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# Rough Prime Bi-Γ-Hyperideals and Fuzzy Prime Bi-Γ-Hyperideals of Γ-Semihypergroups

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**Abstract.** In this paper, we introduce the concept of prime bi-Γ-hyperideals, rough prime bi-Γ-hyperideals and fuzzy prime bi-Γ-hyperideals of Γ-semihypergroups. We prove that the lower approximation of a prime bi-Γ-hyperideal is a prime bi-Γ-hyperideal and the upper approximation of a prime bi-Γ-hyperideal is a prime bi-Γ-hyperideal. Also the rough set theory is applied to prime bi-Γ-hyperideals in the quotient Γ-semihypergroups. In the end, the notion of fuzzy prime bi-Γ-hyperideals of Γ-semihypergroups has been introduced, and we proved that a bi-Γ-hyperideal $B$ of a Γ-semihypergroup $H$ is prime (resp., strongly prime) if and only if the characteristic function $\chi_B$ of $B$ is a fuzzy prime (resp., fuzzy strongly prime) bi-Γ-hyperideal of $H$.

## 1. Introduction

In 1981 and later in 1986, Sen and Saha [35, 36] defined the notion of a Γ-semigroup as a generalization of a semigroup and ternary semigroup. One can see that Γ-semigroups are generalizations of semigroups. Many results of semigroups have been extended to Γ-semigroups directly and via operator semigroups of a Γ-semigroup and a lot of results on Γ-semigroups are published by a lot of mathematicians. For instance, Chattopadhyay [7], Chinram and Jirojkul [8], Chinram and Siammai [9] and Hila [19, 20].

Hyperstructure theory was introduced in 1934, when Marty [30] defined hypergroups, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science and they are studied in many countries of the world. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory, see [10], [11], [12], [13], [39].

In [18], Heidari et al. introduced the notion of Γ-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. They presented many interesting

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examples and obtained a several characterizations of Γ-semihypergroups. In [3], the notion of a Γ-hyperideal of a Γ-semihypergroup is introduced. A Γ-hyperideal of a Γ-semihypergroup is a generalization of an ideal of a semigroup, a generalization of a hyperideal of a semihypergroup and a generalization of a Γ-ideal of a Γ-semigroup. Recently, a lot of other papers [1], [2], [22], [23], [31] studied the structure of Γ-semihypergroups.

The notion of rough sets was introduced by Pawlak in his paper [32]. The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy, or incomplete information. It is turning out to be methodologically sig- nificant to the domains of artificial intelligence and cognitive sciences, especially in the representation of and reasoning with vague and/or imprecise knowledge, data classification, data analysis, machine learning, and knowledge discovery [33], [38]. In connection with algebraic structures, Biswas and Nanda [6] gave the notion of rough subgroups, and Kuroki and Wang [27] introduced the notion of a rough subgroup with respect to a normal subgroup of a group, and gave some properties of the lower and the upper approximations in a group. Kuroki, in [26], introduced the notion of a rough ideal in a semigroup. Jun in [25], applied the roughness to Γ-semigroups. Anvariyeh et al. [4], introduced Pawlak's approximations in Γ-semihypergroups. See also [5, 21, 42–47].

The concept of a fuzzy set, introduced by Zadeh in his classic paper [48], provides a natural framework for generalizing some of the notions of classical algebraic structures. Fuzzy semigroups have been first considered by Kuroki [28]. After the introduction of the concept of fuzzy sets by Zadeh, several researches conducted the researches on the generalizations of the notions of fuzzy sets with huge applications in computer, logics and many branches of pure and applied mathematics. Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. In 1971, Rosenfeld [34] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. A recent book [10] contains a wealth of applications. In [16], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup. A several papers are written on fuzzy sets in several algebraic hyperstructures. The relationships between the fuzzy sets and algebraic hyperstructures have been considered by Corsini, Davvaz, Leoreanu, Zhan, Zahedi, Ameri, Cristea and many other researchers, see [14]. Yaqoob et al. [40, 41], introduced the notion of rough prime bi-ideals and rough fuzzy prime bi-ideals in semigroups. Recently, Davvaz and Fotea introduced structures of fuzzy Γ-hyperideals of Γ-semihypergroups in [15]. In [24], Hila and Gilani obtained some fundamental relationships of a Γ-semihypergroup and its operator semihypergroup in terms of fuzzy subsets, fuzzy hyperideals and fuzzy hyperideal extension highlighting the effectiveness of operator semihypergroups in the study of Γ-semihypergroups in terms of fuzzy subsets.

The notion of bi-ideal was given in 1952 by Good and Hughes [17]. Since then, it was studied in a lot of papers in different algebraic structures. The notion of prime bi-ideals of groupoids was studied by Lee [29]. Further, many authors studied the prime bi-ideals in different structures. Also Shabir et al. [37], studied prime, strongly prime and semiprime bi-ideals of semigroups. In [2], the prime, strongly prime, semiprime, strongly irreducible and irreducible bi-Γ-hyperideals of Γ-semihypergroups are introduced and studied.

In this paper, we introduce the concept of prime bi-Γ-hyperideals, rough prime bi-Γ-hyperideals and fuzzy prime bi-Γ-hyperideals of Γ-semihypergroups. We prove that the lower approximation of a prime bi-Γ-hyperideal is a prime bi-Γ-hyperideal and the upper approximation of a prime bi-Γ-hyperideal is a prime bi-Γ-hyperideal. Also the rough set theory is applied to prime bi-Γ-hyperideals in the quotient Γ-semihypergroups. Also, the notion of fuzzy prime bi-Γ-hyperideals of Γ-semihypergroups is introduced, and we prove among others that a bi-Γ-hyperideal $B$ of a Γ-semihypergroup $H$ is prime (resp. strongly prime) if and only if the characteristic function $\chi_B$ of $B$ is a fuzzy prime (resp. fuzzy strongly prime) bi-Γ-hyperideal of $H$.

