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Applications of Soft Intersection Sets to Hemirings via *SI*-*h*-Bi-Ideals and *SI*-*h*-Quasi-Ideals

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Abstract. The aim of this paper is to lay a foundation for providing a soft algebraic tool in considering many problems that contains uncertainties. In order to provide these soft algebraic structures, we introduce the concepts of *SI*-*h*-bi-ideals and *SI*-*h*-quasi-ideals of hemirings. The relationships between these kinds of soft intersection *h*-ideals are established. Finally, some characterizations of *h*-hemiregular, *h*-intra-hemiregular and *h*-quasi-hemiregular hemirings are investigated by these kinds of soft intersection *h*-ideals.

1. Introduction

In order to model vagueness and uncertainty, Molodtsov [23] introduced soft set theory and it has received much attention since its inception. Since then, especially soft set operations, have undergone tremendous studies. Maji [20] presented some definitions or soft sets. Ali [2, 3] proposed some new operations on soft sets. Sezgin [25] also gave some operations on soft sets. Majumdar [22] investigated some soft mapping. In the same time, this theory has been proven useful in many different fields such as decision making [6, 7, 10, 12, 21], data analysis [32], forecasting and so on. Recently, the algebraic structures of soft sets have been studied increasingly, such as, soft groups [1], soft semigroups [11], soft BCK/BCI-algebras [13], soft hyperstructures [4, 28].

We note that the ideals of semirings play a crucial role in the structure theory, ideals in semirings do not in general coincide with the ideals of a ring. For this reason, the usage of ideals in semirings is somewhat limited. By a hemiring, we mean a special semiring with a zero and with a commutative addition. The properties of *h*-ideals of hemirings were thoroughly investigated by Torre [27] and by using *h*-ideals, Torre established some analogous ring theorems for hemirings. In particular, Jun [14] discussed some properties of hemirings. Zhan et al. [31] discussed *h*-hemiregular hemirings. Some characterizations of *h*-semisimple and *h*-intra-hemiregular hemirings were investigated by Yin et al. [29, 30]. Further, some generalized fuzzy *h*-ideals of hemirings were investigated by Davvaz, Dudek and Ma, for examples, see [8, 9, 15, 17, 18, 24].

Recently, Çağman and Sezgin discussed some important properties on soft intersection groups and soft intersection near-rings, see [5, 26]. By this new idea, Ma et al. [16, 19] introduced the concepts of soft

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intersection hemirings and soft intersection h-ideals(soft intersection h-interior ideals) of hemirings. By soft intersection h-ideals, Ma et al. investigated some characterizations of h-hemiregular hemirings. As a continuation of these two papers, we introduce the concepts of SI-h-bi-ideals and SI-h-quasi-ideals of hemirings. In the same time, we give some characterizations of h-hemiregular, h-intra-hemiregular and h-quasi-hemiregular hemirings.

2. Preliminaries

A semiring is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws:

$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$

are satisfied for all $a, b, c \in S$.

By zero of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S, +)$ is called a hemiring. A one unit 1 on S, we means that $1 \cdot x = x \cdot 1 = x$ for all $x \in S$. For the sake of simplicity, we shall write ab for $a \cdot b(a, b \in S)$.

A subhemiring of S is a subset A of S closed under addition and multiplication. A subset A of S is called a left(right) ideal of S if A is closed under addition and $SA \subseteq A(AS \subseteq A)$. A subset B of S is called a bi ideal of S if B is closed under addition and multiplication such that $BSB \subseteq B$. A subset Q of S is called a quasi-ideal of S if Q is closed under addition and $SQ \cap QS \subseteq Q$.

A subhemiring(left ideal, right ideal, ideal, bi-ideal) A of S is called an h-subhemiring(left h-ideal, right h-ideal, h-ideal, h-bi-ideal), respectively, if for any $x, z \in S$ and $a, b \in A$, $x + a + z = b + z \to x \in A$.

The h-closure $\overline{A}$ of a subset A of S is defined as

$$\overline{A} = \{x \in S | x + a + z = b + z \text{ for some } a, b \in A,\ z \in S\}.$$

A quasi-ideal Q of S is called an h-quasi-ideal of S if $\overline{SQ} \cap \overline{QS} \subseteq Q$ and for any $x, z \in S$ and $a, b \in Q$ from $x + a + z = b + z$, it follows $x \in Q$.

From now on, S is a hemiring, U is an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C \in E$.

Definition 2.1. [23] A soft set f_A over U is a set defined by $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}$. It is clear to see that a soft set is a parameterized family of subsets of the set U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2. [6] Let $f_A, f_B \in S(U)$. Then,

(1) *f_A is said to be a soft subset of f_B and denoted by $f_A \widetilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$, for all $x \in E$. f_A and f_B are said to be soft equal, denoted by $f_A = f_B$, if $f_A \widetilde{\subseteq} f_B$ and $f_A \widetilde{\supseteq} f_B$.*

(2) The union of f_A and f_B, denoted by $f_A \widetilde{\cup} f_B$, is defined as $f_A \widetilde{\cup} f_B = f_{A \cup B}$, where $f_{A \cup B}(x) = f_A(x) \cup f_B(x)$, for all $x \in E$;

(3) the intersection of f_A and f_B, denoted by $f_A \widetilde{\cap} f_B$, is defined as $f_A \widetilde{\cap} f_B = f_{A \cap B}$, where $f_{A \cap B}(x) = f_A(x) \cap f_B(x)$, for all $x \in E$.

Definition 2.3. [5] Let $f_A \in S(U)$ and $\alpha \subseteq U$. Then, upper α-inclusion of f_A, denoted by $U(f_A; \alpha)$, is defined as $U(f_A; \alpha) = \{x \in A | f_A(x) \supseteq \alpha\}$.

Definition 2.4. [5] Let $A \subseteq S$. We denote by $\mathcal{S}_A$ the soft characteristic function of A and define as

$$\mathcal{S}_A(x) = \begin{cases} U & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Definition 2.5. [19] Let $f_S, g_S \in S(U)$. Then,

(1) The soft union-intersection product $f_S \bigstar g_S$ is defined by

$$(f_S \bigstar g_S)(x) = \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(a'_j) \cap g_S(b_i) \cap g_S(b'_j))$$

for all $a_i, a'_j, b_i, b'_j, x, z \in S$, $i = 1, 2, ..., m; j = 1, 2, ..., n.$
and $(f_S \bigstar g_S)(x) = \emptyset$ if x cannot be expressed as $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z;$

(2) The soft union-intersection sum $f_S \boxplus g_S$ is defined by

$$(f_S \boxplus g_S)(x) = \bigcup_{x+a_1+b_1+z=a_2+b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2))$$

for all $a_1, a_2, b_1, b_2, x, z \in S$,$i = 1, 2, ..., m; j = 1, 2, ..., n.$
and $(f_S \boxplus g_S)(x) = \emptyset$ if x cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z.$

The following proposition is obvious.

Proposition 2.6. [19] Let $A, B \subseteq S$. Then,
(1) $A \subseteq B \Rightarrow \mathcal{S}_A \widetilde{\subseteq} \mathcal{S}_B$,
(2) $\mathcal{S}_A \widetilde{\cap} \mathcal{S}_B = \mathcal{S}_{A\cap B}$,
(3) $\mathcal{S}_A \bigstar \mathcal{S}_B = \mathcal{S}_{\overline{AB}}$,
(4) $\mathcal{S}_A \boxplus \mathcal{S}_B = \mathcal{S}_{\overline{A+B}}$.

Definition 2.7. [19]

(1) A soft set f_S over U is called a soft intersection hemiring (briefly, SI-hemiring) if it satisfies:
(SI_1) $f_S(x + y) \supseteq f_S(x) \cap f_S(y)$ for all $x, y \in S$;
(SI_2) $f_S(xy) \supseteq f_S(x) \cap f_S(y)$ for all $x, y \in S$;
(SI_3) $f_S(x) \supseteq f_S(a) \cap f_S(b)$ with $x + a + z = b + z$ for all $x, a, b, z \in S$.

(2) A soft set f_S over U is called a soft intersection left(right) h-ideal(briefly, SI-left(right) h-ideal) of S over U if satisfies (SI_1), (SI_3) and
(SI_4)$f_S(xy) \supseteq f_S(y)(f_S(xy) \supseteq f_S(x))$ for all $x, y \in S$.

It is easy to see that if $f_S(x) = U$ for all $x \in S$, then f_S is an SI-hemiring(SI-left h-ideal, SI-right h-ideal,SI-h-ideal) denoted by $\widetilde{\mathbb{S}}$[19].

Proposition 2.8. [19] Let $A \subseteq S$. Then, A is an h-subhemiring(left h-ideal, right h-ideal, h-ideal) of S if and only if $\mathcal{S}_A$ is an SI-hemiring(SI-left h-ideal, SI-right h-ideal, SI-h-ideal) of S over U.

3. *SI-h*-Bi-Ideals

In this section, we introduce the concept of SI-h-bi-ideals of hemirings and investigate some characterizations.

Definition 3.1. A soft set f_S over U is called a soft intersection h-bi-ideal(briefly, SI-h-bi-ideal) of S over U if it satisfies (SI_1), (SI_2), (SI_3) and
(SI_5) $f_S(xyz) \supseteq f_S(x) \cap f_S(z)$ for all $x, y, z \in S$.

Remark 3.2. If f_S is an SI-h-bi-ideal of S over U, the $f_S(0) \supseteq f_S(x)$ for all $x \in S$.

Example 3.3. Let $U = \{< x, y > |x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$, Dihedral group, be the universal set. Consider the hemiring $S = \mathbb{Z}_4 = \{0, 1, 2, 3\}$, non-negative integers module 4, as the set of paramenters.

Define a soft set f_S over U by

$f_S(0) = \{e, x, y\}$, $f_S(1) = f_S(3) = \{x\}$ and $f_S(2) = \{e, x\}$.
Then, one can easily check that f_S is an *SI*-*h*-bi-ideal of S over U.

