Filomat 30:8 (2016), 2191–2206
DOI 10.2298/FIL1608191N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Periodic and Fixed Point Using Weaker Meir-Keeler Function in Complete Generalized Metric Spaces

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Abstract. We originate some existence results on periodic and fixed point using generalized $(\psi \circ \varphi)$-contractions and generalized (ψ, φ)-contractive mappings with weaker Meir-Keeler function in the setup of a complete generalized metric space in sense of Branciari without Hausdorff assumption. Our results generalize the results of several well-known comparable results in the literature. To illustrate our results, we conclude the paper with some examples.

1. Introduction and Preliminaries

All the way through this paper, by $\mathbb{R}^+$, we designate the set of all real nonnegative numbers, while $\mathbb{N}$ is the set of all natural numbers.

Banach's contraction mapping principle is one of the cornerstone in the development of nonlinear analysis. Fixed point theorems have applications not only in the various branches of mathematics but also in economics, chemistry, biology, computer science, engineering, and others. In particular, such theorems are used to demonstrate the existence and uniqueness of a solution of differential equations, integral equations, functional equations, partial differential equations and others. Due to the importance, generalizations of Banach fixed point theorem have been investigated heavily by many authors (see, e.g., [18] and references cited therein).

In the year 2000, Brianciari in [2] initiated the notion of a generalized metric space as a generalization of a metric space in such a way that the triangle inequality is replaced by the "Quadrilateral inequality", $d(x, y) \leq d(x, a) + d(a, b) + d(b, y)$ for all pairwise distinct points x, y, a and b of X. Then after, many authors initiated and studied many existing fixed point theorems in such spaces. For more details about fixed point theory in generalized metric spaces, we refer the reader to [1],[2], [8]-[13], [15]-[22]; In these papers the authors assumed that the generalized metric space is Hausdorff to get a fixed point.

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25
Keywords. Fixed point; Contractive type mapping, Generalized metric spaces.
Received: 24 May 2014; Accepted: 15 January 2015
Communicated by Ljubomir Ćirić
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The following definitions will be needed in the sequel.

Definition 1.1. *[2] Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ such that for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, one has:*
(p1): $x = y \Longleftrightarrow d(x, y) = 0$,
(p2): $d(x, y) = d(y, x)$,
(p3): $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.
Then (X, d) is called a generalized metric space (or shortly g.m.s).

Any metric space is a generalized metric space, but the converse is not true ([2]).

Definition 1.2. *[2] Let (X, d) be a g.m.s, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is g.m.s convergent to x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.*

Definition 1.3. *[2] Let (X, d) be a g.m.s and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a g.m.s Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number N such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N$.*

Definition 1.4. *[2] Let (X, d) be a g.m.s. Then (X, d) is called a complete g.m.s if every g.m.s Cauchy sequence is g.m.s convergent in X.*

Several publications attempting to generalize fixed point theorems in metric spaces to g.m.s are plagued by the use of some false properties given in [2] (see for example [1], [8], [9], [10]). This was observed first by Samet [20] and then by Sarma, Rao and Rao [22] by assuming that the generalized metric space in Hausdorff.

Recently, Lakzian and Samet [15] proved the following result.

Theorem 1.5. *Let (X, d) be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ is such that for all $x, y \in X$*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{1}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing with $\psi(t) = 0$ if and only if $t = 0$, and $\phi : [0, \infty) \to [0, \infty)$ is continuous and $\phi(t) = 0$ if and only if $t = 0$. Then there exists a unique point $u \in X$ such that $u = Tu$.

Very recently, Chen and Sun [5] improve the Theorem 1.5 by using the notion of weaker (ϕ, φ)-contractive mapping and prove the periodic points and fixed points for this type of contraction. Shatanawi et al. [21] also extended the Theorem 1.5 by replacing in (1) the term $d(x, y)$ by the quantity $\max\{d(x, y), d(x, Tx), d(y, Ty)\}$ and the continuity of ϕ by lower semi-continuity; see also [3]. In the following lemma Z. Kadeburg, S. Radenovič [23] show that the assumption of Hausdorff in the most of fixed point theorems in generalized metric spaces can be omitted.

Lemma 1.6. *[23] Let (X, d) be a generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then the sequence $\{x_n\}$ can converge to at most one point.*

In this paper, we develop some existence results on periodic point using generalized (ψ, φ)-contractive mappings and generalized (ψ, φ)-contractive mappings with weaker Meir-Keeler function in the setup of a complete generalized metric space without the Hausdorffness assumption. Consequently, our results extend, improve and generalize several results in the literature. Our results generalize the results of Chen and Sun [5], Lakzian and Samet [19], Shatanawi et al. [21] as well as, our results generalize several well-known comparable results in the literature. To illustrate our results, we conclude the paper with examples.

2. Periodic and Fixed Point for Generalized ($\psi \circ \varphi$)-Contractions in Generalized Metric Spaces

We will prove periodic and fixed point theorem for self-mappings defined on a complete generalized metric space and satisfying certain weaker Meir-Keeler conditions. To achieve our goal, we recall the notion of a Meir-Keeler function (see [16]).

Definition 2.1. *[16] A function $\psi : [0,+\infty) \to [0,+\infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0,+\infty)$ with $\eta \le t < \eta + \delta$, we have $\psi(t) < \eta$.*

The notion of weaker Meir-Keeler function is as follows:

Definition 2.2. *[4] We call $\psi : [0,+\infty) \to [0,+\infty)$ a weaker Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0,+\infty)$ with $\eta \le t < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.*

As in [4], in this section, we denote by Ψ the set of weaker Meir-Keeler functions $\psi : [0,+1) \to [0,+1)$ satisfying the following hypotheses:

(ψ_0) ψ is non-decreasing;
(ψ_1) $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$;
(ψ_2) for all $t \in [0,\infty)$, $\{\psi^n(t)\}_{n\in N}$ is decreasing;
(ψ_3) for $t_n \in [0,\infty)$, we have that
  (a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \psi(t_n) < \gamma$, and
  (b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \psi(t_n) = 0$.

Also suppose that Φ is the set of non-decreasing and continuous functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying:

(φ_1) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$;
(φ_2) φ is subadditive, that is, for every $\mu_1, \mu_2 \in [0,+\infty)$, $\varphi(\mu_1 + \mu_2) \le \varphi(\mu_1) + \varphi(\mu_2)$;
(φ_3) for all $t \in (0,\infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \varphi(t_n) = 0$.