## 2. Γ-Semihypergroups

In this section, we recall certain definitions and results about Γ-semihypergroups, needed for our purpose.

**Definition 2.1.** *[36] Let $S = \{a, b, c, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then $S$ is called a $\Gamma$-semigroup if there exists a mapping $S\times\Gamma\times S \longrightarrow S$ written as $(a, \gamma, b) \longrightarrow a\gamma b$ satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. Let $K$ be a non-empty subset of $S$. Then $K$ is called a sub $\Gamma$-semigroup of $S$, if $a\gamma b \in K$ for all $a, b \in S$ and $\gamma \in \Gamma$.*

**Definition 2.2.** *A map $\circ : S \times S \rightarrow \mathcal{P}^*(S)$ is called hyperoperation or join operation on the set $S$, where $S$ is a non-empty set and $\mathcal{P}^*(S) = \mathcal{P}(S)\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of $S$. A hypergroupoid is a set $S$ with together a (binary) hyperoperation. A hypergroupoid $(S, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$, $\forall x, y, z \in S$, is called a semihypergroup.*

Let $A$ and $B$ be two non-empty subsets of $S$. Then we define

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, a \circ A = \{a\} \circ A \text{ and } a \circ B = \{a\} \circ B.$$

**Definition 2.3.** *[3] Let $S$ and $\Gamma$ be two non-empty sets. Then $S$ is called a* $\Gamma$-semihypergroup *if every $\gamma \in \Gamma$ is a hyperoperation on $S$, i.e., $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have $x\alpha(y\beta z) = (x\alpha y)\beta z$.*

If every $\gamma \in \Gamma$ is an operation, then $S$ is a $\Gamma$-semigroup. If $(S, \gamma)$ is a hypergroup for every $\gamma \in \Gamma$, then $S$ is called a $\Gamma$-hypergroup.

Let $A$ and $B$ be two non-empty subset of $S$. Then we define

$$A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B = \bigcup \{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}.$$

Let $(S, \circ)$ be a semihypergroup and let $\Gamma = \{\circ\}$. Then $S$ is $\Gamma$-semihypergroup. So every semihypergroup is $\Gamma$-semihypergroup.

Let $S$ be a $\Gamma$-semihypergroup and $\gamma \in \Gamma$. A non-empty subset $A$ of $S$ is called a sub $\Gamma$-semihypergroup of $S$ if $x\gamma y \subseteq A$ for every $x, y \in A$. A $\Gamma$-semihypergroup $S$ is called *commutative* if for all $x, y \in S$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$.

**Example 2.4.** *[3] Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$, we define $\gamma : S \times S \longrightarrow \wp^*(S)$ by $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$. Then $\gamma$ is a hyperoperation. For every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have $(x\alpha y)\beta z = \left[0, \frac{xyz}{\alpha\beta}\right] = x\alpha(y\beta z)$. This mean that $S$ is a $\Gamma$-semihypergroup.*

**Example 2.5.** *[3] Let $(S, \circ)$ be a semihypergroup and $\Gamma$ be a non-empty subset of $S$. We define $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then $S$ is a $\Gamma$-semihypergroup.*

**Definition 2.6.** *[3] A non-empty subset $A$ of a $\Gamma$-semihypergroup $S$ is a right (left) $\Gamma$-hyperideal of $S$ if $A\Gamma S \subseteq A$ ($S\Gamma A \subseteq A$), and is a $\Gamma$-hyperideal of $S$ if it is both a right and a left $\Gamma$-hyperideal.*

**Definition 2.7.** *[3] A non-empty subset $B$ of a $\Gamma$-semihypergroup $S$ is called bi-$\Gamma$-hyperideal of $S$ if*
(1) $B\Gamma B \subseteq B$,
(2) $B\Gamma S\Gamma B \subseteq B$.

A bi-$\Gamma$-hyperideal $B$ of $\Gamma$-semihypergroups $S$ is proper if $B \neq S$.

**Example 2.8.** *[3] Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. Then with hyperoperation $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$ is $\Gamma$-semihypergroup. Let $t \in [0, 1]$ and set $[0, t]$. Then $T$ is left (right, bi)-$\Gamma$-hyperideal of $S$.*

## 3. Prime Bi-Γ-Hyperideals

In this section, we will discuss some properties of prime bi-Γ-hyperideals in a Γ-semihypergroup. $S$ will denote a Γ-semihypergroup throughout in this paper.

**Definition 3.1.** *A bi-*Γ*-hyperideal $B$ of a* Γ*-semihypergroup $S$ is called a prime bi-*Γ*-hyperideal of $S$ if for $x, y \in S$, $x\Gamma S\Gamma y \subseteq B$ (or $x\beta a\gamma y \subseteq B$, for all $a \in S$ and $\beta, \gamma \in \Gamma$) implies $x \in B$ or $y \in B$. A bi-*Γ*-hyperideal $B$ of a* Γ*-semihypergroup $S$ is called semiprime if for $x \in S$, $x\Gamma S\Gamma x \subseteq B$ (or $x\beta a\gamma x \subseteq B$, for all $a \in S$ and $\beta, \gamma \in \Gamma$) implies $x \in B$.*

Here we will denote $(x)_l = \{x\} \cup S\Gamma x$ (resp. $(x)_r = \{x\} \cup x\Gamma S$) is the principal left (resp. right) Γ-hyperideal generated by $x$.

**Theorem 3.2.** *A bi-*Γ*-hyperideal $B$ of a* Γ*-semihypergroup $S$ is prime if and only if for a right* Γ*-hyperideal $R$ and a left* Γ*-hyperideal $L$ of $S$, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

*Proof.* Suppose that $R\Gamma L \subseteq B$ for a right Γ-hyperideal $R$ and a left Γ-hyperideal $L$ of $S$ and $R \nsubseteq B$. Then there exists $x \in R \backslash B$. Let $y \in L$. Then $x\Gamma S\Gamma y \subseteq R\Gamma S\Gamma L \subseteq R\Gamma L \subseteq B$. Since $B$ is a prime bi-Γ-hyperideal and $x \notin B$, we have $y \in B$. Thus $L \subseteq B$.

