# Conjugate Connections with Respect to a Quadratic Endomorphism and Duality

## Cornelia-Livia Bejan[a,b], Mircea Crasmareanu[c]

### *Dedicated to the memory of Academician Mileva Prvanović (1929-2016)*

[a]*Seminarul Matematic "Al. Myller", University "Al. I. Cuza", Iaşi, 700506, Romania*
[b]*Technical University Gh. Asachi Iasi, Romania*
[c]*Faculty of Mathematics, University "Al. I.Cuza", Iaşi, 700506, Romania, http://www.math.uaic.ro/~mcrasm*

**Abstract.** The goal of this paper is to consider the notion of conjugate connection in a unifying setting for both almost complex and almost product geometries, having as model the works of Mileva Prvanović. A main interest is in finding classes of conjugate connections in duality with the initial linear connection; for example in the exponential case of almost complex geometry we arrive at a rule of quantization.

## 1. Introduction and Motivation

Fix $M$ a smooth, $n$-dimensional manifold for which we denote by $C^\infty(M)$–the algebra of smooth real functions on $M$, $\mathfrak{X}(M)$–the Lie algebra of vector fields on $M$, $T^r_s(M)$–the $C^\infty(M)$-module of tensor fields of $(r, s)$-type on $M$. Usually $X, Y, Z, ...$ will be vector fields on $M$ and if $T \to M$ is a vector bundle over $M$, then $\Gamma(T)$ denotes the $C^\infty(M)$-module of sections of $T$; e.g. $\Gamma(TM) = \mathfrak{X}(M)$.

Let $C(M)$ be the set of linear connections on $M$. Since the difference of two linear connections is a tensor field of $(1, 2)$-type, it results that $C(M)$ is a $C^\infty(M)$-affine module associated to the $C^\infty(M)$-linear module $T^1_2(M)$.

Fix now $F$ an endomorphism of the tangent bundle i.e. $F \in \mathcal{T}^1_1(M)$; then the associated linear connections and a special class of such $F$-structures are provided by:

**Definition 1.1.** *i)* $\nabla \in C(M)$ *is an F-connection if F is covariant constant with respect to* $\nabla$, *namely* $\nabla F = 0$. *Let* $C_F(M)$ *be the set of these connections.*
*ii)([7]) F is a quadratic endomorphism if there exists* $\varepsilon \in \{-1, +1\}$ *such that:* $F^2 = \varepsilon I$.

In order to find the above set of connections for a fixed $F$ let us consider according [8, p. 105] the maps:

$$\psi_F : C(M) \to C(M), \quad \chi_F : T^1_2(M) \to T^1_2(M) \tag{1}$$

given by:

$$\psi_F(\nabla) := \frac{1}{2}(\nabla + \varepsilon F \circ \nabla \circ F), \quad \chi_F(\tau) := \frac{1}{2}(\tau + \varepsilon F \circ \tau \circ F). \tag{2}$$

So:

$$\psi_F(\nabla)_X Y = \frac{1}{2}[\nabla_X Y + \varepsilon F(\nabla_X FY)], \quad \chi_F(X,Y) = \frac{1}{2}[\tau(X,Y) + \varepsilon F(\tau(X,FY))]. \tag{3}$$

Then, $\psi_F$ is a $C^\infty(M)$-projector on $C(M)$ associated to the $C^\infty(M)$-linear projector $\chi_F$:

$$\psi_F^2 = \psi_F, \quad \chi_F^2 = \chi_F, \quad \psi_F(\nabla + \tau) = \psi_F(\nabla) + \chi_F(\tau). \tag{4}$$

It follows that $\nabla F = 0$ means $\psi_F(\nabla) = \nabla$ which gives that $C_F(M) = Im\psi_F$. This determines completely $C_F(M)$. Fix $\nabla_0$ arbitrary in $C(M)$ and $\nabla$ in $C_F(M)$. So, $\nabla = \psi_F(\nabla')$ with $\nabla' = \nabla_0 + \tau$. In conclusion, $\nabla = \psi_F(\nabla_0) + \chi_F(\tau)$; in other words, $C_F(M)$ is the affine submodule of $C(M)$ passing through the $F$-connection $\psi_F(\nabla_0)$ and having the direction given by the linear submodule $Im\chi_F$ of $T_2^1(M)$.