The notion of a periodic point of a given mapping $T : X \to X$ is crucial for proving our main theorem. So, we need the following definition.

Definition 2.3. *Let X be a non-empty set. A given mapping $T : X \to X$ admits a periodic point if there exists $u \in X$ such that $u = T^p u$ for some $p \ge 1$. If $p = 1$, u is a fixed point.*
Hence, each fixed point is also a periodic point of T.

Our first main result is the following.

Theorem 2.4. *Let (X,d) be a complete generalized metric space. Suppose that $T : X \to X$ is such that for all $x, y \in X$*

$$\varphi(d(Tx,Ty)) \le \psi(\varphi(M(x,y))) \tag{2}$$

where $\varphi \in \Phi$, $\psi \in \Psi$ and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{3}$$

Then T has a periodic point u in X, that is, there exists $u \in X$ such that $u = T^p u$ for some $p \in \mathbb{N}$. Also T has a unique fixed point, that is, $\mu \in X$ of T such that $\mu = T\mu$.

Proof. First, it is obvious that $M(x,y) = 0$ if and only if $x = y$ is a fixed point of T. Let $x_0 \in X$ an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \text{for all } n \ge 0. \tag{4}$$

Step 1. We claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{5}$$

Substituting $x = x_n$ and $y = x_{n-1}$ in (2) and using properties of functions ψ and φ, we obtain

$$\begin{aligned} \varphi(d(x_{n+1}, x_n)) &= \varphi(d(Tx_n, Tx_{n-1})) \\ &\leq \psi(\varphi(M(x_n, x_{n-1}))), \end{aligned} \tag{6}$$

that is,

$$\varphi(d(x_{n+1}, x_n)) \leq \psi(\varphi(M(x_n, x_{n-1}))) \quad \text{for all } n \geq 1.$$

Note that

$$M(x_n, x_{n-1}) = \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}$$

If for some $n \geq 1$, $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, then $M(x_n, x_{n-1}) = d(x_n, x_{n+1}) > 0$ and $\varphi(d(x_{n+1}, x_n)) > 0$ by a property of φ, so (6) becomes

$$0 < \varphi(d(x_{n+1}, x_n)) \leq \psi(\varphi(d(x_{n+1}, x_n))) < \varphi(d(x_{n+1}, x_n))$$

a contradiction. Thus, for all $n \geq 1$

$$\varphi(d(x_n, x_{n+1})) \leq \psi(\varphi(d(x_{n-1}, x_n))) \tag{7}$$

and by the condition (ψ_0), we have

$$\begin{aligned} \varphi(d(x_n, x_{n+1})) &\leq \psi(\varphi(d(x_{n-1}, x_n))) \\ &\leq \psi(\psi(\varphi(d(x_{n-2}, x_{n-1})))) \\ &= \psi^2(\varphi(d(x_{n-2}, x_{n-1})) \\ &\ \ \vdots \\ &\leq \psi^n(\varphi(d(x_0, x_1))). \end{aligned}$$

Since $\{\psi^n(\varphi(d(x_0, x_1)))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function ψ, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq \varphi(d(x_0, x_1)) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\varphi(d(x_0, x_1))) < \eta$. Since $\lim_{n\to\infty} \psi^n(\varphi(d(x_0, x_1))) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \psi^p(\varphi(d(x_0, x_1))) < \delta + \eta$, for all $p \geq p_0$. Thus, we conclude that $\psi^{p_0+n_0}(\varphi(d(x_0, x_1))) < \eta$. So we get a contradiction. Therefore $\lim_{n\to\infty} \psi^n(\varphi(d(x_0, x_1))) = 0$, hence also

$$\lim_{n\to\infty} \psi^n(\varphi(d(x_n, x_{n+1}))) = 0,$$

that is,

$$\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0,$$

or

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{8}$$

Step 2. We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{9}$$

From (8), there exists $K > 0$ such that $d(x_n, x_{n+1}) \leq K$ for all $n \in \mathbb{N}$. We prove (9) in each of the two following cases:

• If $d(x_{n-1}, x_{n+1}) > K$ for all $n \in \mathbb{N}$, from

$$M(x_{n-1}, x_{n+1}) = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\} = d(x_{n-1}, x_{n+1})$$

and using inequality (2), we have that for each $n \in \mathbb{N}$,

$$\begin{aligned} \varphi(d(x_n, x_{n+2})) &= \varphi(d(Tx_{n-1}, Tx_{n+1})) \\ &\le \psi(\varphi(M(x_{n-1}, x_{n+1}))) \\ &= \psi(\varphi(d(x_{n-1}, x_{n+1}))), \end{aligned} \tag{10}$$

that is,

$$\varphi(d(x_n, x_{n+2})) \le \psi(\varphi(d(x_{n-1}, x_{n+1}))) \tag{11}$$

and from (ψ_0), we have

$$\begin{aligned} \varphi(d(x_n, x_{n+2})) &\le \psi(\varphi(d(x_{n-1}, x_{n+1}))) \\ &\le \psi(\psi(\varphi(d(x_{n-2}, x_n)))) \\ &= \psi^2(\varphi(d(x_{n-2}, x_n))) \\ &\vdots \\ &\le \psi^n(\varphi(d(x_0, x_2))). \end{aligned}$$

Since $\{\psi^n(\varphi(d(x_0, x_2)))\}_{n\in\mathbb{N}}$ is decreasing, by the same proof process, we also conclude