Theorem 3.4. Let $f_S \in S(U)$. Then, f_S is an *SI*-*h*-bi-ideal of S over U if and only if it satisfies(SI_3) and

(SI_6) $f_S \boxplus f_S \widetilde{\subseteq} f_S$;

(SI_7) $f_S \bigstar f_S \widetilde{\subseteq} f_S$;

(SI_8) $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S$.

Proof. Assume that f_S is an *SI*-*h*-bi-ideal of S over U.

(1) Let $x \in S$. If $(f_S \boxplus f_S)(x) = \emptyset$. Then, it is clear that $(f_S \boxplus f_S)(x) \subseteq f_S(x)$. Otherwise, let $a_1, a_2, b_1, b_2, z \in S$ such that $x + a_1 + b_1 + z = a_2 + b_2 + z$.
Then,

$$\begin{aligned}
(f_S \boxplus f_S)(x) &= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} (f_S(a_1) \cap f_S(a_2) \cap f_S(b_1) \cap f_S(b_2)) \\
&\subseteq \bigcup_{x+a_1+b_1+z=a_2+b_2+z} (f_S(a_1 + b_1) \cap f_S(a_2 + b_2)) \\
&\subseteq \bigcup_{x+a_1+b_1+z=a_2+b_2+z} f_S(x) \\
&= f_S(x),
\end{aligned}$$

which implies, $f_S \boxplus f_S \widetilde{\subseteq} f_S$. Thus, ($SI_6$) holds.

(2) Let $x \in S$. If $(f_S \bigstar f_S)(x) = \emptyset$. Then, it is clear that $(f_S \bigstar f_S)(x) \subseteq f_S(x)$. Otherwise, let $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$ with $a_i, a'_j \in S$ and $b_i, b'_j \in S$ for all $i = 1, 2, ..., m$; $j = 1, 2, ..., n$. Thus,

$$\begin{aligned}
(f_S \bigstar f_S)(x) &= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(b_i) \cap f_S(a'_j) \cap f_S(b'_j)) \\
&\subseteq \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(b_i) \cap \widetilde{\mathbb{S}}(a'_j) \cap \widetilde{\mathbb{S}}(b'_j)) \\
&= f_S(x),
\end{aligned}$$

which implies, $f_S \bigstar f_S \widetilde{\subseteq} f_S$. Thus, ($SI_7$) holds.

(3) Let $x \in S$, if $(f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S)(x) = \emptyset$. Then, it is clear that $(f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S)(x) \subseteq f_S(x)$. Otherwise,

$(f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S)(x) = ((f_S \bigstar \widetilde{\mathbb{S}}) \bigstar f_S)(x)$

$$= \bigcup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} ((f_S \bigstar \widetilde{\mathbb{S}})(a_i) \cap (f_S \bigstar \widetilde{\mathbb{S}})(a'_j) \cap f_S(b_i) \cap f_S(b'_j))$$

$$= \bigcup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} (\bigcup_{a_i+\sum_{k=1}^{m_i} a_{ik}b_{ik}+z_i=\sum_{l=1}^{n_i} a'_{jl}b'_{jl}+z_i} (f_S(a_{ik}) \cap f_S(a'_{jl}) \cap \widetilde{\mathbb{S}}(b_{jk}) \cap \widetilde{\mathbb{S}}(b'_{jl}))$$

$$\cap (\bigcup_{a'_j+\sum_{p=1}^{m'_j} a_{ip}b_{ip}+z'_j=\sum_{q=1}^{n'_j} a'_{jq}b'_{jq}+z'_j} (f_S(a_{ip}) \cap f_S(a'_{jq}) \cap \widetilde{\mathbb{S}}(b_{ip}) \cap \widetilde{\mathbb{S}}(b'_{jq})) \cap f_S(b_i) \cap f_S(b'_j))$$

$$= \bigcup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} \bigcup_{a_i+\sum_{k=1}^{m_i} a_{ik}b_{ik}+z_i=\sum_{l=1}^{n_i} a'_{jl}b'_{jl}+z_i} \bigcup_{a'_j+\sum_{p=1}^{m'_j} a_{ip}b_{ip}+z'_j=\sum_{q=1}^{n'_j} a'_{jq}b'_{jq}+z'_j} ((f_S(a_{ik})$$

$$\cap f_S(a'_{jl}) \cap \widetilde{\mathbb{S}}(b_{jk}) \cap \widetilde{\mathbb{S}}(b'_{jl})) \cap ((f_S(a_{ip}) \cap f_S(a'_{jq}) \cap \widetilde{\mathbb{S}}(b_{ip}) \cap \widetilde{\mathbb{S}}(b'_{jq})) \cap f_S(b_i) \cap f_S(b'_j))$$

$$\subseteq \bigcup_{x+\sum_{i=1}^{m'} \tilde{a}_i\tilde{c}_i\tilde{b}_i+\tilde{z}=\sum_{j=1}^{n'} \tilde{a}'_j\tilde{c}'_j\tilde{b}'_j+\tilde{z}} (f_S(a_i) \cap f_S(a'_j) \cap f_S(b_i) \cap f_S(b'_j)) \textbf{ (Please refer to Appendix)}$$

$$\subseteq \bigcup_{x+\sum_{i=1}^{m} a_ic_ib_i+z=\sum_{j=1}^{n} a'_jc'_jb'_j+z} (f_S(\sum_{i=1}^{m'} a_ic_ib_i) \cap f_S(\sum_{j=1}^{n'} a'_jc'_jb'_j))$$

$\subseteq f_S(x)$,

which implies, $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S$. Thus, (SI_8) holds.

Conversely, assume that (SI_3), (SI_6), (SI_7) and (SI_8) hold.

(1)
$$\begin{aligned} f_S(x+y) &\supseteq (f_S \boxplus f_S)(x+y) \\ &= \bigcup_{x+y+a_1+b_1+z=a_2+b_2+z} (f_S(a_1) \cap f_S(a_2) \cap f_S(b_1) \cap f_S(b_2)) \\ &\supseteq f_S(x) \cap f_S(y) \cap f_S(0) \\ &= f_S(x) \cap f_S(y). \end{aligned}$$

Thus, (SI_1) holds.

(2)
$$\begin{aligned} f_S(xy) &\supseteq (f_S \bigstar f_S)(xy) \\ &= \bigcup_{xy+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} (f_S(a_i) \cap f_S(a'_j) \cap f_S(b_i) \cap f_S(b'_j)) \\ &\supseteq f_S(x) \cap f_S(y) \cap f_S(0) \\ &= f_S(x) \cap f_S(y). \end{aligned}$$

Thus, (SI_2) holds.

(3) Let $x, y, z \in S$. Then,

$$\begin{aligned} f_S(xyz) &\supseteq (f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S)(xyz) \\ &= (f_S \bigstar (\widetilde{\mathbb{S}} \bigstar f_S))(xyz) \\ &= \bigcup_{xyz+\sum_{i=1}^{m} a_ib_i+z'=\sum_{j=1}^{n} a'_jb'_j+z'} (f_S(a_i) \cap f_S(a'_j) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(b_i) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(b'_j)) \\ &\supseteq f_S(0) \cap f_S(x) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(0) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(yz) \\ &= f_S(x) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(yz) \\ &= f_S(x) \cap (\bigcup_{yz+\sum_{i=1}^{m} c_id_i+z'=\sum_{j=1}^{n} c'_jd'_j+z'} \widetilde{\mathbb{S}}(c_i) \cap \widetilde{\mathbb{S}}(c'_j) \cap f_S(d_i) \cap f_S(d'_j)) \\ &= f_S(x) \cap f_S(z). \end{aligned}$$

Thus, (SI_5) holds. Hence, f_S is an SI-h-bi-ideal of S over U. □

The following proposition is obvious.

Proposition 3.5. Every SI-$left$ h-ideal($right$ h-ideal, h-ideal) of S over U is an SI-h-bi-ideal of S over U.

Theorem 3.6. Let $f_S, g_S \in S(U)$. If f_S and g_S are an *SI*-*h*-bi-ideal of S over U, then $f_S \bigstar g_S$ and $g_S \bigstar f_S$ are *SI*-*h*-bi-ideals of S over U.

Proof. For all $x, y \in S$, we have

(1) $(f_S \bigstar g_S)(x) \cap (f_S \bigstar g_S)(y)$

$$= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z_1 = \sum_{j=1}^{n} a'_j b'_j + z_1} (f_S(a_i) \cap f_S(a'_j) \cap g_S(b_i) \cap g_S(b'_j))$$

$$\cap \bigcup_{y+\sum_{i=1}^{p} c_i d_i + z_2 = \sum_{j=1}^{q} c'_j d'_j + z_2} (f_S(c_i) \cap f_S(c'_j) \cap g_S(d_i) \cap g_S(d'_j))$$

$$= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z_1 = \sum_{j=1}^{n} a'_j b'_j + z_1} \bigcup_{y+\sum_{i=1}^{p} c_i d_i + z_2 = \sum_{j=1}^{q} c'_j d'_j + z_2} (f_S(a_i) \cap f_S(c_i) \cap f_S(a'_j) \cap f_S(c'_j) \cap g_S(b_i) \cap g_S(d_i) \cap g_S(b'_j) \cap g_S(d'_j))$$

$$\subseteq \bigcup_{x+y+\sum_{i=1}^{k} x_i y_i + z_1 + z_2 = \sum_{j=1}^{l} x'_j y'_j + z_1 + z_2} (f_S(x_i) \cap f_S(x'_j) \cap g_S(y_i) \cap g_S(y'_j))$$

$= (f_S \bigstar g_S)(x + y)$.

This proves that (SI_1) holds, that is, (SI_6) holds.

(2) Similar to (1), we can show that (SI_3) holds.