Since the projector $\psi_F$ is a main tool in finding $C_F(M)$, a careful study of it is necessary. Let us remark a decomposition (of arithmetic mean type) of it [8, p. 106]:

$$\psi_F(\nabla) = \frac{1}{2}(\nabla + C_F(\nabla)) \tag{5}$$

with *the conjugation map $C_F : C(M) \to C(M)$*:

$$C_F(\nabla)_X = \varepsilon F \circ \nabla_X \circ F. \tag{6}$$

Then *the conjugate connection $C_F(\nabla)$* measures how far the connection $\nabla$ is from being an $F$-connection and as it is pointed out in [8, p. 105], $C_F$ is the affine symmetry of the affine module $C(M)$ with respect to the affine submodule $C_F(M)$, made parallel with the linear submodule $\ker \chi_F$.

The present paper is devoted to a carefully study of this connection $C_F(\nabla)$ since all the above computations put in evidence its rôle in the geometry of $F$; recently there are important studies regarding almost complex and almost product geometries together, e.g. [5]. More precisely, the aim of our study is to obtain several properties of it in both the general case and Riemannian geometry. The second section is devoted to this scope and after a general result connecting $\nabla$ and $C_F(\nabla)$, we treat two items:
a) the behavior of the conjugate connection to a linear change of $F$-structures,
b) the use of two tensor fields previously considered in the almost complex geometry.
With respect to a), we arrive at two particular remarkable cases concerning the recurrence of the given quadratic endomorphism $F$ while for b) we derive some useful new identities. Let us pointed out that we follow a similar study for: i) the almost complex geometry in [2]; ii) the almost product geometry from [3]; iii) the (almost) tangent geometry from [4].

In the third section we give some generalizations of the results from the first part by adding an arbitrary tensor field of $(1,2)$-type. All generalized conjugate connections which form a duality with the initial linear connection are determined. The last section is devoted to the exponential conjugate connection, an object introduced in correspondence with a similar one from [1].

We finish this Introduction with the remark that a very interesting paper on a similar subject is [14]. We perform a new study at least from one reason: all our relations are written globally. We meet the distinguished Academician Mileva Prvanović at several international conferences and we dedicate her this work with great honor!

## 2. Properties of the Conjugate Connection

In what follows, for simplicity, we will denote by the superscript $F$ the complex conjugate connection of $\nabla$:

$$\nabla^{(F)} := C_F(\nabla) = \nabla + \varepsilon F \circ \nabla F \tag{7}$$

and then:

$$\nabla_X^{(F)}Y = \nabla_X Y + \varepsilon F(\nabla_X FY - F(\nabla_X Y)) = \varepsilon F(\nabla_X FY). \tag{8}$$

The first properties of the conjugate connection are stated in the next proposition:

**Proposition 2.1.** $\nabla^{(F)}$ *satisfies:*

1. $\nabla^{(F)}F = -\nabla F$; *it results that* $\nabla \in C_F(M)$ *if and only if* $\nabla^{(F)} \in C_F(M)$;
2. $\nabla$ *and* $\nabla^{(F)}$ *are in duality:* $(\nabla^{(F)})^{(F)} = \nabla$;
3. $T_{\nabla^{(F)}} = T_\nabla + \varepsilon F(d^\nabla F)$, *where* $d^\nabla$ *is the exterior covariant derivative induced by* $\nabla$, *namely* $(d^\nabla F)(X, Y) := (\nabla_X F)Y - (\nabla_Y F)X$; *it results that for* $\nabla \in C_F(M)$, *the connections* $\nabla$ *and* $\nabla^{(F)}$ *have the same torsion;*
4. $R_{\nabla^{(F)}}(X, Y, Z) = \varepsilon F(R_\nabla(X, Y, FZ))$; *it results that* $\nabla$ *is flat if and only if* $\nabla^{(F)}$ *is so;*
5. *Assume that* $(M, g, F)$ *is an almost* $\varepsilon$-*Hermitian manifold i.e.* $g(FX, FY) = g(X, Y)$, *[13]-[15]; then* $(\nabla_X^{(F)}g)(FY, FZ) = (\nabla_X g)(Y, Z)$. *It results that* $\nabla$ *is a* $g$-*metric connection if and only if* $\nabla^{(F)}$ *is so.*