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{12}$$

• If, for some $n \in \mathbb{N}$, we have $d(x_{n-1}, x_{n+1}) \le K$ and $d(x_n, x_{n+2}) > K$, then from

$$\begin{aligned} \varphi(d(x_n, x_{n+2})) &= \varphi(d(Tx_{n-1}, Tx_{n+1})) \\ &\le \psi(\varphi(M(x_{n-1}, x_{n+1}))) \\ &\le \varphi(M(x_{n-1}, x_{n+1})) \\ &\le \varphi(K), \end{aligned} \tag{13}$$

and since φ is non-decreasing $d(x_n, x_{n+2}) \le K$, a contradiction. Then $d(x_n, x_{n+2}) > K$ or $d(x_n, x_{n+2}) \le K$ for all $n \in \mathbb{N}$ and in both cases the sequence $\{d(x_n, x_{n+2})\}$ is bounded. Now, if (12) does not hold; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, x_{n_k+2}) \to r > 0$. From

$$d(x_{n_k-1}, x_{n_k+1}) \le d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k+2}) + d(x_{n_k+1}, x_{n_k+2})$$

and

$$d(x_{n_k}, x_{n_k+2}) \le d(x_{n_k-1}, x_{n_k}) + d(x_{n_k-1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k+2})$$

we deduce that

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{n_k+1}) = r.$$

Substituting $x = x_{n_k}$ and $y = x_{n_k+2}$ in (2), we have

$$\begin{aligned} \varphi(d(x_{n_k}, x_{n_k+2})) &= \varphi(d(Tx_{n_k-1}, Tx_{n_k+1})) \\ &\le \psi(\varphi(M(x_{n_k-1}, x_{n_k+1}))); \end{aligned} \tag{14}$$

where $M(x_{n_k-1}, x_{n_k+1}) = \max\{d(x_{n_k-1}, x_{n_k+1}), M(x_{n_k-1}, Tx_{n_k-1}), M(x_{n_k+1}, Tx_{n_k+1})\}$. So

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{n_k+1}) = r.$$

Using by (40) as $k \to \infty$, we get $\varphi(r) \leq \psi(\varphi(r))$. From properties of function ψ, $\psi(\varphi(r)) < \varphi(r)$, a contradiction; so $r = 0$.
Step 3. We claim that $\{x_n\}$ is a g.m.s. Cauchy sequence.

We claim that the following result holds:
Claim: For each $\epsilon > 0$, there is $n_0(\epsilon) \in \mathbb{N}$ such that for all $p, q \geq n_0(\epsilon)$,

$$\varphi(d(x_p, x_q)) < \epsilon, \tag{15}$$

We shall prove (15) by negation. Suppose, on the contrary, that (15) is false.
Then there exists some $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying:
(i) $\varphi(d(x_{p_n}, x_{q_n})) \geq \epsilon$, and
(ii) p_n is the smallest number greater than q_n such that the condition (i) holds.
Since

$$\begin{aligned}
\epsilon &\leq \varphi(d(x_{p_n}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n}, x_{p_n-2}) + d(x_{p_n-2}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n}, x_{p_n-2})) + \varphi(d(x_{p_n-2}, x_{p_n-1})) + \varphi(d(x_{p_n-1}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n}, x_{p_n-2})) + \varphi(d(x_{p_n-2}, x_{p_n-1})) + \epsilon,
\end{aligned}$$

hence we conclude $\lim_{p\to\infty} \varphi(d(x_{p_n}, x_{q_n})) = \epsilon$. Since φ is subadditive and nondecreasing, we conclude

$$\begin{aligned}
\varphi(d(x_{p_n}, x_{q_n})) &\leq \varphi(d(x_{p_n}, x_{p_n+1}) + d(x_{p_n+1}, x_{p_n+2}) + d(x_{p_n+2}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(x_{p_n+1}, x_{p_n+2})) + \varphi(d(x_{p_n+2}, x_{q_n})),
\end{aligned}$$

and so

$$\begin{aligned}
\varphi(d(x_{p_n}, x_{q_n})) - \varphi(d(x_{p_n}, x_{p_n+1})) - \varphi(d(x_{p_n+1}, x_{p_n+2})) &\leq \varphi(d(x_{p_n+1}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n+1}, x_{p_n-1}) + d(x_{p_n-1}, x_{p_n}) + d(x_{p_n}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n+1}, x_{p_n-1})) + \varphi(d(x_{p_n-1}, x_{p_n})) \\
&+ \varphi(d(x_{p_n}, x_{q_n})).
\end{aligned}$$

Passing to the limit as $n \to \infty$, we also have $\lim_{n\to\infty} \varphi(d(x_{p_n+1}, x_{q_n})) = \epsilon$.

Thus, there exists i, $0 \leq i \leq m-1$ such that $p_n - q_n + i = 1$ *mod* m for infinitely many n. If $i = 0$, then we have that for such n,

$$\begin{aligned}
\epsilon &\leq \varphi(d(x_{p_n}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n}, x_{p_n+1}) + d(x_{p_n+1}, x_{q_n+1}) + d(x_{q_n+1}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(x_{p_n+1}, x_{q_n+1})) + \varphi(d(x_{q_n+1}, x_{q_n})) \\
&= \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(Tx_{p_n}, Tx_{q_n})) + \varphi(d(x_{q_n+1}, x_{q_n})) \\
&\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \psi(\varphi(M(x_{p_n}, x_{q_n}))) + \varphi(d(x_{q_n+1}, x_{q_n})).
\end{aligned}$$

where

$$\begin{aligned}
\lim_{n\to\infty} M(x_{p_n}, x_{q_n}) &= \lim_{n\to\infty} \max\{d(x_{p_n}, x_{q_n}), d(Tx_{p_n}, x_{p_n}), d(x_{q_n}, Tx_{q_n}) \\
&= \lim_{n\to\infty} \max\{d(x_{p_n}, x_{q_n}), d(x_{p_{n+1}}, x_{p_n}), d(x_{q_n}, x_{q_{n+1}}) \\
&= \lim_{n\to\infty} d(x_{p_n}, x_{q_n}).
\end{aligned}$$

Passing to the limit as $n \to \infty$. Thus we have

$$\epsilon \leq 0 + \lim_{n\to\infty} \psi(\varphi(d(x_{p_n}, x_{q_n}))) + 0 < \epsilon,$$

a contradiction. Therefore $\lim_{n\to\infty} \varphi(d(x_{p_n}, x_{q_n})) = 0$, by the condition (φ_3), we also have $\lim_{n\to\infty} d(x_{p_n}, x_{q_n}) = 0$. The case $i \neq 0$ is similar. Hence, $\{x_n\}$ is a g.m.s Cauchy sequence.