(3) $(f_S \bigstar g_S) \bigstar (f_S \bigstar g_S)$

$= f_S \bigstar (g_S \bigstar (f_S \bigstar g_S))$

$\widetilde{\subseteq} f_S \bigstar (g_S \bigstar (\widetilde{\mathbb{S}} \bigstar g_S))$ (since $f_S \widetilde{\subseteq} \widetilde{\mathbb{S}}$)

$= f_S \bigstar (g_S \bigstar \widetilde{\mathbb{S}} \bigstar g_S)$

$\widetilde{\subseteq} f_S \bigstar g_S$. (since $g_S \bigstar \widetilde{\mathbb{S}} \bigstar g_S \widetilde{\subseteq} g_S$)

This proves that (SI_7) holds.

(4) $(f_S \bigstar g_S) \bigstar \widetilde{\mathbb{S}} \bigstar (f_S \bigstar g_S)$

$= f_S \bigstar (g_S \bigstar (\widetilde{\mathbb{S}} \bigstar f_S) \bigstar g_S)$ (since $\widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} \widetilde{\mathbb{S}}$)

$\widetilde{\subseteq} f_S \bigstar (g_S \bigstar \widetilde{\mathbb{S}} \bigstar g_S)$

$\widetilde{\subseteq} f_S \bigstar g_S$. (since $g_S \bigstar \widetilde{\mathbb{S}} \bigstar g_S \widetilde{\subseteq} g_S$)

This proves that (SI_8) holds.

It follows from 3.4 that $f_S \bigstar g_S$ is an *SI*-*h*-bi-ideal of S over U. Similarly, we can prove that $g_S \bigstar f_S$ is also an *SI*-*h*-bi-ideal of S over U. □

The following proposition is similar to Proposition 2.8.

Proposition 3.7. Let $A \in S$. Then, A is an *h*-bi-ideal of S if an only if $\mathcal{S}_A$ is an *SI*-*h*-bi-ideal of S over U.

Theorem 3.8. If f_S and h_S are two *SI*-*h*-bi-ideals of S over U, then so is $f_S \widetilde{\cap} h_S$.

Proof. Let $x, y \in S$. Then,

(1)
$$\begin{aligned}(f_S \widetilde{\cap} h_S)(x + y) &= f_S(x + y) \cap h_S(x + y) \\ &\supseteq (f_S(x) \cap f_S(y)) \cap (h_S(x) \cap h_S(y)) \\ &= (f_S(x) \cap h_S(x)) \cap (f_S(y) \cap h_S(y)) \\ &= (f_S \widetilde{\cap} h_S)(x) \cap (f_S \widetilde{\cap} h_S)(y).\end{aligned}$$

(2)
$$\begin{aligned}(f_S \widetilde{\cap} h_S)(xy) &= f_S(xy) \cap h_S(xy) \\ &\supseteq (f_S(x) \cap f_S(y)) \cap (h_S(x) \cap h_S(y)) \\ &= (f_S(x) \cap h_S(x)) \cap (f_S(y) \cap h_S(y)) \\ &= (f_S \widetilde{\cap} h_S)(x) \cap (f_S \widetilde{\cap} h_S)(y).\end{aligned}$$

(3) Now, let $x, z, a, b \in S$ with $x + a + z = b + z$. Then,

$$\begin{aligned}(f_S\widetilde{\cap}h_S)(x) &= f_S(x)\cap h_S(x)\\ &\supseteq (f_S(a)\cap f_S(b))\cap(h_S(a)\cap h_S(b))\\ &= (f_S(a)\cap h_S(a))\cap(f_S(b)\cap h_S(b))\\ &= (f_S\widetilde{\cap}h_S)(a)\cap(f_S\widetilde{\cap}h_S)(b).\end{aligned}$$

(4)
$$\begin{aligned}(f_S\widetilde{\cap}h_S)(xyz) &= f_S(xyz)\cap h_S(xyz)\\ &\supseteq (f_S(x)\cap f_S(z))\cap(h_S(x)\cap h_S(z))\\ &= (f_S(x)\cap h_S(x))\cap(f_S(z)\cap h_S(z))\\ &= (f_S\widetilde{\cap}h_S)(x)\cap(f_S\widetilde{\cap}h_S)(z).\end{aligned}$$

Hence, $f_S\widetilde{\cap}h_S$ is an *SI*-*h*-bi-ideal of S over U. □

Remark 3.9. $f_S\widetilde{\cup}h_S$ may not be an *SI*-*h*-bi-ideal of S over U.

Example 3.10. Assume that $U=\mathbb{Z}^+$, the set of positive integers, is the universal set. Consider two parameter sets $S_1=\mathbb{Z}_4=\{0,1,2,3\}$, non-negative integers module 4, and

$$S_2=\left\{\begin{bmatrix} x & x\\ y & y\end{bmatrix}\Big| x,y\in\mathbb{Z}_2=\{0,1\}\right\},$$

where $\mathbb{Z}_2$ is the set of non-negative integers module 2. Define two soft sets f_{S_1} and f_{S_2} over U by
$f_{S_1}(0)=\mathbb{Z}^+$, $f_{S_1}(1)=f_{S_1}(3)=\{2,3\}$ and $f_{S_1}(2)=\{1,2,3,5\}$.
$f_{S_2}\left(\begin{bmatrix}0&0\\0&0\end{bmatrix}\right)=\mathbb{Z}^+$, $f_{S_2}\left(\begin{bmatrix}0&0\\1&1\end{bmatrix}\right)=\{2,3,5\}$,
$f_{S_2}\left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)=\{1,2,3,5,7\}$ and $f_{S_2}\left(\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)=\{1,2,3,5,7,8\}$.
Then, one can easily check that f_{S_1} and f_{S_2} are both *SI*-*h*-bi-ideals of S over U.
$f_{S_1\cup S_2}\left(\left(3,\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)+\left(2,\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\right)$
$=f_{S_1\cup S_2}\left(1,\begin{bmatrix}0&0\\1&1\end{bmatrix}\right)$
$=f_{S_1}(1)\cup f_{S_2}\left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)$
$=\{2,3\}\cup\{1,2,3,5,7\}$
$=\{1,2,3,5,7\}$,
but $f_{S_1\cup S_2}\left(3,\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)=f_{S_1}(3)\cup f_{S_2}\left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)$
$=\{2,3\}\cup\{1,2,3,5,7\}$
$=\{1,2,3,5,7\}$,
and $f_{S_1\cup S_2}\left(2,\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)=f_{S_1}(2)\cup f_{S_2}\left(\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$
$=\{1,2,3,5\}\cup\{1,2,3,5,7,8\}$
$=\{1,2,3,5,7,8\}$,
which implies, $f_{S_1\cup S_2}\left(3,\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)\cup f_{S_1\cup S_2}\left(2,\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$
$=\{1,2,3,5,7,8\}$.
This implies that
$f_{S_1\cup S_2}\left(\left(3,\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)+\left(2,\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\right)$
$\not\supseteq f_{S_1\cup S_2}\left(3,\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)\cap f_{S_1\cup S_2}\left(2,\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$.
Hence, $f_{S_1}\widetilde{\cup}f_{S_2}$ is not an *SI*-*h*-bi-ideal over U.

4. *SI-h*-Quasi-Ideals

In this section, we introduce the concept of *SI-h*-quasi-ideals and investigate some related properties.

Definition 4.1. A soft set over U is called a soft intersection h-quasi-ideal (briefly, *SI-h*-quasi-ideal) of S over U if it satisfies (SI_1), (SI_3) and
(SI_9) $(f_S \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar f_S) \widetilde{\subseteq} f_S$.

Example 4.2. Assume that $U = \mathbb{Z}^+$, the set of positive integers, is the universal set and $S = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, non-negative positive integers module 6, is the set of parameters. Define a soft set f_S of S over U by
$f_S(0) = \mathbb{Z}^+$, $f_S(1) = f_S(5) = \{6n | n \in \mathbb{Z}^+\}$, $f_S(2) = f_S(4) = \{2n | n \in \mathbb{Z}^+\}$ and $f_S(3) = \{3n | n \in \mathbb{Z}^+\}$.
Then, one can easily check that f_S is an *SI-h*-quasi-ideal of S over U.

The following proposition is obvious.

Proposition 4.3. (1) Every *SI-h*-quasi-ideal of S over U is an *SI*-hemiring of S.
(2) Every *SI-h*-ideal of S over U is an *SI-h*-quasi-ideal of S.
(3) Every *SI-h*-quasi-ideal of S over U is an *SI-h*-bi-ideal of S.

Proof. We only prove (3) and the others are obvious. We only need to show that (SI_7) and (SI_8) hold. By (SI_9), we have
$f_S \bigstar f_S = (f_S \bigstar f_S) \widetilde{\cap} (f_S \bigstar f_S) \widetilde{\subseteq} (f_S \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar \widetilde{f_S}) \widetilde{\subseteq} f_S$.
Thus, (SI_7) holds.

Moveover, we have
$f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \bigstar f_S$ and $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S \bigstar \widetilde{\mathbb{S}} \bigstar \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S \bigstar \widetilde{\mathbb{S}}$, and so
$f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} (f_S \bigstar \widetilde{\mathbb{S}}) \cap (\widetilde{\mathbb{S}} \bigstar f_S) \widetilde{\subseteq} f_S$.

This proves that (SI_8) holds. Hence, f_S is an *SI-h*-bi-ideal of S over U.

Now, we give an important result of uni-int product $f_S \bigstar g_S$. □

Theorem 4.4. Let f_S and g_S be any *SI-h*-quasi-ideals of S over U. Then, $f_S \bigstar g_S$ is an *SI-h*-bi-ideal of S over U.

Proof. Let f_S be an *SI-h*-quasi-ideal of S over U. Then, by Proposition 4.3(3), f_S is an *SI-h*-bi-ideal of S over U. Hence, $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S$.