*Proof.* 1. Other relations we shall use are:

$$\nabla_X^{(F)}FY = F(\nabla_X Y), \quad F(\nabla_X^{(F)}Y) = \nabla_X FY \tag{9}$$

and then:

$$(\nabla_X F)Y = \nabla_X FY - F(\nabla_X Y) = F(\nabla_X^{(F)}Y) - \nabla_X^{(F)}FY = -(\nabla_X^{(F)}F)Y. \tag{10}$$

2. Although a direct proof can be provided by the formula (6), we prefer to give a proof here, in order to use (7):

$$(\nabla^{(F)})^{(F)} = \nabla^{(F)} + \varepsilon F \circ \nabla^{(F)}F = \nabla + \varepsilon F \circ \nabla F + \varepsilon F \circ (-\nabla F) = \nabla. \tag{11}$$

3. A direct computation gives:

$$T_{\nabla^{(F)}}(X, Y) = \nabla_X^{(F)}Y - \nabla_Y^{(F)}X - [X, Y] = \varepsilon F(\nabla_X FY) - \varepsilon F(\nabla_Y FX) - [X, Y] = \tag{12}$$

$$= \varepsilon F(\nabla_X FY - \nabla_Y FX) + T_\nabla(X, Y) - \nabla_X Y + \nabla_Y X = T_\nabla(X, Y) + \varepsilon F((\nabla_X F)Y - (\nabla_Y F)X). \tag{13}$$

4.

$$R_{\nabla^{(F)}}(X, Y, Z) = \nabla_X^{(F)}\nabla_Y^{(F)}Z - \nabla_Y^{(F)}\nabla_X^{(F)}Z - \nabla_{[X,Y]}^{(F)}Z = \tag{14}$$

$$= \varepsilon\left(\nabla_X^{(F)}F(\nabla_Y FZ) - \nabla_Y^{(F)}F(\nabla_X FZ) - F(\nabla_{[X,Y]}FZ)\right) = \tag{15}$$

$$= \varepsilon F\left(\nabla_X \nabla_Y FZ - \nabla_Y \nabla_X FZ - \nabla_{[X,Y]}FZ\right) = \varepsilon F(R_\nabla(X, Y, FZ)). \tag{16}$$

5.

$$(\nabla_X^{(F)}g)(V, W) = X(g(V, W)) - g(\nabla_X^{(F)}V, W) - g(V, \nabla_X^{(F)}W) = \tag{17}$$

$$= X(g(V, W)) - \varepsilon g(F(\nabla_X FV), W) - \varepsilon g(V, F(\nabla_X FW)) \tag{18}$$

for any $X, V$ and $W \in \mathfrak{X}(M)$. With $V := FY$ and $W := FZ$ we get:

$$(\nabla_X^{(F)}g)(FY, FZ) = X(g(FY, FZ)) - g(F(\nabla_X Y), FZ) - g(FY, F(\nabla_X Z)) = \tag{19}$$

$$= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = (\nabla_X g)(Y, Z) \tag{20}$$