Step 4. We claim that T has a periodic point.
We argue by contradiction. Assume that T has no periodic point. Then $\{x_n\}$ is a sequence of distinct points, that is, $x_n \neq x_m$ for all $m \neq n$. By Step 2, since (X, d) is a complete g.m.s, there exists $u \in X$ such that $x_n \to u$. Applying (2) with $x = x_n$ and $y = u$, we obtain

$$\varphi(d(x_{n+1}, Tu)) = \varphi(d(Tx_n, Tu)) \leq \psi(\varphi(M(x_n, u))) \tag{16}$$

where

$$M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}.$$

Since $\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, so we obtain that

$$\lim_{n\to\infty} M(x_n, u) = d(u, Tu). \tag{17}$$

Next we shall find a contradiction of the fact that T has no periodic point in each of the two following cases:

- If for all $n \geq 2$, $x_n \neq u$ and $x_n \neq Tu$.
  Passing to the limit as $\lim_{n\to\infty}$ in (16) and using (17), and the properties of ψ and φ, we obtain

$$\varphi(d(u, Tu)) \leq \psi(\varphi(d(u, Tu))) < \varphi(d(u, Tu))$$

which implies that $d(u, Tu) = 0$, so $u = Tu$, that is, u is a fixed point of T, so u is a periodic point of T. It contradicts the fact that T has no periodic point. Therefore, there exists $u \in X$ such that $u = T^p(u)$ for some $p \in \mathbb{N}$. So T has a periodic point in X.

- Let for some $q \geq 2$, $x_q = u$ or $x_q = Tu$. Since T has no periodic point, then obviously $u \neq x_0$. Indeed, if $x_q = u = x_0$ so $T^q x_0 = x_0$, i.e, x_0 is a periodic point of T, while if $x_q = Tu$ and $x_0 = u$ so $Tx_0 = Tu = x_q = T^q x_0 = T^{q-1}(Tx_0)$, i.e, Tx_0 is a periodic point of T.

For all $n \geq 0$, we have

$$d(T^n u, u) = d(T^n x_q, u) = d(x_{n+q}, u) \quad \text{or} \quad d(T^n u, u) = d(T^{n-1} Tu, u) = d(T^{n-1} x_q, u) = d(x_{n+q-1}, u).$$

In the two precedent identities, the integer $q \geq 2$ is fixed and so $\{x_{n+q}\}$ and $\{x_{n+q-1}\}$ are subsequences from $\{x_n\}$ and since $\{x_n\}$ g.m.s. converges to u in (X, d). On the other hand, we can see that $x_n \neq x_m$ for whenever $m \neq n$. Indeed if for some $m, n \in \mathbb{N}$, $x_n = x_m$ and $n < m$, then by (7), we have

$$\varphi(d(x_n, x_{n+1})) = \varphi(d(x_m, x_{m+1})) \leq \psi^{m-n}\varphi(d(x_n, x_{n+1})) < \varphi(d(x_n, x_{n+1})), \tag{18}$$

a contradiction. So by Lemma 1.6 and (18), the two subsequences g.m.s. converge to same unique limit u, i.e,

$$\lim_{n\to\infty} d(x_{n+q}, u) = \lim_{n\to\infty} d(x_{n+q-1}, u) = 0.$$

Thus,

$$\lim_{n\to\infty} d(T^n u, u) = 0. \tag{19}$$

By quadrilateral inequality, then by (19),

$$\lim_{n\to\infty} d(T^{n+2} u, u) = 0. \tag{20}$$

On the other hand, since T has no periodic point, it follows that

$$T^s u \neq T^r u \quad \text{for any } s, r \in \mathbb{N}, \quad s \neq r. \tag{21}$$

Using (21) and the quadrilateral inequality, we may write

$$|d(T^{n+1}u, Tu) - d(u, Tu)| \le d(T^{n+1}u, T^{n+2}u) + d(T^{n+2}u, u).$$

Passing to the limit as $n \to \infty$ in the above and proceeding as (5) (since the point x_0 is arbitrary), using (20) we obtain

$$\lim_{n\to\infty} d(T^{n+1}u, Tu) = d(u, Tu). \tag{22}$$

Now, by (2)

$$\varphi(d(T^{n+1}u, Tu)) \le \psi(\varphi(M(T^n u, u))) \tag{23}$$

where

$$M(T^n u, u) = \max\{d(T^n u, u), d(T^n u, T^{n+1}u), d(u, Tu)\} \to d(u, Tu) \text{ as } n \to \infty.$$

Passing to the limit as $n \to \infty$ in (23) and using (22) and the above limit, we get that

$$\varphi(d(u, Tu)) \le \psi(\varphi(d(u, Tu))) < \varphi(d(u, Tu))$$

which holds only if $d(u, Tu) = 0$, i.e, $Tu = u$, which implies that u is a periodic point of T. This contradicts the fact that T has no periodic point.
Consequently T admits a periodic point, that is, there exists $u \in X$ such that $u = T^p u$ for some $p \ge 1$.

Step 5. Existence of a fixed point of T.
If $p = 1$, then $u = Tu$, that is, u is a fixed point of T. Suppose now that $p > 1$. We will prove that $\mu = T^{p-1}u$ is a fixed point of T. Suppose that it is not the case, that is, $T^{p-1}u \neq T^p u$. Then $d(T^{p-1}u, T^p u) > 0$ and $\psi(\varphi(d(T^{p-1}u, T^p u))) > 0$, which implies that $\psi(\varphi(M(T^{p-1}u, T^p u))) > 0$. Now, using inequality (2), we obtain

$$\begin{aligned}\varphi(d(u, Tu)) =& \varphi(d(T^p u, T^{p+1}u))\\ =& \varphi(d(T(T^{p-1}u), T(T^p u)))\\ \le& \psi(\varphi(M(T^{p-1}u, T^p u)))\end{aligned}$$

which by the property of ψ implies

$$\varphi(d(u, Tu)) < \varphi(M(T^{p-1}u, T^p u)), \tag{24}$$

where

$$\begin{aligned}M(T^{p-1}u, T^p u) =& \max\{d(T^{p-1}u, T^p u), d(T^{p-1}u, T^p u), d(T^p u, T^{p+1}u)\}\\ =& \max\{d(T^{p-1}u, T^p u), d(u, Tu)\} = d(T^{p-1}u, T^p u)\end{aligned}$$

because otherwise we get a contradiction with (24). Thus, (24) becomes

$$\varphi(d(u, Tu)) < \varphi(d(T^{p-1}u, T^p u)). \tag{25}$$

Again, using (2) we have

$$\begin{aligned}\varphi(d(T^{p-1}u, T^p u)) =& \varphi d(T(T^{p-2}u), T(T^{p-1}u)))\\ \le& \psi(\varphi(M(T^{p-2}u, T^{p-1}u))).\end{aligned}$$