Conversely, suppose that for a right Γ-hyperideal $R$ and a left Γ-hyperideal $L$ of $S$, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$. If $x\Gamma S\Gamma y \subseteq B$ for $x, y \in S$, then $(x\Gamma S)\Gamma(S\Gamma y) \subseteq x\Gamma S\Gamma y \subseteq B$. Since $x\Gamma S$ is a right Γ-hyperideal and $S\Gamma y$ is a left Γ-hyperideal of $S$, by hypothesis, $x\Gamma S \subseteq B$ or $S\Gamma y \subseteq B$.

If $x\Gamma S \subseteq B$, $x^2 \subseteq x\Gamma S \subseteq B$. Thus

$$(x)_r\Gamma(x)_l = (\{x\} \cup x\Gamma S)\Gamma(\{x\} \cup S\Gamma x) = x^2 \cup x\Gamma S\Gamma x \cup x\Gamma S^2\Gamma x \subseteq x^2 \cup x\Gamma S \subseteq B.$$

Since $(x)_r$ is a right Γ-hyperideal and $(x)_l$ is a left Γ-hyperideal of $S$ containing $x$, $(x)_r \subseteq B$ or $(x)_l \subseteq B$ by hypothesis. Hence $x \in B$. If $S\Gamma y \subseteq B$, $y \in B$ by the similar method. Therefore $B$ is prime. □

**Proposition 3.3.** *If a bi-*Γ*-hyperideal $B$ of a* Γ*-semihypergroup $S$ is prime, then $B$ is a left or a right* Γ*-hyperideal of $S$.*

*Proof.* Since $B\Gamma S$ is a right Γ-hyperideal of $S$ and $S\Gamma B$ a left Γ-hyperideal of $S$ such that $(B\Gamma S)\Gamma(S\Gamma B) \subseteq B\Gamma S\Gamma B \subseteq B$, we get $B\Gamma S \subseteq B$ or $S\Gamma B \subseteq B$ by Theorem 3.2. Hence $B$ is a left hyperideal or a right hyperideal of $S$. □

Now let $L_B$, $R_B$, $I_L$ and $I_R$ for a bi-Γ-hyperideal $B$ of a Γ-semihypergroup $S$ as follows:

$$L_B = \{x \in B : S\Gamma x \subseteq B\} \text{ and } R_B = \{x \in B : x\Gamma S \subseteq B\}$$

$$I_L = \{y \in L_B : y\Gamma S \subseteq L_B\} \text{ and } I_R = \{x \in R_B : S\Gamma y \subseteq R_B\}.$$

Then we have the following results.

**Proposition 3.4.** *Let $B$ be a prime bi-*Γ*-hyperideal of a* Γ*-semihypergroup $S$. Then $L_B$ (resp. $R_B$) is a left (resp. right)* Γ*-hyperideal of $S$ contained in $B$ if $L_B$ (resp. $R_B$) is non-empty.*

*Proof.* Let $x \in L_B \neq \emptyset$ and $z \in S$. Then for $\gamma \in \Gamma$, $z\gamma x \subseteq S\Gamma x \subseteq B$. Since $S\Gamma z\Gamma x \subseteq S^2\Gamma x \subseteq S\Gamma x \subseteq B$, we have $z\gamma x \subseteq L_B$. Thus $S\Gamma L_B \subseteq L_B$, so $L_B$ is a left Γ-hyperideal of $S$.

Also, by the similar method we can prove $R_B$ is a right Γ-hyperideal of $S$ contained in $B$. □

**Theorem 3.5.** *Let $B$ be a bi-*Γ*-hyperideal of a* Γ*-semihypergroup $S$. Then $I_L$ (resp. $I_R$) is the largest* Γ*-hyperideal of $S$ contained in $B$ if $I_L$ (resp. $I_R$) is non-empty. Furthermore $I_L$ coincides with $I_R$.*

*Proof.* Let $x \in I_L$. Then $x\Gamma S \subseteq L_B$. Thus for any $z \in S$ and $\gamma \in \Gamma$, $x\gamma z \subseteq x\Gamma S \subseteq L_B$. Hence $x\Gamma z\Gamma S \subseteq x\Gamma S^2 \subseteq x\Gamma S \subseteq L_B$. Therefore $x\gamma z \subseteq I_L$, and so $I_L\Gamma G \subseteq I_L$. So $I_L$ is a right $\Gamma$-hyperideal of $S$. Since $I_L \subseteq L_B \subseteq B$, we have $x \in L_B$ and $x \in B$. Thus $S\Gamma x \subseteq B$ and $S\Gamma z\Gamma x \subseteq S^2\Gamma x \subseteq S\Gamma x \subseteq B$ for any $z \in G$. Hence $z\gamma x \subseteq L_B$. Since $L_B$ is a left $\Gamma$-hyperideal of $S$ by Proposition 3.4, and $x\Gamma G \subseteq L_B$, we get $z\Gamma x\Gamma S \subseteq S\Gamma L_B \subseteq L_B$. Therefore $z\gamma x \subseteq I_L$, and so $S\Gamma I_L \subseteq I_L$. So $I_L$ is a left $\Gamma$-hyperideal of $S$. It follows that $I_L$ is a $\Gamma$-hyperideal of $S$ contained in $B$.

Let $A$ be any $\Gamma$-hyperideal of $S$ contained in $B$. Then $S\Gamma A \subseteq A \subseteq B$, and so $A \subseteq L_B$. Since $A\Gamma S \subseteq A \subseteq L_B$, we get $A \subseteq I_L$. Therefore $I_L$ is the largest $\Gamma$-hyperideal of $S$ contained in $B$.