which completes the proof. $\square$

There are some direct consequences of these formulae:
i) If the pair $(\nabla, F)$ is *special* i.e. $(\nabla_X F)Y = (\nabla_Y F)X$ (according to [1] or [16, p. 1003]), then $d^\nabla F = 0$ and again, the connections $\nabla$ and $\nabla^{(F)}$ have the same torsion. If $(M, F, \nabla)$ is $\varepsilon$-*nearly Kähler*, which means $(\nabla_X F)Y + (\nabla_Y F)X = 0$ (see [16, p. 1003]), then $F(d^\nabla F) = 2\varepsilon(\nabla^{(F)} - \nabla)$.
ii) If $\nabla$ is the Levi-Civita connection of $g$ (see [12]), then $\nabla^{(F)}$ is also metric with respect to $g$.
iii) If $\nabla$ is the Levi-Civita connection of $g$ and in addition, $\nabla \in C_F(M)$, then $\nabla^{(F)} = \nabla$ as the unique symmetric $g$-metric connection.

More generally, let $f \in Diff(M)$ be an automorphism of the $G$-structure defined by $J$ i.e. $f_* \circ F = F \circ f_*$. If $f$ is an affine transformation for $\nabla$, namely $f_*(\nabla_X Y) = \nabla_{f_*X} f_* Y$, then $f$ is also an affine transformation for $\nabla^{(F)}$.

Let also recall that in a Hermitian geometry various choices of nice connections are obtained by requiring additional less stringent conditions on the torsion; for example, the Chern and Bismut connections are discussed in details in [9]. Two natural generalizations of the case $\nabla \in C_F(M)$ are given in our framework by:

**Proposition 2.2.** *Let $\nabla$ be a symmetric linear connection.*
*i) Assume that $F$ is $\nabla$-recurrent i.e. $\nabla F = \eta \otimes F$, where $\eta$ is a 1-form. Then $\nabla^{(F)}$ is a semi-symmetric connection.*
*ii) Assume that $\nabla F = \varepsilon \eta \otimes I_{\mathfrak{X}(M)}$. Then $\nabla^{(F)}$ is a quarter-symmetric connection.*

*Proof.* Recall that a non-torsionfree linear connection is called:
-*semi-symmetric* if there exists a 1-form $\pi$ such that its torsion is, [6]:

$$T(X, Y) = \pi(Y)X - \pi(X)Y, \tag{21}$$

-*quarter-symmetric* if in addition there exists a tensor field $F$ of $(1, 1)$-type such that, [2, p. 122]:

$$T(X, Y) = \pi(Y)FX - \pi(X)FY. \tag{22}$$

i) We have $\nabla^{(F)} = \nabla + \eta \otimes I$ and from the item 3 of previous Proposition we get $T_{\nabla^{(F)}} = \eta \otimes I - I \otimes \eta = \eta \wedge I$.
ii) It results that $\nabla^{(F)} = \nabla + \eta \otimes F$ and, as above, we get $T_{\nabla^{(F)}} = \eta \otimes F - F \otimes \eta = \eta \wedge F$. $\square$

The next subject consists of the behavior of $\nabla^{(\cdot)}$ for families of quadratic endomorphisms. Let $F_1$ and $F_2$ be corresponding to the same $\varepsilon$ and consider the pencil of $(1, 1)$-tensor fields $F_{\alpha,\beta} := \alpha F_1 + \beta F_2$, with $\alpha$ and $\beta \in \mathbb{R}$. In order that $F_{\alpha,\beta}$ to be a quadratic endomorphism with the same $\varepsilon$, there are necessary two conditions:
1) $F_1$ and $F_2$ be skew-commuting structures: $F_1 F_2 = -F_2 F_1$; for $\varepsilon = -1$ this condition implies that the dimension of $M$ is $n = 4m$ and the triple $(J_1, J_2, J_3 := J_1 J_2)$ is a *quaternionic structure* on $M$ conform [10];
2) $(\alpha, \beta)$ belongs to the unit circle $S^1$: $\alpha^2 + \beta^2 = 1$.
Then:

$$\nabla_X^{(F_{\alpha,\beta})} Y = \alpha^2 \nabla_X^{(F_1)} Y + \beta^2 \nabla_X^{(F_2)} Y + \varepsilon \alpha\beta [F_1(\nabla_X F_2 Y) + F_2(\nabla_X F_1 Y)] \tag{23}$$

and there are two remarkable particular cases:
i) if $F_1$ and $F_2$ are both recurrent with respect to $\nabla$ with the same 1-form of recurrence: $\nabla F_i = \eta \otimes F_i$, then the conjugate connections coincide $\nabla^{(F_1)} \equiv \nabla^{(F_2)} =: \nabla^{(F_{12})}$ and it follows the invariance of $\nabla^{(F.)}$:

$$\nabla^{(F_{\alpha,\beta})} = \nabla^{(F_{12})}; \tag{24}$$

ii) assume that the triple $(\nabla, F_1, F_2)$ is a mixed-recurrent structure: $\nabla F_i = \eta \otimes F_j$ with $i \neq j$. Then $\nabla$ is the average of the two conjugate connections, $\nabla = \frac{1}{2}(\nabla^{(F_1)} + \nabla^{(F_2)})$ and:

$$\nabla^{(F_{\alpha,\beta})} = \nabla + \varepsilon(\alpha^2 - \beta^2)\eta \otimes F_1 F_2 + 2\alpha\beta\eta \otimes I. \tag{25}$$

The last subject of this section treats two tensor fields associated to a pair (almost complex structure, linear connection) in [11]:
1) *the structural tensor field*:

$$C_\nabla^F(X, Y) := \frac{1}{2}[(\nabla_{FX}F)Y + (\nabla_X F)FY] \tag{26}$$

2) *the virtual tensor field*:

$$B_\nabla^F(X, Y) := \frac{1}{2}[(\nabla_{FX}F)Y - (\nabla_X F)FY]. \tag{27}$$

From the item 1 of the first Proposition it results that both these tensor fields are skew-symmetric with respect to the conjugation of connections:

$$C_{\nabla^{(F)}}^F = -C_\nabla^F, \quad B_{\nabla^{(F)}}^F = -B_\nabla^F. \tag{28}$$

Also:

$$C_\nabla^F(FX, FY) = \varepsilon C_\nabla^F(X, Y), \quad B_\nabla^F(FX, FY) = \varepsilon B_\nabla^F(X, Y). \tag{29}$$

The importance of these tensor fields for our study is given by the following straightforward relation:

$$\nabla^{(F)} = \nabla + \varepsilon(C_\nabla^F - B_\nabla^F). \tag{30}$$

Recall after [2] or [3] that two linear connections are called *projectively equivalent* if there exists a 1-form $\tau$ such that:

$$\nabla' = \nabla + \tau \otimes I + I \otimes \tau. \tag{31}$$

A straightforward calculation gives that $C^F$ is invariant for projectively changes (31), while for $B$ we have:

$$(B_{\nabla'}^F - B_\nabla^F)(X, Y) = \tau(FY)FX - \varepsilon\tau(Y)X. \tag{32}$$

Unfortunately, the conjugation of connections is not invariant under projectively equivalence since:

$$(\nabla')^{(F)} = \nabla^{(F)} + \tau \otimes I + \varepsilon F \otimes (\tau \circ F). \tag{33}$$

## 3. Generalized Conjugate Connections and Duality

In this section we present a natural generalization of the complex conjugate connection.

**Definition 3.1.** *A generalized complex conjugate connection of $\nabla$ is:*

$$\nabla^{(F,C)} := \nabla^{(F)} + C \tag{34}$$

*with $C \in T_2^1(M)$ arbitrary.*

Since the duality $\nabla \leftrightarrow \nabla^{(F)}$ is a main feature of $\nabla^{(F)}$, let us search for tensor fields $C$ such that $(\nabla^{(F,C)})^{(F,C)} = \nabla$. From:

$$(\nabla^{(F,C)})_X^{(F,C)}Y = \nabla_X Y + \varepsilon F(C(X, FY)) + C(X, Y), \tag{35}$$