For any $x, y \in S$, if x or y cannot be expressed $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$ or $y + \sum_{i=1}^{p} c_i d_i + z' = \sum_{j=1}^{q} c'_j d'_j + z'$, then $(f_S \bigstar h_S)(x) = \emptyset$ or $(f_S \bigstar h_S)(y) = \emptyset$, and so $(f_S \bigstar h_S)(x) \cap (f_S \bigstar h_S)(y) \subseteq (f_S \bigstar h_S)(x + y)$. Otherwise, we have

$$
\begin{aligned}
&(f_S \bigstar h_S)(x) \cap (f_S \bigstar h_S)(y) \\
= & \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=i}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(a'_j) \cap h_S(b_i) \cap h_S(b'_i)) \\
\cap & \bigcup_{y+\sum_{i=1}^{p} c_i d_i + z' = \sum_{j=i}^{q} c'_j d'_j + z'} (f_S(c_i) \cap f_S(c'_j) \cap h_S(d_i) \cap h_S(d'_i)) \\
= & \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=i}^{n} a'_j b'_j + z} \\
& \bigcup_{y+\sum_{i=1}^{p} c_i d_i + z' = \sum_{j=i}^{q} c'_j d'_j + z'} (f_S(a_i) \cap f_S(a'_j) \cap f_S(c_i) \cap f_S(c'_j) \cap h_S(b_i) \cap h_S(b'_j) \cap h_S(d_i) \cap h_S(d'_j)) \\
\subseteq & \bigcup_{x+y+\sum_{i=1}^{k} x_i y_i + z + z' = \sum_{j=1}^{l} x'_j y'_j + z + z'} (f_S(x_i) \cap f_S(x'_j) \cap h_S(y_i) \cap h_S(y'_j)) \\
= & (f_S \bigstar h_S)(x + y).
\end{aligned}
$$

Thus, (SI_1) holds, that is (SI_6) holds.

Similarly, we can prove that (SI_3) holds.

$(f_S \bigstar h_S) \bigstar (f_S \bigstar h_S)$
$= (f_S \bigstar h_S \bigstar f_S) \bigstar h_S$
$\widetilde{\subseteq} (f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S) \bigstar h_S$ (since $h_S \widetilde{\subseteq} \widetilde{\mathbb{S}}$)
$\widetilde{\subseteq} f_S \bigstar h_S.$ (since $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S$)

Thus, (SI_7) holds.

Finally,

$(f_S \bigstar h_S) \bigstar \widetilde{\mathbb{S}} \bigstar (f_S \bigstar h_S)$
$= (f_S \bigstar (h_S \bigstar \widetilde{\mathbb{S}}) \bigstar f_S) \bigstar h_S$
(since $h_S \widetilde{\subseteq} \widetilde{\mathbb{S}}$)
$\widetilde{\subseteq} (f_S \bigstar (\widetilde{\mathbb{S}} \bigstar \widetilde{\mathbb{S}}) \bigstar f_S) \bigstar h_S$
$\widetilde{\subseteq} (f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S) \bigstar h_S \widetilde{\subseteq} f_S \bigstar h_S.$ (since $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S$)

Thus, (SI_8) holds. It follows from Theorem 3.4 that $f_S \bigstar g_S$ is an *SI-h*-bi-ideal of S over U. □

Proposition 4.5. (1) Let f_S and g_S be any *SI-right h*-ideal and *SI-left h*-ideal of S over U, respectively. Then, $f_S \widetilde{\cap} g_S$ is an *SI-h*-quasi-ideal of S over U.
(2) Let f_S and g_S be two *SI-h*-quasi-ideals of S over U. Then, so is $f_S \widetilde{\cap} g_S$.

Proof. By similar proof of Theorem 3.8, we can prove that (SI_1) and (SI_3) hold.
(1) If f_S and g_S are any *SI-right h*-ideal and *SI-left h*-ideal of S over U, respectively, then

$((f_S \widetilde{\cap} g_S) \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar (f_S \widetilde{\cap} g_S)) \widetilde{\subseteq} (f_S \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar g_S) \widetilde{\subseteq} f_S \widetilde{\cap} g_S.$

Thus, (SI_9) holds. Hence, $f_S \widetilde{\cap} g_S$ is an *SI-h*-quasi-ideal of S over U.

(2) If f_S and g_S are two *SI-h*-quasi-ideals of S over U,

$((f_S \widetilde{\cap} g_S) \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar (f_S \widetilde{\cap} g_S)) \widetilde{\subseteq} (f_S \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar f_S) \widetilde{\subseteq} f_S,$

and

$((f_S \widetilde{\cap} g_S) \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar (f_S \widetilde{\cap} g_S)) \widetilde{\subseteq} (g_S \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar g_S) \widetilde{\subseteq} g_S,$

and so

$((f_S \widetilde{\cap} g_S) \bigstar \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \bigstar (f_S \widetilde{\cap} g_S)) \widetilde{\subseteq} f_S \widetilde{\cap} g_S.$

Then, $f_S \widetilde{\cap} g_S$ is an *SI-h*-quasi-ideal of S over U. □

Similar to Proposition 2.8, we can get the following proposition.

Proposition 4.6. Let $A \subseteq S$. Then, A is an *h*-quasi-ideal of S if and only if $\mathcal{S}_A$ is an *SI-h*-quasi-ideal of S over U.

Finally, we give the following important result:

Theorem 4.7. (1) Let $f_S \in S(U)$ and $\alpha \subseteq U$ such that $\alpha \in Im(f_S)$. If f_S is an *SI-h*-quasi-ideal of S over U, then $U(f_S; \alpha)$ is an *h*-quasi-ideal of S.

(2) Let $f_S \in S(U)$. If $U(f_S; \alpha)$ is an *h*-quasi-ideal of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ is a totally ordered set by inclusion f_S is an *SI-h*-quasi-ideal of S over U.

Proof. (1) Since $f_S(x) = \alpha$ for some $x \in S$, then $\emptyset \neq U(f_S; \alpha) \subseteq \alpha$.

(*i*) Let $x, y \in U(f_S; \alpha)$. Then, $f_S(x) \supseteq \alpha$ and $f_S(y) \supseteq \alpha$. Since f_S is an *SI-h*-quasi-ideal of S over U, then $f_S(x + y) \supseteq f_S(x) \cap f_S(y) \supseteq \alpha \cap \alpha = \alpha$, and so $x + y \in U(f_S; \alpha)$.

(*ii*) Let $a, b \in U(f_S; \alpha)$ and $x, z \in S$ such that $x + a + z = b + z$. Then, $f_S(a) \supseteq \alpha$ and $f_S(b) \supseteq \alpha$. Since f_S is an *SI-h*-quasi-ideal of S over U, then $f_S(x) \supseteq f_S(a) \cap f_S(b) \supseteq \alpha \cap \alpha = \alpha$, and so $x \in U(f_S; \alpha)$.

(*iii*) Let $a \in \overline{S \cdot U(f_S; \alpha)} \cap \overline{U(f_S; \alpha) \cdot S}$. Then, there exist $x_1, x_2, y_1, y_2 \in U(f_S; \alpha)$ and $s_1, s_2, t_1, t_2, z_1, z_2 \in S$ such that $a + s_1 x_1 + z_1 = s_2 x_2 + z_1$ and $a + y_1 t_1 + z_2 = y_2 t_2 + z_2$. Hence, $f_S(x_1) \supseteq \alpha$, $f_S(x_2) \supseteq \alpha$, $f_S(y_1) \supseteq \alpha$ and $f_S(y_2) \supseteq \alpha$.

Moreover, we have

$$
\begin{aligned}
(\widetilde{\mathbb{S}}\bigstar f_S)(a) &= \bigcup_{a+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (\widetilde{\mathbb{S}}(a_i) \cap \widetilde{\mathbb{S}}(a'_j) \cap f_S(b_i) \cap f_S(b'_j)) \\
&\supseteq \widetilde{\mathbb{S}}(s_1) \cap \widetilde{\mathbb{S}}(s_2) \cap f_S(x_1) \cap f_S(x_2) \\
&= f_S(x_1) \cap f_S(x_2) \\
&\supseteq \alpha \cap \alpha \\
&= \alpha,
\end{aligned}
$$

and

$$
\begin{aligned}
(f_S\bigstar\widetilde{\mathbb{S}})(a) &= \bigcup_{a+\sum_{i=1}^{m} c_i d_i + z' = \sum_{j=1}^{n} c'_j d'_j + z'} (f_S(c_i) \cap f_S(c'_j) \cap \widetilde{\mathbb{S}}(d_i) \cap \widetilde{\mathbb{S}}(d'_j)) \\
&\supseteq f_S(y_1) \cap f_S(y_2) \cap \widetilde{\mathbb{S}}(t_1) \cap \widetilde{\mathbb{S}}(t_2) \\
&= f_S(y_1) \cap f_S(y_2) \\
&\supseteq \alpha \cap \alpha \\
&= \alpha.
\end{aligned}
$$

Since f_S is an *SI*-*h*-quasi-ideal of S over U, then $f_S(a) \supseteq (\widetilde{\mathbb{S}}\bigstar f_S)(a) \cap (f_S\bigstar\widetilde{\mathbb{S}})(a) \supseteq \alpha \cap \alpha = \alpha$, which implies that $a \in U(f_S;\alpha)$. This proves that $U(f_S;\alpha)$ is an *h*-quasi-ideal of S.

(2) Let $f_S \in S(U)$. Then,

(*i*) Let $x, y \in S$ be such that $f_S(x) = \alpha_1$ and $f_S(y) = \alpha_2$, where it may be assumed $\alpha_1 \subseteq \alpha_2$. Then, $x \in U(f_S;\alpha_1)$ and $y \in U(f_S;\alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, then $y \in U(f_S;\alpha_1)$. Since $U(f_S;\alpha)$ is an *h*-quasi-ideal of f_S for each $\alpha \subseteq U$, then $x + y \in U(f_S;\alpha_1)$. Hence, $f_S(x + y) \supseteq \alpha_1 = \alpha_1 \cap \alpha_2 = f_S(x) \cap f_S(y)$.