Also we can prove that $I_R$ is the largest $\Gamma$-hyperideal of $S$ contained in $B$ by the similar method. Furthermore, since $I_L$ and $I_R$ are the largest $\Gamma$-hyperideals of $S$ contained in $B$, $I_L$ coincides with $I_R$. □

**Remark 3.6.** *By Theorem 3.5 we denote $I_B$ as $I_B = I_L = I_R$.*

**Proposition 3.7.** *Let $S$ be a $\Gamma$-semihypergroup. If $B$ is a prime bi-$\Gamma$-hyperideal of $S$, then $I_B$ is a prime $\Gamma$-hyperideal of $S$ contained in $B$.*

*Proof.* Let $B$ be a bi-$\Gamma$-hyperideal of $S$. Then by Theorem 3.5, $I_B$ is a $\Gamma$-hyperideal of $S$. Let us suppose that $X\Gamma Y \subseteq I_B$ for any $\Gamma$-hyperideal $X, Y$ of $S$. Since $X$ is a left $\Gamma$-hyperideal, $Y$ a right $\Gamma$-hyperideal of $S$ and $I_B \subseteq L_B \subseteq B$, by Theorem 3.2, we get $X \subseteq B$ or $Y \subseteq B$. Since $I_B$ is the largest $\Gamma$-hyperideal contained in $B$, we get $X \subseteq I_B$ or $Y \subseteq I_B$. It follows that $I_B$ is a prime $\Gamma$-hyperideal of $S$. □

**Corollary 3.8.** *Let $S$ be a $\Gamma$-semihypergroup. If $B$ is a semi-prime bi-$\Gamma$-hyperideal of $S$, then $I_B$ is a semi-prime $\Gamma$-hyperideal of $S$ if $I_B$ is non-empty.*

**Proposition 3.9.** *If a bi-$\Gamma$-hyperideal $B$ of a $\Gamma$-semihypergroup $S$ is semiprime, then for any left (resp. right) $\Gamma$-hyperideal $L$ (resp. $R$) of $S$, $L^2 \subseteq B$ implies $L \subseteq B$ (resp. $R^2 \subseteq B$ implies $R \subseteq B$).*

*Proof.* Suppose that $L^2 \subseteq B$ for a left $\Gamma$-hyperideal $L$ of $S$. If $L \nsubseteq B$, then there exists $x \in L\backslash B$. Since $x\Gamma S\Gamma x \subseteq L\Gamma S\Gamma L \subseteq L^2 \subseteq B$ and $B$ is semiprime, we get $x \in B$. It is impossible. Hence $L \subseteq B$. We can prove that if $R^2 \subseteq B$, then $R \subseteq B$ by the similar method. □

A non-empty subset $A$ of a $\Gamma$-semihypergroup $S$ is called quasi-$\Gamma$-hyperideal of $S$ if $A\Gamma S \cap S\Gamma A \subseteq A$.

**Proposition 3.10.** *If a bi-$\Gamma$-hyperideal $B$ of a $\Gamma$-semihypergroup $S$ is semiprime, then $B$ is a quasi-$\Gamma$-hyperideal of $S$.*

*Proof.* Let $y \in B\Gamma S \cap S\Gamma B$. Then $y\Gamma S\Gamma y \subseteq (B\Gamma S)\Gamma S\Gamma(S\Gamma B) \subseteq B\Gamma S\Gamma B \subseteq B$. Since $B$ is semiprime, we have $y \in B$. Thus $B\Gamma S \cap S\Gamma B \subseteq B$. Therefore $B$ is a quasi-$\Gamma$-hyperideal. □

**Proposition 3.11.** *Let $S$ be a $\Gamma$-semihypergroup. A prime bi-$\Gamma$-hyperideal $B$ of $S$ is a prime one-sided $\Gamma$-hyperideal.*

*Proof.* It will be sufficient to show that $B$ is a one-sided $\Gamma$-hyperideal. So, let us suppose that $B$ is not a one-sided $\Gamma$-hyperideal of $S$. Then $B\Gamma S \nsubseteq B$ and $S\Gamma B \nsubseteq S$. Since $B$ is prime, $(B\Gamma S)\Gamma S\Gamma(S\Gamma B) \nsubseteq S$, and since $B$ is a bi-$\Gamma$-hyperideal of $S$, $B\Gamma S\Gamma B \subseteq B$, then $(B\Gamma S)\Gamma S\Gamma(S\Gamma B) \subseteq B$, a contradiction. Hence, $B\Gamma S \subseteq B$ or $S\Gamma B \subseteq B$, that is, $B$ is a one-sided $\Gamma$-hyperideal of $S$. □

For a bi-$\Gamma$-hyperideal $B$, let us put

$$J(B) = \{x \in S | S\Gamma x\Gamma S \subseteq B\}.$$

It is not difficult to show that $J(B)$ is the (unique) largest two-sided $\Gamma$-hyperideal contained in $B$.

**Proposition 3.12.** *Let $S$ be a $\Gamma$-semihypergroup. If the bi-$\Gamma$-hyperideal $B$ is prime, then $J(B)$ is a prime $\Gamma$-hyperideal.*

*Proof.* Let $B$ be prime and suppose that $I_1\Gamma I_2 \subseteq J(B)$ for (two-sided) $\Gamma$-hyperideals $I_1$ and $I_2$. Then, since $I_1\Gamma I_2 \subseteq B$, by Theorem 3.1, $I_1 \subseteq B$ or $I_2 \subseteq B$. Now $J(B)$ is the largest $\Gamma$-hyperideal in $B$, so this implies $I_1 \subseteq J(B)$ or $I_2 \subseteq J(B)$. Therefore $J(B)$ is prime. □

## 4. Rough Prime Bi-Γ-Hyperideals in Γ-Semihypergroups

In what follows, let $S$ denote a Γ-semihypergroup unless otherwise specified.