it results that we are interested in finding solutions $C$ to:

$$F(C(X, FY)) = -\varepsilon C(X, Y). \tag{36}$$

Let us remark that:
i) $C_0 = -\varepsilon \nabla F$ is a particular solution of (36),
ii) if $C$ is a solution, then $F \circ C$ is again a solution.
So, let us search the duality property for:

$$\nabla^{(F,\lambda,\mu)} = \nabla^{(F)} + \lambda \nabla F + \mu F(\nabla F) \tag{37}$$

with $\lambda, \mu \in \mathbb{R}$.

**Proposition 3.2.** *The duality* $\nabla \leftrightarrow \nabla^{(F,\lambda,\mu)}$ *holds only for the pairs:*
*i)* $(\lambda, \mu) \in \{(0,0), (0,-\varepsilon)\}$;
*ii) in addition to the case* $\varepsilon = +1$ *for:* $(\lambda, \mu) = (\pm\frac{1}{2}, -\frac{1}{2})$.

*Proof.* From:

$$(\nabla^{(F,\lambda,\mu)})_X^{(F,\lambda,\mu)} Y = [(1 + \varepsilon\mu)^2 + \mu^2 + 2\varepsilon\lambda^2]\nabla_X Y + \tag{38}$$

$$+2\lambda(2\varepsilon\mu + 1)F(\nabla_X Y) - 2\lambda(2\varepsilon\mu + 1)\nabla_X FY - 2[\mu(1 + \varepsilon\mu) + \lambda^2]F(\nabla_X FY) \tag{39}$$

we obtain the system:

$$\begin{cases} (1 + \varepsilon\mu)^2 + \mu^2 + 2\varepsilon\lambda^2 = 1 \\ \lambda(1 + 2\varepsilon\mu) = 0 \\ \mu(1 + \varepsilon\mu) + \lambda^2 = 0. \end{cases} \tag{40}$$

From the second equation it results two cases:
i) $\lambda = 0$ which together with the third equation yields $\mu_1 = 0$ and $\mu_2 = -\varepsilon$. Both these solution satisfy also the first equation; ii) $\mu = -\frac{\varepsilon}{2}$ which replaced in the third equation gives $\lambda^2 = \frac{\varepsilon}{4}$. It follows $\varepsilon > 0$ which means the almost product case with $\mu = -\frac{\varepsilon}{2}$ and $\lambda = \pm\frac{1}{2}$.
Let us pointed out that:

$$\nabla^{(F,0,0)} = \nabla^{(F)}, \quad \nabla^{(F,0,-\varepsilon)} = \nabla, \quad \nabla_X^{(F,\pm 1/2,-1/2)} Y = \frac{1}{2}[\nabla_X Y \pm (\nabla_X F)Y + F(\nabla_X FY)] \tag{41}$$

which confirms our result. $\square$

Returning to the general case (34), let us present the generalizations of some relations from Proposition 2.1:
1. $\nabla^{(F,C)} F = -\nabla F + C(\cdot, F\cdot) - F \circ C$. Then $\nabla \in C_F(M)$ if and only if $\nabla^{(F,\lambda\nabla F + \mu F \circ \nabla F)} \in C_F(M)$ with $\lambda$ and $\mu$ arbitrary real numbers;
2. the discussion above;
3. $T_{\nabla^{(F,C)}} = T_\nabla + \varepsilon F(d^\nabla F) + 2C_{skew}$, where $C_{skew}$ is the skew-symmetric part of $C$ i.e. $2C_{skew}(X,Y) = C(X,Y) - C(Y,X)$. So, if $C$ is symmetric and $\nabla \in C_F(M)$, then $\nabla$ and $\nabla^{(F,C)}$ have the same torsion;
4. $R_{\nabla^{(F,C)}}(X,Y)Z = \varepsilon F(R_\nabla(X,Y)FZ) - C(X, F(\nabla_Y FZ)) + C(Y, F(\nabla_X FZ)) - C([X,Y], Z) - F(\nabla_X F(C(Y,Z))) + F(\nabla_Y F(C(X,Z))) + C(X, C(Y,Z)) - C(Y, C(X,Z))$.