Again, this implies that

$$\varphi(d(T^{p-1}u, T^p u)) < \varphi(M(T^{p-2}u, T^{p-1}u)), \tag{26}$$

where

$$\begin{aligned}M(T^{p-2}u, T^{p-1}u) &= \max\{d(T^{p-2}u, T^{p-1}u), d(T^{p-2}u, T^{p-1}u), d(T^{p-1}u, T^{p}u)\}\\ &= \max\{d(T^{p-2}u, T^{p-1}u), d(T^{p-1}u, T^{p}u)\} = d(T^{p-2}u, T^{p-1}u)\end{aligned}$$

because of (26). Thus, from (26)

$$\varphi(d(T^{p-1}u, T^{p}u)) < \varphi(d(T^{p-2}u, T^{p-1}u)). \tag{27}$$

Continuing this process as (25) and (27), we find that

$$\varphi(d(u, Tu)) < \varphi(d(T^{p-1}u, T^{p}u)) < \varphi(d(T^{p-2}u, T^{p-1}u)) < \ldots < \varphi(d(u, Tu))$$

which is a contradiction. We deduce that $\mu = T^{p-1}u$ is a fixed point of T.

Step 6. Uniqueness of the fixed point of T.
Suppose that there are two distinct points $\mu, \nu \in X$ such that $T\mu = \mu$ and $T\nu = \nu$. Then, $M(\mu, \nu) = \max\{d(\mu, \nu), d(\mu, T\mu), d(\nu, T\nu)\} = d(\mu, \nu)$ and $\varphi(d(\mu, \nu)) > 0$. By (2), we obtain

$$\begin{aligned}\varphi(d(\mu, \nu)) &= \varphi(d(T\mu, T\nu)) \leq \psi(\varphi(M(\mu, \nu)))\\ &= \psi(\varphi(d(\mu, \nu))) < \varphi(d(\mu, \nu))\end{aligned}$$

a contradiction. Thus, T has a unique fixed point. This completes the proof of theorem. □

We illustrate Theorem 2.4 by two examples which are obtained by modifying the one from [1, 10].

Example 2.5. *Let $X = \{a, b, c, e\}$ and we define $d : X \times X \to \mathbb{R}^+$ by $d(a, b) = 0.25$, $d(a, c) = d(b, c) = 0.1$, $d(a, e) = d(b, e) = d(c, e) = 0.2$ and $d(x, x) = 0$ for all $x \in X$.*

Then (X, d) be a generalized metric space, but it is not a metric space, because

$$d(a, b) = 0.25 > d(a, c) + d(b, c) = 0.2.$$

Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(t) = t$ and $\varphi(t) = \frac{3}{4}t$.
We next define $T : X \to X$ by

$$Tx = \begin{cases} c, & \text{if } x \neq e,\\ a, & \text{if } x = e.\end{cases}$$

Then all conditions of Theorem 2.4 are satisfied and c is a unique fixed point of T.

Example 2.6. *Let $X = \{t, 2t, 3t, 4t\}$. Define $d : X \times X \to \mathbb{R}$ as follows:*
$d(t, 2t) = d(2t, t) = 3$,
$d(2t, 3t) = d(3t, 2t) = d(t, 3t) = d(3t, t) = 1$,
$d(t, 4t) = d(4t, t) = d(2t, 4t) = d(4t, 2t) = d(3t, 4t) = d(4t, 3t) = 4$.
Then (X, d) is a complete generalized metric space but (X, d) is not a metric space because it lacks the triangular property:

$$3 = d(t, 2t) > d(t, 3t) + d(3t, 2t) = 1 + 1 = 2.$$

Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(t) = t$ and $\varphi(t) = \frac{1}{2}t$.
Next we define a mapping $T : X \to X$ as follows:

$$Tx = \begin{cases} 3t, & \text{if } x \neq 4t,\\ t, & \text{if } x = 4t.\end{cases}$$

Note that

$$d(T(t), T(2t)) = d(T(t), T(3t)) = d(T(2t), T(3t)) = 0$$

and in all other cases

$$d(Tx, Ty) = 1, d(x, Tx) = d(y, Ty) = 4.$$

Then all conditions of Theorem 2.4 are satisfied and $3t$ is a unique fixed point of T.

3. Periodic and Fixed Point for Generalized (ψ, φ)-Contractions in Generalized Metric Spaces

We assume Ψ the set of functions $\psi : [0, +\infty) \to [0, +\infty)$ to be a weaker Meir-Keeler function satisfying conditions (ψ_0), (ψ_1), (ψ_2) and (ψ_3) from Section 2. Also consider Φ the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ to be a non-decreasing and continuous function satisfying (φ_1) ((φ_2) and (φ_3) are not needed).

Our second main result is the following.

Theorem 3.1. *Let (X, d) be a complete generalized metric space. Suppose that $T : X \to X$ is such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{28}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \tag{29}$$

Then T has a periodic point u in X, that is, there exists $u \in X$ such that $u = T^p u$ for some $p \in \mathbb{N}$. Also if $0 < \psi(t) < t$ for $t > 0$, then there exists a unique fixed point $u \in X$ of T such that $u = Tu$.