Let $S$ be a Γ-semihypergroup and $\gamma \in \Gamma$. An equivalence relation $\rho$ on $S$ is said to be regular on $S$ if for every $x \in S$, we have

$$(a, b) \in \rho \text{ implies } (a\gamma x, b\gamma x) \in \rho \text{ and } (x\gamma a, x\gamma b) \in \rho.$$

If $\rho$ is a regular relation on $S$, then, for every $x \in S$, $[x]_\rho$ stands for the class of $x$ with the represent $\rho$. A regular relation $\rho$ on $S$ is called complete if $[a]_\rho\gamma[b]_\rho = [a\gamma b]_\rho$ for all $a, b \in S$ and $\gamma \in \Gamma$. In addition, $\rho$ on $S$ is called congruence if, for every $(a, b) \in S$ and $\gamma \in \Gamma$, we have

$$c \in [a]_\rho\gamma[b]_\rho \Longrightarrow [c]_\rho \subseteq [a]_\rho\gamma[b]_\rho.$$

Let $A$ be a non-empty subset of a Γ-semihypergroup $S$ and $\rho$ be a regular relation on $S$. Then the sets

$$\underline{Apr}_\rho(A) = \left\{x \in S : [x]_\rho \subseteq A\right\} \text{ and } \overline{Apr}_\rho(A) = \left\{x \in S : [x]_\rho \cap A \neq \emptyset\right\}$$

are called $\rho$-lower and $\rho$-upper approximations of $A$ respectively. For a non-empty subset $A$ of $S$, $\rho(A) = (\underline{Apr}_\rho(A), \overline{Apr}_\rho(A))$ is called a rough set with respect to $\rho$ if $\underline{Apr}_\rho(A) \neq \overline{Apr}_\rho(A)$.

A Γ-hyperideal $A$ of a Γ-semihypergroup $S$ is prime if $x\Gamma y \subseteq A$ for some $x, y \in S$ implies $x \in A$ or $y \in A$. Let $\rho$ be a regular relation on a Γ-semihypergroup $S$. Then a subset $A$ of $S$ is called a $\rho$-upper [$\rho$-lower] rough prime Γ-hyperideal of $S$, if $\overline{Apr}_\rho(A)$ $\left[\underline{Apr}_\rho(A)\right]$ is a prime Γ-hyperideal of $S$.

**Theorem 4.1.** *[5] Let $\rho$ be a complete regular relation on a Γ-semihypergroup $S$ and $A$ be a prime Γ-hyperideal of $S$. Then $\overline{Apr}_\rho(A)$ is a prime Γ-hyperideal of $S$.*

**Theorem 4.2.** *[5] Let $\rho$ be a complete regular relation on a Γ-semihypergroup $S$ and $A$ be a prime Γ-hyperideal of $S$. Then $\underline{Apr}_\rho(A)$ is, if it is non-empty, a prime Γ-hyperideal of $S$.*

A subset $A$ of a Γ-semihypergroup $S$ is called a $\rho$-upper [$\rho$-lower] rough bi-Γ-hyperideal of $S$ if $\overline{Apr}_\rho(A)$ [$\underline{Apr}_\rho(A)$] is a bi-Γ-hyperideal of $S$.

**Theorem 4.3.** *[4] Let $\rho$ be a regular relation on $S$ and let $A$ be a bi-Γ-hyperideal of $S$. Then*

(1) *$A$ is a $\rho$-upper rough bi-Γ-hyperideal of $S$.*

(2) *If $\rho$ is complete such that the $\rho$-lower approximation of $A$ is non-empty, then $A$ is a $\rho$-lower rough bi-Γ-hyperideal of $S$.*

Let $\rho$ be a regular relation on a Γ-semihypergroup $S$. Then a subset $A$ of $S$ is called a $\rho$-lower rough prime bi-Γ-hyperideal of $S$ if $\underline{Apr}_\rho(A)$ is a prime bi-Γ-hyperideal of $S$. A $\rho$-upper rough prime bi-Γ-hyperideal of $S$ is defined analogously. A is called a rough prime bi-Γ-hyperideal of $S$ if $A$ is a $\rho$-lower and a $\rho$-upper rough prime bi-Γ-hyperideal of $S$.

**Theorem 4.4.** *Let $\rho$ be a complete regular relation on a Γ-semihypergroup $S$ and $A$ be a prime bi-Γ-hyperideal of $S$. Then $A$ is a $\rho$-upper rough prime bi-Γ-hyperideal of $S$.*

*Proof.* Since $A$ is a bi-Γ-hyperideal of $S$, then by Theorem 4.3(1), $\overline{Apr}_\rho(A)$ is bi-Γ-hyperideal of $S$. Let $a$ be any arbitrary element of $S$, then for $\beta, \gamma \in \Gamma$, we have

$$x\beta a\gamma y \subseteq \overline{Apr}_\rho(A), \text{ for some } x, y \in S.$$

Then

$$[x]_\rho\beta[a]_\rho\gamma[y]_\rho \cap A = [x\beta a\gamma y]_\rho \cap A \neq \phi.$$

Thus there exist $x' \in [x]_\rho$, $a' \in [a]_\rho$ and $y' \in [y]_\rho$ such that $x'\beta a'\gamma y' \subseteq A$. Since $A$ is a prime bi-Γ-hyperideal, we have $x' \in A$ or $y' \in A$. Thus

$$x' \in [x]_\rho \cap A \text{ or } y' \in [y]_\rho \cap A.$$

Thus

$$[x]_\rho \cap A \neq \phi \quad \text{or} \quad [y]_\rho \cap A \neq \phi,$$

and so $x \in \overline{Apr}_\rho(A)$ or $y \in \overline{Apr}_\rho(A)$. Therefore $\overline{Apr}_\rho(A)$ is a prime bi-Γ-hyperideal of $S$. Here $a$ was any arbitrary element of $S$, so this theorem holds for all $a \in S$. □

**Theorem 4.5.** *Let $\rho$ be a complete regular relation on a Γ-semihypergroup $S$ and $A$ be a prime bi-Γ-hyperideal of $S$. Then $\underline{Apr}_\rho(A)$ is, if it is non-empty, a prime bi-Γ-hyperideal of $S$.*