(*ii*) Let $x, a, b, z \in S$ with $x + a + z = b + z$ such that $f_S(a) = \alpha_1$ and $f_S(b) = \alpha_2$, where $\alpha_1 \subseteq \alpha_2$. Then, $a \in U(f_S;\alpha_1)$ and $b \in U(f_S;\alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, then $b \in U(f_S;\alpha_1)$. Since $U(f_S;\alpha)$ is an *h*-quasi-ideal of f_S for each $\alpha \subseteq U$. Then, $x \in U(f_S;\alpha_1)$. Hence,

$f_S(x) \supseteq \alpha_1 = \alpha_1 \cap \alpha_2 = f_S(a) \cap f_S(b)$.

(*iii*) Let $x \in S$ be such that $(\widetilde{\mathbb{S}}\bigstar f_S)(x) = \alpha_1$ and $(f_S\bigstar\widetilde{\mathbb{S}})(x) = \alpha_2$, where $\alpha_1 \subseteq \alpha_2$. Then, $x \in U(\widetilde{\mathbb{S}}\bigstar f_S;\alpha_1)$ and $x \in U(f_S\bigstar\widetilde{\mathbb{S}};\alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $x \in U(f_S\bigstar\widetilde{\mathbb{S}};\alpha_1)$. From $(\widetilde{\mathbb{S}}\bigstar f_S)(x) = \alpha_1$, then there exist $s_1, s_2, z_1 \in S$ and $k_1, k_2 \in U(f_S;\alpha_1)$ such that $x + s_1k_1 + z_1 = s_2k_2 + z_1$, that is, $x \in \overline{S \cdot U(f_S;\alpha_1)}$. Similarly, we can prove that $x \in \overline{U(f_S;\alpha_1) \cdot S}$. Hence, $x \in \overline{S \cdot U(f_S;\alpha_1)} \cap \overline{U(f_S;\alpha_1) \cdot S}$. Since $U(f_S;\alpha_1)$ is an *h*-quasi-ideal of f_S, then $x \in U(f_S;\alpha_1)$. Thus, we have

$f_S(x) \supseteq \alpha_1 = \alpha_1 \cap \alpha_2 = (\widetilde{\mathbb{S}}\bigstar f_S)(x) \cap (f_S\bigstar\widetilde{\mathbb{S}})(x)$.

Hence, f_S is an *SI*-*h*-quasi-ideal of S over U. □

5. *h*-Hemiregular Hemirings

In this section, we investigate some characterizations by means of *SI*-*h*-ideals, *SI*-*h*-bi-ideals and *SI*-*h*-quasi-ideals.

Definition 5.1. [31] A hemiring S is called *h*-hemiregular if for each $a \in S$, there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$.

Lemma 5.2. [31] If A and B, are respectively, a right *h*-ideal and a left *h*-ideal of S, then $\overline{AB} \subseteq A \cap B$.

Lemma 5.3. [31] *A* hemiring S is *h*-hemiregular if and only if for any right *h*-ideal A and left *h*-ideal B, we have $\overline{AB} = A \cap B$.

Theorem 5.4. [19] For any hemiring S, the following conditions are equivalent:

(1) S is *h*-hemiregular;

(2) $f_S\bigstar g_S = f_S\widetilde{\cap} g_S$ for any *SI*-right *h*-ideal f_S and any *SI*-left *h*-ideal g_S of S over U.

Lemma 5.5. [30] Let S be a hemiring. Then, the following conditions are equivalent:

(1) S is *h*-hemiregular;

(2) $B = \overline{BSB}$ for every h-bi-ideal B of S;

(3) $Q = \overline{QSQ}$ for every h-quasi-ideal Q of S.

Theorem 5.6. For any hemiring S, the following conditions are equivalent:

(1) S is h-hemiregular;

(2) $f_S = f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S$ for every SI-h-bi-ideal f_S of S over U;

(3) $f_S = f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S$ for every SI-h-quasi-ideal f_S of S over U.

Proof. (1)⇒(2) Let S be an h-hemiregular hemiring, f_S an SI-h-bi-ideal of S over U. For any $x \in S$. There exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular. Thus, we have

$$
\begin{aligned}
&(f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S)(x)\\
&= ((f_S \bigstar \widetilde{\mathbb{S}}) \bigstar f_S)(x)\\
&= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} ((f_S \bigstar \widetilde{\mathbb{S}})(a_i) \cap (f_S \bigstar \widetilde{\mathbb{S}})(a'_j) \cap f_S(b_i) \cap f_S(b'_j))\\
&\supseteq (f_S \bigstar \widetilde{\mathbb{S}})(xa) \cap (f_S \bigstar \widetilde{\mathbb{S}})(xa') \cap f_S(x)\\
&= \bigcup_{xa+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(a'_j) \cap \widetilde{\mathbb{S}}(b_i) \cap \widetilde{\mathbb{S}}(b'_j))\\
&\cap \bigcup_{xa'+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=i}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(a'_j) \cap \widetilde{\mathbb{S}}(b_i) \cap \widetilde{\mathbb{S}}(b'_j)) \cap f_S(x)\\
&\supseteq (f_S(xax) \cap f_S(xa'x)) \cap (f_S(xax) \cap f_S(xa'x)) \cap f_S(x)\\
&\qquad (\text{Since } xa + xaxa + za = xa'xa + za \text{ and } xa' + xaxa' + za' = xa'xa' + za')\\
&\supseteq f_S(x) \cap f_S(x) \cap f_S(x)\\
&= f_S(x),
\end{aligned}
$$

which implies, $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\supseteq} f_S$. Since f_S is an SI-h-bi-ideal of S over U, then $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S$. Thus, we have $f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S = f_S$.

(2)⇒(3) This is straightforward by Proposition 4.3.

(3)⇒(1) Let Q be any h-quasi-ideal of S. Then, by Proposition 4.6, the soft characteristic function $\mathcal{S}_A$ of A is an SI-h-quasi-ideal of S over U.

Thus, by the assumption and Proposition 2.6(3), we have

$$\mathcal{S}_A = \mathcal{S}_A \bigstar \widetilde{\mathbb{S}} \bigstar \mathcal{S}_A = \mathcal{S}_A \bigstar \mathcal{S}_S \bigstar \mathcal{S}_A = \mathcal{S}_{\overline{ASA}}.$$

It follows from Proposition 2.6(1), we have $A = \overline{ASA}$. Thus, by Lemma 5.5, S is h-hemiregular. □

Theorem 5.7. Let f_S be a soft set of an h-hemiregular hemiring S. Then, the following conditions are equivalent:

(1) f_S may be presented in the form $f_S = g_S \bigstar h_S$, where g_S is an SI-$right$ h-ideal and h_S is an SI-$left$ h-ideal of S over U;

(2) f_S is an SI-h-bi-ideal of S over U;

(3) f_S is an SI-h-quasi-ideal of S over U.

Proof. (1)⇒(2) If there exist an SI-$right$ h-ideal g_S and an SI-$left$ h-ideal h_S of S such that $f_S = g_S \bigstar h_S$, then by Proposition 4.3, every SI-$left$($right$) h-ideal of S is an SI-h-bi-ideal of S. Thus, g_S and h_S are SI-h-bi-ideals of S over U. It follows from Theorem 3.6 that $g_S \bigstar h_S = f_S$ is an SI-h-bi-ideal of S.

(2)⇒(3) This is straightforward by Proposition 4.3.

(3)⇒(1) Since S is h-hemiregular, then by Theorem 5.6, $f_S = f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S$, where f_S is an SI-h-quasi-ideal of S over U. Thus,

$$f_S = f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S = f_S \bigstar (\widetilde{\mathbb{S}} \bigstar \widetilde{\mathbb{S}}) \bigstar f_S = (f_S \bigstar \widetilde{\mathbb{S}}) \bigstar (\widetilde{\mathbb{S}} \bigstar f_S).$$

Hence, we can easily show that $f_S \bigstar \widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{S}} \bigstar f_S$ are an SI-$right$ h-ideal and an SI-$left$ h-ideal of S over U, respectively. In fact,

$$(f_S \bigstar \widetilde{\mathbb{S}}) \bigstar \widetilde{\mathbb{S}} = f_S \bigstar (\widetilde{\mathbb{S}} \bigstar \widetilde{\mathbb{S}}) \widetilde{\subseteq} f_S \bigstar \widetilde{\mathbb{S}} \text{ and } \widetilde{\mathbb{S}} \bigstar (\widetilde{\mathbb{S}} \bigstar f_S) = (\widetilde{\mathbb{S}} \bigstar \widetilde{\mathbb{S}}) \bigstar f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \bigstar f_S.$$ □

Theorem 5.8. For any hemiring S, the following conditions are equivalent:

(1) S is h-hemiregular;

(2) $f_S \widetilde{\cap} g_S = f_S \bigstar g_S \bigstar f_S$ for every SI-h-bi-ideal f_S and every SI-h-ideal g_S of S over U;

(3) $f_S \widetilde{\cap} g_S = f_S \bigstar g_S \bigstar f_S$ for every SI-h-quasi-ideal f_S and every SI-h-ideal g_S of S over U.

Proof. (1)⇒(2) Let f_S and g_S be any SI-h-bi-ideal and SI-h-ideal of S over U, respectively. Then,

$f_S \bigstar g_S \bigstar f_S \widetilde{\subseteq} f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S$ and $f_S \bigstar g_S \bigstar f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \bigstar (g_S \bigstar \widetilde{\mathbb{S}}) \widetilde{\subseteq} \widetilde{\mathbb{S}} \bigstar g_S \widetilde{\subseteq} g_S$.

Then, $f_S \bigstar g_S \bigstar f_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$.

For any $x \in S$, there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular.