Proof. First, it is obvious that $M(x, y) = 0$ if and only if $x = y$ is a fixed point of T. Let $x_0 \in X$ an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \text{for all } n \geq 0. \tag{30}$$

Step 1. We claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{31}$$

Substituting $x = x_n$ and $y = x_{n-1}$ in (28) and using properties of functions ψ and φ, we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \psi(M(x_n, x_{n-1})) - \varphi(M(x_n, x_{n-1})) \\ &\leq \psi(M(x_n, x_{n-1})), \end{aligned} \tag{32}$$

that is,

$$d(x_{n+1}, x_n) \leq \psi(M(x_n, x_{n-1})) \quad \text{for all } n \geq 1.$$

Note that

$$M(x_n, x_{n-1}) = \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}$$

If for some $n \geq 1$, $d(x_n, x_{n-1}) < d(x_n, x_{n+1})$, then $M(x_n, x_{n-1}) = d(x_n, x_{n+1}) > 0$ and so by the condition (φ_1), $\varphi(d(x_{n+1}, x_n)) > 0$; so (32) becomes

$$0 < d(x_{n+1}, x_n) \leq \psi(d(x_{n+1}, x_n)) - \varphi(d(x_{n+1}, x_n)) < \psi(d(x_{n+1}, x_n))$$

a contradiction. Thus, for all $n \geq 1$

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1})) \tag{33}$$

and by the condition (ψ_0), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi(d(x_{n-1}, x_n)) \\ &\leq \psi(\psi(d(x_{n-2}, x_{n-1}))) \\ &= \psi^2(d(x_{n-2}, x_{n-1})) \\ &\;\;\vdots \\ &\leq \psi^n(d(x_0, x_1)). \end{aligned}$$

Since $\{\psi^n(d(x_0,x_1))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function ψ, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq d(x_0,x_1) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(d(x_0,x_1)) < \eta$. Since $\lim_{n\to\infty}\psi^n(d(x_0,x_1)) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \psi^p(d(x_0,x_1)) < \delta + \eta$, for all $p \geq p_0$. Thus, we conclude that $\psi^{p_0+n_0}(d(x_0,x_1)) < \eta$. So we get a contradiction. Therefore $\lim_{n\to\infty}\psi^n(d(x_0,x_1)) = 0$, that is,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

Step 2. We shall prove that

$$\lim_{n\to\infty} d(x_n,x_{n+2}) = 0. \tag{34}$$

Using inequality (28), we also have that for each $n \in \mathbb{N}$,

$$d(x_n,x_{n+2}) = d(Tx_{n-1},Tx_{n+1}) \leq \psi(M(x_{n-1},x_{n+1})) - \varphi(M(x_{n-1},x_{n+1})). \tag{35}$$

From (31), there exists $K > 0$ such that $d(x_n,x_{n+1}) \leq K$ for all $n \in \mathbb{N}$. We prove (34) in each of the two following cases:

• If $d(x_{n-1},x_{n+1}) > K$ for all $n \in \mathbb{N}$, from

$$M(x_{n-1},x_{n+1}) = \max\{d(x_{n-1},x_{n+1}), d(x_{n-1},Tx_{n-1}), d(x_{n+1},Tx_{n+1})\} = d(x_{n-1},x_{n+1})$$

and using inequality (28), we have that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(x_n,x_{n+2}) &= d(Tx_{n-1},Tx_{n+1}) \\ &\leq \psi(M(x_{n-1},x_{n+1})) - \varphi(M(x_{n-1},x_{n+1})) \\ &= \psi(d(x_{n-1},x_{n+1})) - \varphi(d(x_{n-1},x_{n+1})), \end{aligned} \tag{36}$$

that is,

$$d(x_n,x_{n+2}) \leq \psi(d(x_{n-1},x_{n+1})) - \varphi(d(x_{n-1},x_{n+1})), \tag{37}$$

and from (ψ_0), we have

$$\begin{aligned} d(x_n,x_{n+2}) &\leq \psi(d(x_{n-1},x_{n+1})) - \varphi(d(x_{n-1},x_{n+1})) \\ &\leq \psi(d(x_{n-1},x_{n+1})) \\ &\leq \psi(\psi(d(x_{n-2},x_n))) \\ &= \psi^2(d(x_{n-2},x_n)) \\ &\vdots \\ &\leq \psi^n(d(x_0,x_2)). \end{aligned}$$

Since $\{\psi^n(d(x_0,x_2))\}_{n\in\mathbb{N}}$ is decreasing, by the same proof process, we also conclude

$$\lim_{n\to\infty} d(x_n,x_{n+2}) = 0. \tag{38}$$

• If, for some $n \in \mathbb{N}$, we have $d(x_{n-1},x_{n+1}) \leq K$ and $d(x_n,x_{n+2}) > K$, then from (28) and properties of function ψ,

$$\begin{aligned} d(x_n,x_{n+2}) &= d(Tx_{n-1},Tx_{n+1}) \\ &\leq \psi(M(x_{n-1},x_{n+1})) - \varphi(M(x_{n-1},x_{n+1})) \\ &\leq \psi(M(x_{n-1},x_{n+1})) \\ &\leq \psi(K) \\ &\leq K, \end{aligned} \tag{39}$$

that is, $d(x_n, x_{n+2}) \leq K$, a contradiction. Then $d(x_n, x_{n+2}) > K$ or $d(x_n, x_{n+2}) \leq K$ for all $n \in \mathbb{N}$ and in both cases the sequence $\{d(x_n, x_{n+2})\}$ is bounded. Now, if (12) does not hold; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, x_{n_k+2}) \to r > 0$. From

$$d(x_{n_k-1}, x_{n_k+1}) \leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k+2}) + d(x_{n_k+1}, x_{n_k+2})$$

and

$$d(x_{n_k}, x_{n_k+2}) \leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k-1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k+2})$$

we deduce that

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{n_k+1}) = r.$$

Substituting $x = x_{n_k}$ and $y = x_{n_k+2}$ in (2), we have

$$\begin{aligned} d(x_{n_k}, x_{n_k+2}) &= d(Tx_{n_k-1}, Tx_{n_k+1}) \\ &\leq \psi(M(x_{n_k-1}, x_{n_k+1})) - \varphi(M(x_{n_k-1}, x_{n_k+1})); \end{aligned} \tag{40}$$

where $M(x_{n_k-1}, x_{n_k+1}) = \max\{d(x_{n_k-1}, x_{n_k+1}), d(x_{n_k-1}, Tx_{n_k-1}), d(x_{n_k+1}, Tx_{n_k+1})\}$. So

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{n_k+1}) = r.$$

From (40) as $k \to \infty$ and by using the condition (ψ_3) of the function ψ, we get $r \leq r - \varphi(r)$. So $\varphi(r) = 0$. Thus $r = 0$, a contradiction.

Step 3. We claim that $\{x_n\}$ is a g.m.s. Cauchy sequence.