*Proof.* Since $A$ is a bi-Γ-hyperideal of $S$, by Theorem 4.3(2), we know that $\underline{Apr}_\rho(A)$ is a bi-Γ-hyperideal of $S$. Let $a$ be any arbitrary element of $S$, then for $\beta, \gamma \in \Gamma$, we have

$$x\beta a\gamma y \subseteq \underline{Apr}_\rho(A), \text{ for some } x, y \in S.$$

Then

$$[x]_\rho\beta[a]_\rho\gamma[y]_\rho = [x\beta a\gamma y]_\rho \subseteq A.$$

We suppose that $\underline{Apr}_\rho(A)$ is not a prime bi-Γ-hyperideal, then there exist $x, y \in S$ and an arbitrary element $a \in S$, such that $x\beta a\gamma y \in \underline{Apr}_\rho(A)$ but $x \notin \underline{Apr}_\rho(A)$ and $y \notin \underline{Apr}_\rho(A)$ for $\beta, \gamma \in \Gamma$. Thus

$$[x]_\rho \nsubseteq A \text{ and } [y]_\rho \nsubseteq A.$$

Then there exist

$$x' \in [x]_\rho,\ x' \notin A \text{ and } y' \in [y]_\rho,\ y' \notin A.$$

Thus for $\beta, \gamma \in \Gamma$,

$$x'\beta a\gamma y' \subseteq [x]_\rho\beta[a]_\rho\gamma[y]_\rho \subseteq A.$$

Since $A$ is a prime bi-Γ-hyperideal, we have $x' \in A$ or $y' \in A$. It contradicts the supposition. This means that $\underline{Apr}_\rho(A)$ is, if it is non-empty, a prime bi-Γ-hyperideal of $S$. Here $a$ was any arbitrary element of $S$, so this theorem holds for all $a \in S$. □

The following example shows that the converse of Theorem 4.4 and Theorem 4.5 does not hold.

**Example 4.6.** *Let $S = \{1, 2, 3, 4\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

| $\beta$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | $\{1,3\}$ | 4 |
| 2 | 2 | 2 | 2 | 4 |
| 3 | 1 | 2 | $\{1,3\}$ | 4 |
| 4 | 4 | 4 | 4 | 4 |

| $\gamma$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | $\{1,3\}$ | 4 |
| 2 | 2 | 2 | 2 | 4 |
| 3 | $\{1,3\}$ | 2 | 3 | 4 |
| 4 | 4 | 4 | 4 | 4 |

*Clearly $S$ is a Γ-semihypergroup. Let $\rho$ be a complete regular relation on $S$ such that the $\rho$-regular classes are the subsets $\{1, 2, 3\}$, $\{4\}$. Now for $A = \{2, 3, 4\} \subseteq S$, $\overline{Apr}_\rho(A) = \{1, 2, 3, 4\}$ and $\underline{Apr}_\rho(A) = \{4\}$. It is clear that $\overline{Apr}_\rho(A)$ and $\underline{Apr}_\rho(A)$ are prime bi-Γ-hyperideals of $S$, but $A$ is not a prime bi-Γ-hyperideal of $S$.*

## 5. Rough Prime Bi-Γ-Hyperideals in the Quotient Γ-Semihypergroups

Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$. We put $\widehat{\Gamma} = \{\widehat{\gamma} : \gamma \in \Gamma\}$. For every $[a]_\rho, [b]_\rho \in S/\rho$, we define $[a]_\rho \widehat{\gamma} [b]_\rho = \{[z]_\rho : z \in a\gamma b\}$.

**Theorem 5.1.** *[4, Theorem 4.1] If $S$ is a $\Gamma$-semihypergroup, then $S/\rho$ is a $\widehat{\Gamma}$-semihypergroup.*

**Definition 5.2.** *Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$. The $\rho$-upper approximation and $\rho$-lower approximation of a non-empty subset $A$ of $S$ can be presented in an equivalent form as shown below:*

$$\overline{\overline{Apr}}_\rho(A) = \left\{[x]_\rho \in S/\rho : [x]_\rho \cap A \neq \emptyset\right\} \text{ and } \underline{\underline{Apr}}_\rho(A) = \left\{[x]_\rho \in S/\rho : [x]_\rho \subseteq A\right\},$$

*respectively.*

**Theorem 5.3.** *[4, Theorems 4.3, 4.4] Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$. If $A$ is a sub $\Gamma$-semihypergroup of $S$. Then,*

*(1) $\overline{\overline{Apr}}_\rho(A)$ is a sub $\widehat{\Gamma}$-semihypergroup of $S/\rho$.*

*(2) $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a sub $\widehat{\Gamma}$-semihypergroup of $S/\rho$.*

**Theorem 5.4.** *[5] Let $\rho$ be a regular relation a $\Gamma$-semihypergroup $S$. If $A$ is a left (right, two-sided) $\Gamma$-hyperideal of $S$, then $\overline{\overline{Apr}}_\rho(A)$ is a left (right, two-sided) $\widehat{\Gamma}$-hyperideal of $S/\rho$.*

**Theorem 5.5.** *[5] Let $\rho$ be a regular relation a $\Gamma$-semihypergroup $S$. If $A$ is a left (right, two-sided) $\Gamma$-hyperideal of $S$, then $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a left (right, two-sided) $\widehat{\Gamma}$-hyperideal of $S/\rho$.*

**Theorem 5.6.** *Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$ and $A$ be a prime $\Gamma$-hyperideal of $S$. Then $\overline{\overline{Apr}}_\rho(A)$ is a prime $\widehat{\Gamma}$-hyperideal of $S/\rho$.*

*Proof.* Since $A$ is a $\Gamma$-hyperideal of $S$, by Theorem 5.4, we know that $\overline{\overline{Apr}}_\rho(A)$ is a $\widehat{\Gamma}$-hyperideal of $S/\rho$. Suppose

$$[x]_\rho \widehat{\gamma} [y]_\rho \subseteq \overline{\overline{Apr}}_\rho(A), \text{ for some } [x]_\rho, [y]_\rho \in S/\rho \text{ and } \widehat{\gamma} \in \widehat{\Gamma}.$$