Thus, we have

$$
\begin{aligned}
& (f_S \bigstar g_S \bigstar f_S)(x) \\
&= ((f_S \bigstar g_S) \bigstar f_S)(x) \\
&= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} ((f_S \bigstar g_S)(a_i) \cap (f_S \bigstar g_S)(a'_j) \cap f_S(b_i) \cap f_S(b'_j)) \\
&\supseteq (f_S \bigstar g_S)(xa) \cap (f_S \bigstar g_S)(xa') \cap f_S(x) \\
&= \bigcup_{xa+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(a'_j) \cap g_S(b_i) \cap g_S(b'_j)) \\
&\cap \bigcup_{xa'+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(a'_j) \cap g_S(b_i) \cap g_S(b'_j)) \cap f_S(x) \\
&\supseteq (f_S(x) \cap g_S(axa) \cap g_S(a'xa)) \cap (f_S(x) \cap g_S(axa') \cap g_S(a'xa')) \cap f_S(x) \\
&\quad (xa + xaxa + za = xa'xa + za \text{ and } xa' + xaxa' + za' = xa'xa' + za') \\
&\supseteq f_S(x) \cap g_S(x) \\
&= (f_S \widetilde{\cap} g_S)(x),
\end{aligned}
$$

which implies, $f_S \cap g_S \widetilde{\subseteq} f_S \bigstar g_S \bigstar f_S$. Thus, we have

$f_S \bigstar g_S \bigstar f_S = f_S \widetilde{\cap} g_S$.

(2)⇒(3) This is straightforward by Proposition 4.3.

(3)⇒(1) Since $\widetilde{\mathbb{S}}$ is an SI-h-ideal of S over U, then by the assumption, we have

$f_S = f_S \widetilde{\cap} \widetilde{\mathbb{S}} = f_S \bigstar \widetilde{\mathbb{S}} \bigstar f_S$.

It follows from Theorem 5.6 that S is h-hemiregular. □

Theorem 5.9. Let S be a hemiring. Then, the following conditions are equivalent:

(1) S is h-hemiregular;

(2) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every SI-h-bi-ideal f_S and every SI-$left$ h-ideal g_S of S over U;

(3) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every SI-h-quasi-ideal f_S and every SI-$left$ h-ideal g_S of S over U;

(4) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every SI-*right* h-ideal f_S and every SI-h-bi-ideal of S over U;

(5) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every SI-*right* h-ideal f_S and every SI-h-quasi-ideal of S over U;

(6) $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \bigstar g_S \bigstar h_S$ for every SI-*right* h-ideal f_S, every SI-h-bi-ideal g_S and every SI-$left$ h-ideal h_S of S over U;

(7) $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \bigstar g_S \bigstar h_S$ for every SI-*right* h-ideal f_S, every SI-h-quasi-ideal g_S and every SI-$left$ h-ideal h_S of S over U.

Proof. (1)⇒(2) Let f_S and g_S be any SI-h-bi-ideal and any SI-$left$ h ideal of S over U, respectively. For any $x \in S$, there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular. Then,

$$
\begin{aligned}
& (f_S \bigstar g_S)(x) \\
&= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(a'_j) \cap g_S(b_i) \cap g_S(b'_j)) \\
&\supseteq f_S(x) \cap g_S(ax) \cap g_S(a'x) \\
&\supseteq f_S(x) \cap g_S(x) \\
&= (f_S \cap g_S)(x),
\end{aligned}
$$

which implies, $f_S\widetilde{\cap}g_S\widetilde{\subseteq}f_S\bigstar g_S$.

(2)⇒(1) Let f_S and g_S be any *SI-right h*-ideal and any *SI-left h*-ideal of S over U, respectively. Then, it is easy to see that f_S is an *SI-h*-bi-ideal of S over U. By the assumption, we have

$f_S\widetilde{\cap}g_S\widetilde{\subseteq}f_S\bigstar g_S\widetilde{\subseteq}(f_S\bigstar\widetilde{\mathbb{S}})\widetilde{\cap}(\widetilde{\mathbb{S}}\bigstar g_S)\widetilde{\subseteq}f_S\widetilde{\cap}g_S$. Hence, $f_S\widetilde{\cap}g_S = f_S\bigstar g_S$. It follows from Theorem 5.4 that S is h-hemiregular.

Similarly, we can show that (1)⇒(3), (1)⇒(4), (1)⇒(5).

(1)⇒(6) Let f_S, g_S and h_S be any *SI-right h*-ideal, any *SI-h*-bi-ideal and any *SI-left h*-ideal of S over U, respectively. For any $x \in S$, there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular. Then, we have

$$\begin{aligned}
&(f_S\bigstar g_S\bigstar h_S)(x)\\
&= \bigcup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} ((f_S\bigstar g_S)(a_i) \cap (f_S\bigstar g_S)(a'_j) \cap h_S(b_i) \cap h_S(b'_j))\\
&\supseteq (f_S\bigstar g_S)(x) \cap h_S(ax) \cap h_S(a'x)\\
&= \bigcup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} (f_S(a_i) \cap f_S(a'_j) \cap g_S(b_i) \cap g_S(b'_j)) \cap h_S(ax) \cap h_S(a'x)\\
&\supseteq f_S(xa) \cap f_S(xa') \cap g_S(x) \cap h_S(ax) \cap h_S(a'x)\\
&\supseteq f_S(x) \cap g_S(x) \cap h_S(x)\\
&= (f_S\widetilde{\cap}g_S\widetilde{\cap}h_S)(x),
\end{aligned}$$

which implies, $f_S\widetilde{\cap}g_S\widetilde{\cap}h_S\widetilde{\subseteq}f_S\bigstar g_S\bigstar h_S$.

(6)⇒(7) This is straightforward by Proposition 4.3.

(7)⇒(1) Let f_S and h_S be any *SI-right h*-ideal and any *SI-left h*-ideal of S over U, respectively. Since $\widetilde{\mathbb{S}}$ is an *SI-h*-quasi-ideal of S over U, then by the assumption, we have

$f_S\widetilde{\cap}h_S = f_S\widetilde{\cap}\widetilde{\mathbb{S}}\widetilde{\cap}h_S\widetilde{\subseteq}f_S\bigstar\widetilde{\mathbb{S}}\bigstar h_S\widetilde{\subseteq}f_S\bigstar h_S\widetilde{\subseteq}(f_S\bigstar\widetilde{\mathbb{S}})\widetilde{\cap}(\widetilde{\mathbb{S}}\bigstar h_S) \subseteq f_S\widetilde{\cap}h_S$.

Then, $f_S\widetilde{\cap}h_S = f_S\bigstar h_S$. It follows from Theorem 5.4 that S is h-hemiregular. □

6. *h-intra*-Hemiregular Hemirings

In this section, we investigate some characterizations by means of *SI-h*-ideals, *SI-h*-bi-ideal and *SI-h*-quasi-ideals.

Definition 6.1. [30] A hemiring S is called *h-intra*-hemiregular if for each $x \in S$, there exist $a_i, a'_i, b_j, b'_j, z \in S$ such that $x + \sum_{i=1}^{m} a_ix^2a'_i + z = \sum_{j=1}^{n} b_jx^2b'_j + z$. Equivalent definitions:
(1) $x \in \overline{Sx^2S}, \forall x \in S$; (2) $A \subseteq \overline{SA^2S}, \forall A \subseteq S$.

Lemma 6.2. [30] Let S be a hemiring. Then, the following conditions are equivalent:
(1) S is *h-intra*-hemiregular;
(2) $L \cap R \subseteq \overline{LR}$ for every left h-ideal L and every right h-ideal R of S.

Theorem 6.3. Let S be a hemiring. Then, the following conditions are equivalent:
(1) S is *h-intra*-hemiregular;
(2) $f_S\widetilde{\cap}g_S\widetilde{\subseteq}f_S\bigstar g_S$ for every *SI-left h*-ideal f_S and every *SI-right h*-ideal of S over U.

Proof. (1)⇒(2) Let f_S and g_S be any *SI-left h*-ideal and any *SI-right h*-ideal of S over U, respectively.For any $x \in S$. Then, there exist $a_i, a'_i, b_j, b'_j, z \in S$ such that

$$x + \sum_{i=1}^{m} a_ix^2a'_i + z = \sum_{j=1}^{n} b_jx^2b'_j + z.$$

Then, we have

$$
\begin{aligned}
&(f_S \bigstar g_S)(x)\\
&= \bigcup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} (f_S(a_i) \cap f_S(a'_j) \cap g_S(b_i) \cap g_S(b'_j))\\
&\supseteq f_S(a_ix) \cap f_S(b_jx) \cap g_S(xa'_i) \cap g_S(xb'_j)\\
&\supseteq f_S(x) \cap g_S(x)\\
&= f_S \widetilde{\cap} g_S(x),
\end{aligned}
$$

which implies, $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$.

(2)⇒(1) Let L and R be any left h-ideal and any *right* h-ideal of S, respectively. Then, by Proposition 2.8, $\mathcal{S}_L$ and $\mathcal{S}_R$ are an *SI-left* h-ideal and an *SI-right* h-ideal of S over U, respectively. Now, by Proposition 2.6 and assumption, we have

$$\mathcal{S}_{L\cap R} = \mathcal{S}_L \widetilde{\cap} \mathcal{S}_R \subseteq \mathcal{S}_L \bigstar \mathcal{S}_L = \mathcal{S}_{\overline{LR}}.$$

By Proposition 2.6, we have $L \cap R \subseteq \overline{LR}$. Thus, it follows from Lemma 6.2 that S is *h-intra*-hemiregular. □

Lemma 6.4. [30] Let S be a hemiring. Then, the following are equivalent:

(1) S is both h-hemiregular and *h-intra*-hemiregular.

(2) $B = \overline{B^2}$ for every h-bi-ideal B of S.

(3) $Q = \overline{Q^2}$ for every h-quasi-ideal Q of S.