We claim that the following result holds:

Claim: For each $\epsilon > 0$, there is $n_0(\epsilon) \in \mathbb{N}$ such that for all $p, q \geq n_0(\epsilon)$,

$$d(x_p, x_q) < \epsilon, \tag{41}$$

We shall prove (41) by negation. Suppose, on the contrary, that (41) is false. Then there exists some $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying:

$$d(x_{p_n}, x_{q_n}) \geq \epsilon.$$

Now, we let $n > m$. Then corresponding to $q_n \geq n$ use, we can choose p_n in such a way, that it is the smallest integer with $p_n > q_n \geq n$ satisfying $d(x_{q_n}, x_{p_n}) \geq \epsilon$. Therefore $d(x_{q_n}, x_{p_n-1}) < \epsilon$ and

$$\begin{aligned} \epsilon &\leq d(x_{q_n}, x_{p_n}) \\ &\leq d(x_{q_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{p_n-2}) + d(x_{p_n-2}, x_{p_n}) \\ &< \epsilon + d(x_{p_n-1}, x_{p_n-2}) + d(x_{p_n-2}, x_{p_n}). \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(x_{q_n}, x_{p_n}) = \epsilon.$$

On the other hand, we can conclude that

$$\begin{aligned} \epsilon &\leq d(x_{q_n}, x_{p_n}) \\ &\leq d(x_{q_n}, x_{q_n+1}) + d(x_{q_n+1}, x_{p_n+1}) + d(x_{p_n+1}, x_{p_n}) \\ &\leq d(x_{q_n}, x_{q_n+1}) + d(x_{q_n+1}, x_{q_n}) + d(x_{q_n}, x_{p_n}) + d(x_{p_n}, x_{p_n+1}) + d(x_{p_n+1}, x_{p_n}). \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(x_{q_n+1}, x_{p_n+1}) = \epsilon.$$

Using (28)

$$d(x_{q_n+1}, x_{p_n+1}) = d(fx_{q_n}, fx_{p_n}) \quad \leq \quad \psi(M(x_{q_n}, x_{p_n})) - \varphi(M(x_{q_n}, x_{p_n})),$$

where $M(x_{q_n}, x_{p_n}) = \max\{d(x_{q_n}, x_{p_n}), d(x_{q_n}, Tx_{q_n}), d(x_{p_n}, Tx_{p_n})\}$. Passing to the limit as $n \to \infty$, by using the condition (ψ_3) of the function ψ, we obtain that

$$\epsilon \leq \epsilon - \varphi(\epsilon),$$

and consequently, $\varphi(\epsilon) = 0$. By the definition of the function φ, we get $\epsilon = 0$ which is contraction. Hence, $\{x_n\}$ is g.m.s Cauchy.

Step 4. We claim that T has a periodic point.

We argue by contradiction. Assume that T has no periodic point. Then $\{x_n\}$ is a sequence of distinct points, that is, $x_n \neq x_m$ for all $m \neq n$. By Step 2, since (X, d) is a complete g.m.s, there exists $u \in X$ such that $x_n \to u$. Applying (28) with $x = x_n$ and $y = u$, we obtain

$$d(x_{n+1}, Tu) = d(Tx_n, Tu) \leq \psi(M(x_n, u)) - \varphi(M(x_n, u)) \leq \psi(M(x_n, u)), \tag{42}$$

that is,

$$d(x_{n+1}, Tu) \leq \psi(M(x_n, u))$$

where

$$M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}.$$

Since $\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, so we obtain that

$$\lim_{n\to\infty} M(x_n, u) = d(u, Tu). \tag{43}$$

Next we shall find a contradiction of the fact that T has no periodic point in each of the two following cases:

- If for all $n \geq 2$, $x_n \neq u$ and $x_n \neq Tu$.
  Taking the $\lim_{n\to\infty}$ in (42) and using (43), and the properties of ψ and φ, we obtain

$$d(u, Tu) \leq \psi(d(u, Tu)) - \varphi(d(u, Tu))$$

which implies that $d(u, Tu) = 0$, so $u = Tu$, that is, u is a fixed point of T, so u is a periodic point of T. It contradicts the fact that T has no periodic point. Therefore, there exists $u \in X$ such that $u = T^p(u)$ for some $p \in \mathbb{N}$. So T has a periodic point in X.

- Let for some $q \geq 2$, $x_q = u$ or $x_q = Tu$. Since T has no periodic point, then obviously $u \neq x_0$. Indeed, if $x_q = u = x_0$ so $T^q x_0 = x_0$, i.e, x_0 is a periodic point of T, while if $x_q = Tu$ and $x_0 = u$ so $Tx_0 = Tu = x_q = T^q x_0 = T^{q-1}(Tx_0)$, i.e, Tx_0 is a periodic point of T.

For all $n \geq 0$, we have

$$d(T^n u, u) = d(T^n x_q, u) = d(x_{n+q}, u) \quad \text{or} \quad d(T^n u, u) = d(T^{n-1} Tu, u) = d(T^{n-1} x_q, u) = d(x_{n+q-1}, u).$$

In the two precedent identities, the integer $q \geq 2$ is fixed and so $\{x_{n+q}\}$ and $\{x_{n+q-1}\}$ are subsequences from $\{x_n\}$ and since $\{x_n\}$ g.m.s. converges to u in (X, d). On the other hand, we can easily see that $x_n \neq x_m$ for whenever $m \neq n$. Indeed if for some $m, n \in \mathbb{N}$, $x_n = x_m$ and $n < m$, then by (33), we have

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) \leq \psi^{m-n} d(x_n, x_{n+1}) < d(x_n, x_{n+1}), \tag{44}$$

a contradiction. So by Lemma 1.6 and (44), the two subsequences g.m.s. converge to same unique limit u, i.e,

$$\lim_{n\to\infty} d(x_{n+q}, u) = \lim_{n\to\infty} d(x_{n+q-1}, u) = 0.$$

Thus,

$$\lim_{n\to\infty} d(T^n u, u) = 0. \tag{45}$$

By quadrilateral inequality, then by (45),

$$\lim_{n\to\infty} d(T^{n+2} u, u) = 0. \tag{46}$$

On the other hand, since T has no periodic point, it follows that

$$T^s u \neq T^r u \quad \text{for any } s, r \in \mathbb{N}, \quad s \neq r. \tag{47}$$

