representation of $\mathbb{L}(E)$ into $\mathbb{L}(H)$. We still put it (H, π). Set $e \in H$ with $\|e\| = 1$ and p the projection from H onto $span\{e\}$.

If there exists $q \in \mathbb{L}(E)$ such that $\pi(q) = p$, then clearly q is a minimal projection in $\mathbb{L}(E)$, i.e. $q\mathbb{L}(E)q = \mathbb{C}q$. Thus

$$0 \neq q\mathbb{K}(E)q \subseteq q\mathbb{L}(E)q = \mathbb{C}q,$$

which follows $q \in \mathbb{K}(E)$ and $\pi[\mathbb{K}(E)]$ contains a compact operator. By [3, Theorem 2.4.9], $\pi[\mathbb{K}(E)] \supseteq \mathbb{K}(H)$. Since those two algebras are simple, $\mathbb{K}(E)$ is * isomorphic to $\mathbb{K}(H)$ through π.

Now suppose that p is not in $\pi[\mathbb{L}(E)]$. We claim that for any $T \in \mathbb{K}(E) \setminus \{0\}$, there must exist some element y of $(1-p)H$ with $\pi(T)(y) \neq 0$. In fact if $T \in \mathbb{K}(E) \setminus \{0\}$ and $\pi(T)[(1-p)H] = 0$, then $\pi(T^*T)$ has to be $\|T\|^2 \cdot p$, which will cause $p \in \pi[\mathbb{L}(E)]$.

Set Δ be the set $\{a \in \mathbb{L}(E)_+ | \pi(a) \leq 1 - p\}$ and let $\mathcal{A}$ be the hereditary C^*–subalgebra of $\mathbb{L}(E)$ generated by Δ. By Kadison's theorem ([3, Theorem 5.2.2]), for each $x \in (1-p)H \setminus \{0\}$, there exists $a \in \mathbb{L}(E)_+$ such that $\pi(a)(e) = 0$ and $\pi(a)(x) = x$, which follows Δ is non-zero and $\pi(\mathcal{A})H = (1-p)H$. Moreover by [3, Theorem 5.5.2], $\pi(\mathcal{A})$ acts irreducible on $(1-p)H$. Let $T \in \mathbb{K}(E) \setminus \{0\}$, there exists $y \in (1-p)H$ such that $z = \pi(T)(y) \neq 0$. Then $\pi(\mathcal{A})z = (1-p)H$. Now if $T' \in (\mathcal{A}T)^\perp$, then $\pi(T')(1-p)H = 0$, which means T' has to be 0. Thanks to Lemma 3.3, we have shown $\mathcal{A}$ acts essentially irreducibly on E. By hypothesis, $\mathcal{A}$ acts topologically irreducibly on E. As a result of Lemma 3.2, $\overline{\mathcal{A}\mathbb{K}(E)} = \mathbb{K}(E)$. Then

$$(1-p)H = \overline{\pi(\mathcal{A})H} = \overline{\pi(\mathcal{A})\pi[\mathbb{K}(E)]H} = \overline{\pi[\mathcal{A}\mathbb{K}(E)]H} = \overline{\pi[\mathbb{K}(E)]H} = H,$$

which is a contradiction. So p must be contained in $\pi[\mathbb{L}(E)]$ and thus $\mathcal{B} \sim_M \mathbb{C}$. □

Combining Proposition 3.1 and Proposition 3.4, finally we get a characterization of Hilbert modules over compact operators.

Theorem 3.5. *Let E be a Hilbert $\mathcal{B}$–module and $\mathcal{A}$ be a C^*–subalgebra of $\mathbb{L}(E)$. The following conditions are all equivalent if and only if $\mathcal{B} \sim_M \mathbb{C}$.*

(1) $\mathcal{A}$ is strictly dense in $\mathbb{L}(E)$;
(2) $\mathcal{A}$ acts topologically irreducibly on E;
(2) $\mathcal{A}$ acts essentially irreducibly on E;
(4) $\mathcal{A}$ has minimal commutant.

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