Theorem 6.5. Let S be a hemiring. Then, the following conditions are equivalent:

(1) S is both h-hemiregular and *h-intra*-hemiregular;

(2) $f_S = f_S \bigstar f_S$ for every *SI-h*-bi-ideal f_S of S over U(that is, every *SI-h*-bi-ideal of S is idempotent);

(3) $f_S = f_S \bigstar f_S$ for every *SI-h*-quasi-ideal f_S of S over U(that is, every *SI-h*-quasi-ideal of S is idempotent).

Proof. (1)⇒(2) Let f_S be any *SI-h*-bi-ideal of S over U. Then, it is clear that $f_S \bigstar f_S \widetilde{\subseteq} f_S$. For any $x \in S$, there exist $a_1, a_2, p_i, p'_i, q_j, q'_j, z \in S$ such that

$$
\begin{aligned}
&x + \sum_{j=1}^{n}(xa_2q_jx)(xq'_ja_1x) + \sum_{j=1}^{n}(xa_1q_jx)(xq'_ja_2x) + \sum_{i=1}^{m}(xa_1p_ix)(xp'_ia_1x) + \sum_{i=1}^{m}(xa_2p_ix)(xp'_ia_2x) + z\\
&= \sum_{i=1}^{m}(xa_2p_ix)(xp'_ia_1x) + \sum_{i=1}^{m}(xa_1p_ix)(xp'_ia_2x) + \sum_{j=1}^{n}(xa_1q_jx)(xq'_ja_1x) + \sum_{j=1}^{n}(xa_2q_jx)(xq'_ja_2x) + z.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&(f_S \bigstar f_S)(x)\\
&= \bigcup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} (f_S(a_i) \cap f_S(a'_j) \cap f_S(b_i) \cap f_S(b'_j))\\
&\supseteq f_S(xa_2q_jx) \cap f_S(xq'_ja_1x) \cap f_S(xa_1q_jx) \cap f_S(xq'_ja_2x) \cap f_S(xa_1p_ix) \cap f_S(xp'_ia_1x) \cap f_S(xa_ip_ix)\\
&\cap f_S(xp'_ia_2x)\\
&\supseteq f_S(x),
\end{aligned}
$$

which implies, $f_S \widetilde{\subseteq} f_S \bigstar f_S$. Thus, $f_S = f_S \bigstar f_S$.

(2)⇒(3) This is straightforward by Proposition 4.3.

(3)⇒(1) Let Q be any h-quasi-ideal of S. Then, by Proposition 2.8, $\mathcal{S}_Q$ is an *SI-h*-quasi-ideal of S over U. Now, by the assumption and Proposition 2.6, we have

$$\mathcal{S}_Q = \mathcal{S}_Q \bigstar \mathcal{S}_Q = S_{\overline{Q^2}}.$$

Then, by Proposition 2.6, $Q = \overline{Q^2}$. It follows from Lemma 6.4 that S is both h-hemiregular and *h-intra*-hemiregular. □

Similarly, we can get the following theorem:

Theorem 6.6. Let S be a hemiring. Then, the following conditions are equivalent:

(1) S is both h-hemiregular and *h-intra*-hemiregular;

(2) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for all *SI-h*-bi-ideals f_S and g_S of S over U;

(3) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for all *SI-h*-quasi-ideals f_S and g_S of S over U;

(4) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every *SI-h*-bi-ideal f_S and every *SI-h*-quasi-ideal g_S of S over U;
(5) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every *SI-h*-quasi-ideal f_S and every *SI-h*-bi-ideal g_S of S over U;
(6) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for all *SI-h*-quasi-ideals f_S and g_S of S over U.

7. *h*-Quasi-Hemiregular Hemirings

In this section, we investigate some characterizations of *h*-quasi-hemiregular hemirings by means of three *SI-h*-ideals.

Definition 7.1. [15] A subset A of S is called idempotent if $A = \overline{A^2}$. A hemiring S is called $left(right)$ *h*-quasi-hemiregular if every left(right) *h*-ideal is idempotent and is called *h*-quasi-hemiregular if every left *h*-ideal and every *right h*-ideal is idempotent.

Lemma 7.2. [15] *A hemiring S is $left$ h-quasi-hemiregular if and only if one of the following conditions holds:*
(1) There exist $c_i, d_i, c'_j, d'_j, z \in S$ such that
$x + \sum_{i=1}^{m} c_i x d_i x + z = \sum_{j=1}^{n} c'_j x d'_j x + z$ for all $x \in S$;
(2) $x \in \overline{SxSx}$ for all $x \in S$;
(3) $A \subseteq \overline{SASA}$ for all $A \in S$;
(4) $I \cap L = \overline{IL}$ for every h-ideal I an every $left$ h-ideal L of S.

Theorem 7.3. A hemiring is $left(right)$ *h*-quasi-hemiregular if and only if every *SI*-$left(right)$ *h*-ideal of S is idempotent.

Proof. Let S be a $left$ *h*-quasi-hemiregular hemiring, f_S any *SI*-$left$ *h*-ideal of S over U. For any $x \in S$, there exist $c_i, c'_j, d_i, d'_j, z \in S$ such that

$$x + \sum_{i=1}^{m} c_i x d_i x + z = \sum_{j=1}^{n} c'_j x d'_j x + z.$$

Then, we have

$$\begin{aligned}(f_S \bigstar f_S)(x) &= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a'_j b'_j + z'} (f_S(a_i) \cap f_S(a'_j) \cap f_S(b_i) \cap f_S(b'_j)) \\ &\supseteq f_S(c_i x) \cap f_S(c'_j x) \cap f_S(d_i x) \cap f_S(d'_j x) \\ &\supseteq f_S(x),\end{aligned}$$

which implies, $f_S \widetilde{\subseteq} f_S \bigstar f_S$. Since f_S is an *SI*-$left$ *h*-ideal of S over U, then $f_S \bigstar f_S \widetilde{\subseteq} f_S$. Thus, we have $f_S \bigstar f_S = f_S$.

Conversely, let L be any $left$ *h*-ideal of S. Then, $\mathcal{S}_L$ is an *SI*-$left$ *h*-ideal of S over U by Proposition 2.8. Then, by Proposition 2.6, we have

$$\mathcal{S}_L = \mathcal{S}_L \bigstar \mathcal{S}_L = \mathcal{S}_{\overline{L^2}},$$

which implies, $L = \overline{L^2}$. Hence, S is $left$ *h*-quasi-hemiregular. Similarly, we can prove the case for *SI-right h*-ideals. □

Theorem 7.4. Let S be a hemiring. Then, the following conditions are equivalent:
(1) S is $left$ *h*-quasi-hemiregular;
(2) $f_S \widetilde{\cap} g_S = f_S \bigstar g_S$ for every *SI-h*-ideal f_S and every *SI*-$left$ *h*-ideal g_S of S over U;
(3) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every *SI-h*-ideal f_S and every *SI-h*-bi-ideal g_S of S over U;
(4) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every *SI-h*-ideal f_S and every *SI-h*-quasi-ideal g_S of S over U.

Proof. (1)⇒(3) Let f_S and g_S be any *SI-h*-ideal and *SI-h*-bi-ideal of S over U, respectively. For $x \in S$, by Lemma 7.2, we have $x \in \overline{SxSx} \subseteq \overline{S\overline{SxSxS}x} \subseteq \overline{SxSxSx}$, and so there exist $c_i, c'_j, d_i, d'_j, e_i, e'_j, z \in S$ such that $x + \sum_{j=1}^{m'} c_i x d_i x e_i x + z = \sum_{j=1}^{n'} c'_j x d'_j x e'_j x + z$.
Then, we have

$$\begin{aligned}&(f_S \bigstar f_S)(x)\\ &= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cap f_S(a'_j) \cap g_S(b_i) \cap g_S(b'_j))\\ &\supseteq f_S(c_i x d_i) \cap f_S(c'_j x d'_j) \cap g_S(x e_i x) \cap g_S(x e'_j x)\\ &\supseteq f_S(x) \cap g_S(x)\\ &= f_S \widetilde{\cap} g_S(x),\end{aligned}$$

which implies, $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar f_S$.

(3)⇒(4)⇒(2) It is clear.

(2)⇒(1) Let I and L be any h-ideal and any *left* h-ideal of S, respectively. Then, $\mathcal{S}_I$ and $\mathcal{S}_L$ are an *SI-h*-ideal and *SI-left h*-ideal of S, respectively. Then,

$$\mathcal{S}_{I\cap L} = \mathcal{S}_I \widetilde{\cap} \mathcal{S}_L = \mathcal{S}_I \bigstar \mathcal{S}_L = \mathcal{S}_{\overline{IL}},$$

which implies, $I \cap L = \overline{IL}$. It follows from Lemma 7.2 that S is h-quasi-hemiregular. □

Similarly, we can get the following theorem.

Theorem 7.5. Let S be a hemiring. Then, the following conditions are equivalent:

(1) S is *left* h-quasi-hemiregular;

(2) $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \bigstar g_S \bigstar h_S$ for every *SI-h*-ideal f_S, every *SI-right* h-ideal g_S and every *SI-h*-bi-ideal h_S of S over U;

(3) $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \bigstar g_S \bigstar h_S$ for every *SI-h*-ideal f_S, every *SI-right* h-ideal g_S and every *SI-h*-quasi-ideal h_S of S over U.

Now, we can describe the characterization of h-quasi-hemiregular hemirings.

Theorem 7.6. *A* hemiring S is h-quasi-hemiregular if and only if $f_S = (\widetilde{\mathbb{S}} \bigstar f_S)^2 \widetilde{\cap} (f_S \bigstar \widetilde{\mathbb{S}})^2$ for every *SI-h*-quasi-ideal of S over U.