Using (47) and the quadrilateral inequality, we may write

$$|d(T^{n+1}u, Tu) - d(u, Tu)| \leq d(T^{n+1}u, T^{n+2}u) + d(T^{n+2}u, u).$$

Passing to the limit as $n \to \infty$ in the above limit and proceeding as (31) (since the point x_0 is arbitrary), using (46) we obtain

$$\lim_{n\to\infty} d(T^{n+1}u, Tu) = d(u, Tu). \tag{48}$$

Now, by (28)

$$d(T^{n+1}u, Tu) \leq \psi(M(T^n u, u)) - \varphi(M(T^n u, u)) \tag{49}$$

where

$$M(T^n u, u) = \max\{d(T^n u, u), d(T^n u, T^{n+1}u), d(u, Tu)\} \to d(u, Tu) \text{ as } n \to \infty.$$

Passing to the limit as $n \to \infty$ in (49) and using (48) and the above limit, we get that

$$d(u, Tu) \leq \psi(d(u, Tu)) - \varphi(d(u, Tu))$$

which holds only if $d(u, Tu) = 0$, i.e, $Tu = u$, which implies that u is a periodic point of T. This contradicts the fact that T has no periodic point.
Consequently T admits a periodic point, that is, there exists $u \in X$ such that $u = T^p u$ for some $p \geq 1$.

Step 5. Existence of a fixed point of T.
If $p = 1$, then $u = Tu$, that is, u is a fixed point of T. Suppose now that $p > 1$. We will prove that $\mu = T^{p-1}u$ is a fixed point of T. Suppose that it is not the case, that is, $T^{p-1}u \neq T^p u$. Then $d(T^{p-1}u, T^p u) > 0$ and $\psi(d(T^{p-1}u, T^p u)) > 0$, which implies that $\psi(M(T^{p-1}u, T^p u)) > 0$. Now, using inequality (28), we obtain

$$\begin{aligned} d(u, Tu) &= d(T^p u, T^{p+1}u) \\ &= d(T(T^{p-1}u), T(T^p u)) \\ &\leq \psi(M(T^{p-1}u, T^p u)) - \varphi(M(T^{p-1}u, T^p u)) \\ &< \psi(M(T^{p-1}u, T^p u)) \end{aligned}$$

which by the property of ψ implies

$$d(u, Tu) < M(T^{p-1}u, T^p u), \tag{50}$$

where

$$\begin{aligned} M(T^{p-1}u, T^pu) &= \max\{d(T^{p-1}u, T^pu), d(T^{p-1}u, T^pu), d(T^pu, T^{p+1}u)\} \\ &= \max\{d(T^{p-1}u, T^pu), d(u, Tu)\} = d(T^{p-1}u, T^pu) \end{aligned}$$

because otherwise we get a contradiction with (50). Thus, (50) becomes

$$d(u, Tu) < d(T^{p-1}u, T^pu). \tag{51}$$

Again, using (28) we have

$$\begin{aligned} d(T^{p-1}u, T^pu) =& d(T(T^{p-2}u), T(T^{p-1}u)) \\ \le& \psi(M(T^{p-2}u, T^{p-1}u)) - \varphi(M(T^{p-2}u, T^{p-1}u)) \\ <& \psi(M(T^{p-2}u, T^{p-1}u)). \end{aligned}$$

Again, this implies that

$$d(T^{p-1}u, T^pu) < M(T^{p-2}u, T^{p-1}u), \tag{52}$$

where

$$\begin{aligned} M(T^{p-2}u, T^{p-1}u) &= \max\{d(T^{p-2}u, T^{p-1}u), d(T^{p-2}u, T^{p-1}u), d(T^{p-1}u, T^pu)\} \\ &= \max\{d(T^{p-2}u, T^{p-1}u), d(T^{p-1}u, T^pu)\} = d(T^{p-2}u, T^{p-1}u) \end{aligned}$$

because of (52). Thus, from (52)

$$d(T^{p-1}u, T^pu) < d(T^{p-2}u, T^{p-1}u). \tag{53}$$

Continuing this process as (51) and (53), we find that

$$d(u, Tu) < d(T^{p-1}u, T^pu) < d(T^{p-2}u, T^{p-1}u) < \ldots < d(u, Tu)$$

which is a contradiction. We deduce that $\mu = T^{p-1}u$ is a fixed point of T.

Step 6. Uniqueness of the fixed point of T.
Suppose that there are two distinct points $\mu, \nu \in X$ such that $T\mu = \mu$ and $T\nu = \nu$. Then, $M(\mu, \nu) = \max\{d(\mu, \nu), d(\mu, T\mu), d(\nu, T\nu)\} = d(\mu, \nu)$ and $\psi(d(\mu, \nu)) > 0$. By (28), we obtain

$$\begin{aligned} d(\mu, \nu) =& d(T\mu, T\nu) \le \psi(M(\mu, \nu)) - \varphi(M(\mu, \nu)) \\ =& \psi(d(\mu, \nu)) - \varphi(d(\mu, \nu)) < \psi(d(\mu, \nu)) \end{aligned}$$

a contradiction. Thus, T has a unique fixed point. This completes the proof of theorem. □

Remark 3.2. *Theorem 2.4 extends Theorem 2.1 and Theorem 2.2 of Chen and Sun [5] while Theorem 3.1 extends Theorem 3.1 of Lakzian and Samet [15], and Theorem 2.2 of Shatanawi et al. [21].*

We illustrate Theorem 3.1 by an example which is obtained by modifying the one from [6].

Example 3.3. *Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ is a constant, and we define $d : X \times X \to [0, 1)$ by*

(1) $d(x, x) = 0$, $d(x, y) = d(y, x)$, for all $x, y \in X$;

(2) $d(t, 2t) = 3a$, $d(t, 3t) = d(2t, 3t) = a$ $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = 2a$; $d(t, 5t) = d(3t, 5t) = a$, and $d(2t, 5t) = d(4t, 5t) = 2a$, where $a > 0$ is a constant.

Then (X, d) be a generalized metric space, but it is not a metric space, because

$$d(t, 2t) = 3a > d(t, 3t) + d(3t, 2t) = 2a.$$

If $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(t) = 4t$ and $\varphi(t) = 3t$.

We next define $f : X \to X$ by

$$Tx = \begin{cases} 3t, & \text{if } x \neq 4t, \\ t, & \text{if } x = 4t. \end{cases}$$

Then all conditions of Theorem 3.1 are satisfied and $3t$ is a unique fixed point of T.

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