Now, there exist $z$, such that $z \in x\gamma y \subseteq \overline{Apr}_\rho(A)$. We obtain $[x\gamma y]_\rho \cap A \neq \phi$. Since $A$ is a prime $\Gamma$-hyperideal of $S$. Then by Theorem 4.1, $\overline{Apr}_\rho(A)$ is a prime $\Gamma$-hyperideal of $S$. So

$$x \in \overline{Apr}_\rho(A) \text{ or } y \in \overline{Apr}_\rho(A).$$

Now, as $z \in x\gamma y$, we obtain

$$[z]_\rho \in [x]_\rho \widehat{\gamma} [y]_\rho.$$

On the other hand, since $z \in \overline{Apr}_\rho(A)$, we have $[z]_\rho \cap A \neq \emptyset$. Thus,

$$[x]_\rho \in \overline{\overline{Apr}}_\rho(A) \text{ or } [y]_\rho \in \overline{\overline{Apr}}_\rho(A).$$

Therefore $\overline{\overline{Apr}}_\rho(A)$ is a prime $\widehat{\Gamma}$-hyperideal of $S/\rho$. □

**Theorem 5.7.** *Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$ and $A$ be a prime $\Gamma$-hyperideal of $S$. Then $\underline{\underline{Apr}}_{\rho}(A)$ is, if it is non-empty, a prime $\widehat{\Gamma}$-hyperideal of $S/\rho$.*

*Proof.* Since $A$ is a $\Gamma$-hyperideal of $S$, by Theorem 5.5, we know that $\underline{\underline{Apr}}_{\rho}(A)$ is a $\widehat{\Gamma}$-hyperideal of $S/\rho$. Suppose

$$[x]_\rho\widehat{\gamma}[y]_\rho \subseteq \underline{\underline{Apr}}_{\rho}(A), \text{ for some } [x]_\rho, [y]_\rho \in S/\rho \text{ and } \widehat{\gamma} \in \widehat{\Gamma}.$$

Now, there exist $z$, such that $z \in x\gamma y \subseteq \underline{Apr}_{\rho}(A)$. We obtain $[x\gamma y]_\rho \subseteq A$. Since $A$ is a prime $\Gamma$-hyperideal of $S$. Then by Theorem 4.2, $\underline{Apr}_{\rho}(A)$ is prime $\Gamma$-hyperideal of $S$. So

$$x \in \underline{Apr}_{\rho}(A) \text{ or } y \in \underline{Apr}_{\rho}(A).$$

Now, as $z \in x\gamma y$, we obtain

$$[z]_\rho \in [x]_\rho\widehat{\gamma}[y]_\rho.$$

On the other hand, since $z \in \underline{Apr}_{\rho}(A)$, we have $[z]_\rho \subseteq A$. Thus,

$$[x]_\rho \in \underline{\underline{Apr}}_{\rho}(A) \text{ or } [y]_\rho \in \underline{\underline{Apr}}_{\rho}(A).$$

Therefore $\underline{\underline{Apr}}_{\rho}(A)$ is a prime $\widehat{\Gamma}$-hyperideal of $S/\rho$. □

**Theorem 5.8.** *Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$ and $A$ be a bi-$\Gamma$-hyperideal of $S$. Then $\overline{\overline{Apr}}_{\rho}(A)$ is a bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$.*

*Proof.* Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\overline{\overline{Apr}}_{\rho}(A)$ and $[s]_\rho$ be any element of $S/\rho$. Then,

$$[x]_\rho \cap A \neq \emptyset \quad \text{and} \quad [y]_\rho \cap A \neq \emptyset.$$

Hence, $x \in \overline{Apr}_{\rho}(A)$ and $y \in \overline{Apr}_{\rho}(A)$. By Theorem 4.3(1), $\overline{Apr}_{\rho}(A)$ is a bi-$\Gamma$-hyperideal of $S$. So, for every $\alpha, \beta \in \Gamma$, we have

$$x\alpha s\beta y \subseteq \overline{Apr}_{\rho}(A).$$

Now, for every $t \in x\alpha s\beta y$, we obtain

$$[t]_\rho \in [x]_\rho\widehat{\alpha}s\widehat{\beta}[y]_\rho.$$

On the other hand, since $t \in \overline{Apr}_{\rho}(A)$, we have $[t]_\rho \cap A \neq \emptyset$. Thus,

$$[x]_\rho\widehat{\alpha}s\widehat{\beta}[y]_\rho \subseteq \overline{\overline{Apr}}_{\rho}(A).$$

Therefore, $\overline{\overline{Apr}}_{\rho}(A)$ is a bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$. □

**Theorem 5.9.** *Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$ and $A$ be a bi-$\Gamma$-hyperideal of $S$. Then $\underline{\underline{Apr}}_{\rho}(A)$ is, if it is non-empty, a bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$.*

*Proof.* Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\underline{\underline{Apr}}_\rho(A)$ and $[s]_\rho$ be any element of $S/\rho$. Then,

$$[x]_\rho \subseteq A \text{ and } [y]_\rho \subseteq A.$$

Hence, $x \in \underline{Apr}_\rho(A)$ and $y \in \underline{Apr}_\rho(A)$. By Theorem 4.3(2), $\underline{Apr}_\rho(A)$ is a bi-$\Gamma$-hyperideal of $S$. So, for every $\alpha, \beta \in \Gamma$, we have

$$x\alpha s\beta y \subseteq \underline{Apr}_\rho(A).$$

Then, for every $t \in x\alpha s\beta y$, we obtain

$$[t]_\rho \in [x]_\rho\widehat{\alpha}a\widehat{\beta}[y]_\rho.$$

On the other hand, since $t \in \underline{Apr}_\rho(A)$, we have $[t]_\rho \subseteq A$. So,

$$[x]^m_\rho\widehat{\alpha}a\widehat{\beta}[y]^n_\rho \subseteq \underline{\underline{Apr}}_\rho(A).$$