Proof. Let S be an h-quasi-hemiregular hemiring, f_S an *SI-h*-quasi-ideal of S over U. we know that $\widetilde{\mathbb{S}} \bigstar f_S$ and $f_S \bigstar \widetilde{\mathbb{S}}$ are an *SI-left h*-ideal and an *SI-right h*-ideal of S over U, respectively, and so both $\widetilde{\mathbb{S}} \bigstar f_S$ and $f_S \bigstar \widetilde{\mathbb{S}}$ are idempotent by Theorem 7.3. Hence, we have

$$(\widetilde{\mathbb{S}} \bigstar f_S)^2 \widetilde{\cap} (f_S \bigstar \widetilde{\mathbb{S}})^2 = (\widetilde{\mathbb{S}} \bigstar f_S) \widetilde{\cap} (f_S \bigstar \widetilde{\mathbb{S}}) \widetilde{\subseteq} f_S.$$

For any $x \in S$, there exist $c_i, c'_j, d_i, d'_j, z \in S$ such that $x + \sum_{i=1}^{m'} c_i x d_i x + z = \sum_{j=1}^{n'} c'_j x d'_j x + z$ since S is *left* h-quasi-hemiregular. Then, we have

$$\begin{aligned}&(\widetilde{\mathbb{S}} \bigstar f_S)^2(x)\\ &= \bigcup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} ((\widetilde{\mathbb{S}} \bigstar f_S)(a_i) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(a'_j) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(b_i) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(b'_j))\\ &\supseteq (\widetilde{\mathbb{S}} \bigstar f_S)(c_i x) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(c'_j x) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(d_i x) \cap (\widetilde{\mathbb{S}} \bigstar f_S)(d'_j x)\\ &\supseteq f_S(x),\end{aligned}$$

which implies, $f_S \widetilde{\subseteq} (\widetilde{\mathbb{S}} \bigstar f_S)^2$. Similarly, we can prove $f_S \widetilde{\subseteq} (f_S \bigstar \widetilde{\mathbb{S}})^2$, and so, $f_S \widetilde{\subseteq} (\widetilde{\mathbb{S}} \bigstar f_S)^2 \widetilde{\cap} (f_S \bigstar \widetilde{\mathbb{S}})^2$. Thus, $f_S = (\widetilde{\mathbb{S}} \bigstar f_S)^2 \widetilde{\cap} (f_S \bigstar \widetilde{\mathbb{S}})^2$.

Conversely, let f_S be any *SI-left h*-ideal of S over U. Then, by Proposition 4.3, we have f_S is an *SI-h*-quasi-ideal of S over U. Then,

$f_S = (\widetilde{\mathbb{S}} \bigstar f_S)^2 \widetilde{\cap} (f_S \bigstar \widetilde{\mathbb{S}})^2 \widetilde{\subseteq} (\widetilde{\mathbb{S}} \bigstar f_S)^2 \widetilde{\subseteq} f_S \bigstar f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \bigstar f_S \widetilde{\subseteq} f_S$.
Thus, $f_S = f_S \bigstar f_S$. Then, by Theorem 4.3, S is $left$ h-quasi-hemiregular. Similarly, we can prove S is a *right* h-quasi-hemiregular. Therefore, S is h-quasi-hemiregular. □

Lemma 7.7. [15] *A* hemiring S is both $left$ h-quasi-hemiregular and h-*intra*-hemiregular if and only if for any $x \in S$, there exist $c_i, d_i, c'_j, d'_j, z \in S$ such that $x + \sum_{i=1}^{m} c_i x^2 d_i x + z = \sum_{j=1}^{n} c'_j x^2 d'_j x + z$.

Similar to Theorems 7.4 and 7.5, we can get the following theorem.

Theorem 7.8. Let S be a hemiring. Then, the following conditions are equivalent:
(1) S is both $left$ h-hemiregular and h-*intra*-hemiregular;
(2) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every *SI*-$left$ h-ideal f_S and every *SI*-h-bi-ideal g_S of S over U;
(3) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \bigstar g_S$ for every *SI*-$left$ h-ideal f_S and every *SI*-h-quasi-ideal g_S of S over U.

8. Conclusions

The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, we make a new approach to hemirings by means of soft set theory, with the concepts of *SI*-hemirings, *SI*-h-ideals, *SI*-h-bi-ideals and *SI*-h-quasi-ideals. Finally, we investigate the characterizations of h-hemiregular hemirings, h-*intra*-hemiregular hemirings and h-quasi-hemiregular hemirings.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of hemirings. In the future study of soft hemirings, we can consider to apply this kind of new soft hemirings to some applied fields, such as decision making, data analysis and forecasting and so on.

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Appendix

It suffices to show that when $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$ and for each $i = 1, \cdots, m$ and $j = 1, \cdots, n$, $a_i + \sum_{k=1}^{m_i} a_{ik} b_{ik} + z_i = \sum_{l=1}^{n_i} a'_{jl} b'_{jl} + z_i$ and $a'_j + \sum_{p=1}^{m'_j} a_{ip} b_{ip} + z'_j = \sum_{q=1}^{n'_j} a'_{jq} b'_{jq} + z'_j$, we have $x + \sum_{i=1}^{m'} \tilde{a}_i \tilde{c}_i \tilde{b}_i + \tilde{z} = \sum_{j=1}^{n'} \tilde{a}'_j \tilde{c}'_j \tilde{b}'_j + \tilde{z}$ for some $m', n', \tilde{a}_i, \tilde{b}_i, \tilde{c}_i, \tilde{a}'_j, \tilde{b}'_j$ and $\tilde{c}'_j$.

For each i, we have

$$a_i + \sum_{k=1}^{m_i} a_{ik} b_{ik} + z_i = \sum_{l=1}^{n_i} a'_{jl} b'_{jl} + z_i \tag{1}$$

Multiplying two side of Eq. (1) by b_i, we have

$$a_i b_i + \sum_{k=1}^{m_i} a_{ik} b_{ik} b_i + z_i b_i = \sum_{l=1}^{n_i} a'_{jl} b'_{jl} b_i + z_i b_i \tag{2}$$

Summing for all i ranging from 1 to m, we have

$$\sum_{i=1}^{m} a_i b_i + \sum_{i=1}^{m}\sum_{k=1}^{m_i} a_{ik} b_{ik} b_i + \sum_{i=1}^{m} z_i b_i = \sum_{i=1}^{m}\sum_{l=1}^{n_i} a'_{jl} b'_{jl} b_i + \sum_{i=1}^{m} z_i b_i \tag{3}$$

Similarly, we have

$$\sum_{j=1}^{n}\sum_{q=1}^{n'_j} a'_{jq} b'_{jq} b'_j + \sum_{j=1}^{n} z'_j b'_j = \sum_{j=1}^{n} a'_j b'_j + \sum_{j=1}^{n}\sum_{p=1}^{m'_j} a_{ip} b_{ip} b'_j + \sum_{j=1}^{n} z'_j b'_j \tag{4}$$

Adding Eqs. (3) and (4), we have

$$\sum_{i=1}^{m} a_i b_i + \sum_{i=1}^{m}\sum_{k=1}^{m_i} a_{ik} b_{ik} b_i + \sum_{i=1}^{m} z_i b_i + \sum_{j=1}^{n}\sum_{q=1}^{n'_j} a'_{jq} b'_{jq} b'_j + \sum_{j=1}^{n} z'_j b'_j = \sum_{j=1}^{n} a'_j b'_j + \sum_{j=1}^{n}\sum_{p=1}^{m'_j} a_{ip} b_{ip} b'_j + \sum_{j=1}^{n} z'_j b'_j + \sum_{i=1}^{m}\sum_{l=1}^{n_i} a'_{jl} b'_{jl} b_i + \sum_{i=1}^{m} z_i b_i \tag{5}$$

Adding two side of Eq. (1) by $x + z$, we have

$$(x + \sum_{i=1}^{m} a_i b_i + z) + \sum_{i=1}^{m}\sum_{k=1}^{m_i} a_{ik} b_{ik} b_i + \sum_{i=1}^{m} z_i b_i + \sum_{j=1}^{n}\sum_{q=1}^{n'_j} a'_{jq} b'_{jq} b'_j + \sum_{j=1}^{n} z'_j b'_j = x + \sum_{j=1}^{n} a'_j b'_j + z + \sum_{j=1}^{n}\sum_{p=1}^{m'_j} a_{ip} b_{ip} b'_j + \sum_{j=1}^{n} z'_j b'_j + \sum_{i=1}^{m}\sum_{l=1}^{n_i} a'_{jl} b'_{jl} b_i + \sum_{i=1}^{m} z_i b_i \tag{6}$$

By $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$, we have

$$\sum_{j=1}^{n} a'_j b'_j + z + \sum_{i=1}^{m}\sum_{k=1}^{m_i} a_{ik} b_{ik} b_i + \sum_{i=1}^{m} z_i b_i + \sum_{j=1}^{n}\sum_{q=1}^{n'_j} a'_{jq} b'_{jq} b'_j + \sum_{j=1}^{n} z'_j b'_j = x + \sum_{j=1}^{n} a'_j b'_j + z + \sum_{j=1}^{n}\sum_{p=1}^{m'_j} a_{ip} b_{ip} b'_j + \sum_{j=1}^{n} z'_j b'_j + \sum_{i=1}^{m}\sum_{l=1}^{n_i} a'_{jl} b'_{jl} b_i + \sum_{i=1}^{m} z_i b_i \tag{7}$$

Hence, Eq. (7) can be reformulated as the following form:

$$\sum_{i=1}^{m}\sum_{k=1}^{m_i} a_{ik} b_{ik} b_i + \sum_{j=1}^{n}\sum_{q=1}^{n'_j} a'_{jq} b'_{jq} b'_j + z' = x + \sum_{j=1}^{n}\sum_{p=1}^{m'_j} a_{ip} b_{ip} b'_j + \sum_{i=1}^{m}\sum_{l=1}^{n_i} a'_{jl} b'_{jl} b_i + z', \tag{8}$$

where $z' = \sum_{j=1}^{n} a'_j b'_j + z + \sum_{i=1}^{m} z_i b_i + \sum_{j=1}^{n} z'_j b'_j$.

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