**Example 3.3.** *After Theorem 1 of [16, p. 1003], in the almost complex case the tensor field* $C = -\frac{1}{2}J(\nabla J)$ *is involved in the tt\*-geometry of* $(M, J)$.

The rest of this Section is devoted to other two facts concerning the conjugate connection: firstly an iterated conjugation of $\nabla$ with respect to the given pair $(F_1, F_2)$ of previous Section is considered and secondly the constant covariance of a given endomorphism with respect to $\nabla$ versus $\nabla^{(F)}$. More precisely, we have:

**Proposition 3.4.** *Let $F_1$ and $F_2$ quadratic endomorphisms with the corresponding $\varepsilon_1$ and $\varepsilon_2$. Then:*

$$\left(\nabla^{(F_1)}\right)_X^{(F_2)} Y = \varepsilon_1 \varepsilon_2 F_2 F_1 (\nabla_X F_1 F_2 Y) \tag{42}$$

*Hence, if $F_2 F_1 = \pm F_1 F_2$ we have the symmetry of conjugation:* $\left(\nabla^{(F_1)}\right)^{(F_2)} = \left(\nabla^{(F_2)}\right)^{(F_1)}$.

For $F_1 = F_2$ we reobtain on this way the duality from Proposition 2.1. Also, it follows directly the equivalence of the following statements in the case of $\varepsilon_1 = \varepsilon_2$:

i) $\left(\nabla^{(F_1)}\right)^{(F_2)} (= \left(\nabla^{(F_2)}\right)^{(F_1)}) = \nabla$,
ii) $F_1 F_2 (= \pm F_2 F_1)$ is parallel with respect to $\nabla$.

Fix now another endomorphism $T$ not necessary of quadratic type but commuting with the given $F$. Then:

**Proposition 3.5.** *The covariant derivative of $T$ with respect to $\nabla^{(F)}$ is:*

$$\left(\nabla_X^{(F)} T\right) Y = \varepsilon F((\nabla_X T) F Y) \tag{43}$$

*and then $T$ is parallel with respect to $\nabla^{(F)}$ if and only if it is parallel with respect to $\nabla$.*

*Proof.* We have:

$$\left(\nabla_X^{(F)} T\right) Y = \nabla_X^{(F)} T Y - T(\nabla_X^{(F)} Y) = \varepsilon F(\nabla_X F T Y) - \varepsilon T F(\nabla_X F Y) = \varepsilon F(\nabla_X T F Y - T(\nabla X Y)) \tag{44}$$

which means the claimed relation. The second part follows from the the fact that $F$ is admits the inverse $F^{-1} = \varepsilon F$. □

## 4. Exponential Conjugate Connections

Let us consider the functions $(cs_\varepsilon, sn_\varepsilon)$ given by: i) $(\cos, \sin)$ for $\varepsilon = -1$, ii) $(\cosh, \sinh)$ for $\varepsilon = +1$. Hence: $cs_\varepsilon^2 - \varepsilon sn_\varepsilon^2 = 1$. Following [1] we consider:

**Definition 4.1.** *The exponential conjugate connection of $\nabla$ is:*

$$\nabla^{(F,\theta)} := \exp(-\theta F) \circ \nabla \circ \exp(\theta F) \tag{45}$$

*where:*

$$\exp(\theta F) = cs_\varepsilon \theta \cdot I + sn_\varepsilon \theta \cdot F, \quad \exp(-\theta F) = cs_\varepsilon \theta \cdot I - sn_\varepsilon \theta \cdot F \tag{46}$$

*with $\theta$ an arbitrary real number, possible on the 1-dimensional torus $S^1 = \mathbb{R}/\mathbb{Z}$.*

Our conventions in (45) is the reverse of the choice of [1] and this fact is motivated by the usual conjugation of $b \in G$ with respect to the element $a$ of a group $(G, \cdot)$ as $a^{-1} \cdot b \cdot a$. Also, in [1] the exponential complex conjugate connection is parametrized by the projective line $P^1 = S^1/\pi$. The classical conjugate $\nabla^{(F)}$ in the almost complex case corresponds to $\theta = \frac{\pi}{2}$.

**Proposition 4.2.** $\nabla^{(F,\theta)}$ *is in duality with $\nabla$ only for:*
i) *($\varepsilon = -1$) $\theta = \frac{k}{2}\pi$ with $k$ an integer;*
ii) *($\varepsilon = +1$) $\theta = 0$.*

*Proof.* Is a consequence of: $(\nabla^{(F,\theta)})^{(F,\theta)} = \nabla^{(F,2\theta)}$ which results after a straightforward computation. □

Let us remark that:
a) the almost product case means the reduction $\nabla^{(F,0)} = \nabla$,
b) the almost complex case i) above is characterized by *the quantization rule*: $\theta \in \frac{\pi}{2}\mathbb{Z}$.

# References

[1] D. V. Alekseevsky, V. Cortés, C. Devchand, Special complex manifolds, J. Geom. Phys. 42(1-2) (2002) 85–105.
[2] A. M. Blaga, M. Crasmareanu, The geometry of complex conjugate connections, Hacet. J. Math. Stat. 41(1) (2012) 119–126.
[3] A. M. Blaga, M. Crasmareanu, The geometry of product conjugate connections, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) 59(1) (2013) 73–84.
[4] A. M. Blaga, M. Crasmareanu, The geometry of tangent conjugate connections, Hacet. J. Math. Stat. 44(4) (2015) 767–774.
[5] A. Borowiec, M. Ferraris, M. Francaviglia, I. Volovich, Almost-complex and almost-product Einstein manifolds from a variational principle, J. Math. Phys. 40(7) (1999) 3446–3464.
[6] S. K. Chaubey, R. H. Ojha, On a semi-symmetric non-metric connection, Filomat 26(2) (2012) 269–275.
[7] M. Crasmareanu, C. Ida, Almost analytic forms with respect to a quadratic endomorphism and their cohomology, Turkish J. Math. 39(3) (2015) 322–334.
[8] V. Cruceanu, Almost product bicomplex structures on manifolds, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) 51(1) (2005) 99–118.
[9] P. Gauduchon, Hermitian connections and Dirac operators, Boll. Un. Mat. Ital. B (7) 11 (1997) 257–288.
[10] S. Ishihara, Quaternion Kählerian manifolds, J. Diff. Geom. 9 (1974) 483–500.
[11] V. F. Kirichenko, Method of generalized Hermitian geometry in the theory of almost contact manifold, Itogi Nauki i Tekhniki, Problems of geometry, 18, 25–71, 1986; translated in J. Soviet. Math., 42(5) (1988) 1885–1919.
[12] S. M. Minčić, Lj. S. Velimirović, M. S. Stanković, New integrability conditions of derivational equations of a submanifold in a generalized Riemannian space, Filomat 24(4) (2010) 137–146.
[13] M. Prvanović, Bianchi-type identities on almost Hermitian manifolds, Bull. Cl. Sci. Math. Nat. Sci. Math. 121(25) (2000) 71–88.
[14] M. Prvanović, Conjugate connections on manifolds with complex or product structure, in Differential geometry (Budapest, 1979), 563–582, Colloq. Math. Soc. János Bolyai, 31, North-Holland, Amsterdam-New York, 1982.
[15] M. Prvanović, The conformal change of the metric of an almost Hermitian manifold applied to the antiholomorphic curvature tensor, Commun. Math. 21(1) (2013) 77–90.
[16] L. Schäfer, $tt^*$-geometry on the tangent bundle of an almost complex manifold, J. Geom. Phys. 57(3) (2007) 999–1014.