Therefore, $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$. □

**Theorem 5.10.** *Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$ and $A$ be a prime bi-$\Gamma$-hyperideal of $S$. Then $\overline{\overline{Apr}}_\rho(A)$ is a prime bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$.*

*Proof.* Since $A$ is a bi-$\Gamma$-hyperideal of $S$, by Theorem 5.8, we know that $\overline{\overline{Apr}}_\rho(A)$ is a bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$. For all $s \in S$, suppose

$$[x]_\rho\widehat{\beta}s\widehat{\gamma}[y]_\rho \subseteq \overline{\overline{Apr}}_\rho(A), \text{ for some } [x]_\rho, [s]_\rho, [y]_\rho \in S/\rho \text{ and } \widehat{\beta}, \widehat{\gamma} \in \widehat{\Gamma}.$$

Now, there exist $z$, such that $z \in x\beta s\gamma y \subseteq \overline{Apr}_\rho(A)$. We obtain $[x\beta s\gamma y]_\rho \cap A \neq \phi$. Since $A$ is a prime bi-$\Gamma$-hyperideal of $S$. Then by Theorem 4.4, $\overline{Apr}_\rho(A)$ is a prime bi-$\Gamma$-hyperideal of $S$. So

$$x \in \overline{Apr}_\rho(A) \text{ or } y \in \overline{Apr}_\rho(A).$$

Now, as $z \in x\beta s\gamma y$, we obtain

$$[z]_\rho \in [x]_\rho\widehat{\beta}s\widehat{\gamma}[y]_\rho.$$

On the other hand, since $z \in \overline{Apr}_\rho(A)$, we have $[z]_\rho \cap A \neq \emptyset$. Thus,

$$[x]_\rho \in \overline{\overline{Apr}}_\rho(A) \text{ or } [y]_\rho \in \overline{\overline{Apr}}_\rho(A).$$

Therefore $\overline{\overline{Apr}}_\rho(A)$ is a prime bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$. □

**Theorem 5.11.** *Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$ and $A$ be a prime bi-$\Gamma$-hyperideal of $S$. Then $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a prime bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$.*

*Proof.* Since $A$ is a bi-$\Gamma$-hyperideal of $S$, by Theorem 5.9, we know that $\underline{\underline{Apr}}_{\rho}(A)$ is a bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$. For all $s \in S$, suppose

$$[x]_{\rho}\widehat{\beta}s\widehat{\gamma}[y]_{\rho} \subseteq \underline{\underline{Apr}}_{\rho}(A), \text{ for some } [x]_{\rho}, [s]_{\rho}, [y]_{\rho} \in S/\rho \text{ and } \widehat{\beta}, \widehat{\gamma} \in \widehat{\Gamma}.$$

Now, there exist $z$, such that $z \in x\beta s\gamma y \subseteq \underline{Apr}_{\rho}(A)$. We obtain $[x\beta s\gamma y]_{\rho} \subseteq A$. Since $A$ is prime bi-$\Gamma$-hyperideal of $S$. Then by Theorem 4.5, $\underline{Apr}_{\rho}(A)$ is prime bi-$\Gamma$-hyperideal of $S$. So

$$x \in \underline{Apr}_{\rho}(A) \text{ or } y \in \underline{Apr}_{\rho}(A).$$

Now, as $z \in x\beta s\gamma y$, we obtain

$$[z]_{\rho} \in [x]_{\rho}\widehat{\beta}s\widehat{\gamma}[y]_{\rho}.$$

On the other hand, since $z \in \underline{Apr}_{\rho}(A)$, we have $[z]_{\rho} \subseteq A$. Thus,

$$[x]_{\rho} \in \underline{\underline{Apr}}_{\rho}(A) \text{ or } [y]_{\rho} \in \underline{\underline{Apr}}_{\rho}(A).$$

Therefore $\underline{\underline{Apr}}_{\rho}(A)$ is a prime bi-$\widehat{\Gamma}$-hyperideal of $S/\rho$. $\square$

## 6. Fuzzy Prime Bi-Γ-Hyperideals of Γ-Semihypergroups

In this section, we introduce the concept fuzzy prime bi-$\Gamma$-hyperideals of $\Gamma$-semihypergroups and their structure is investigated.

**Definition 6.1.** *Let $X$ be a non-empty set. A fuzzy subset $\mu$ of $X$ is a function $\mu : X \to [0,1]$. Let $\mu, \lambda$ be two fuzzy subsets of $X$, we say that $\mu$ is contained in $\lambda$ if $\mu(x) \leq \lambda(x)$, $\forall x \in X$.*

**Definition 6.2.** *Let $\mu$ be a fuzzy subset of a set $X$ and let $t \in [0,1]$. The set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset of $\mu$.*

**Definition 6.3.** *Let $H$ be a $\Gamma$-semihypergroup. A fuzzy subset $\mu$ of $H$ is called a fuzzy sub $\Gamma$-semihypergroup of $H$ if for all $x, y \in H$, $\alpha \in \Gamma$,*

$$\inf_{t \in x\alpha y}\{\mu(t)\} \geq \min\{\mu(x), \mu(y)\}.$$

**Definition 6.4.** *Let $H$ be a $\Gamma$-semihypergroup and $\mu$ be a fuzzy subset of $H$. Then $\mu$ is called*
*(1) a fuzzy right $\Gamma$-hyperideal of $H$ if $\mu(x) \leq \inf\limits_{h \in x\gamma y} \mu(h)$, $\forall x, y \in H, \gamma \in \Gamma$.*
*(2) a fuzzy left $\Gamma$-hyperideal of $H$ if $\mu(y) \leq \inf\limits_{h \in x\gamma y} \mu(h)$, $\forall x, y \in H, \gamma \in \Gamma$.*
*(3) a fuzzy $\Gamma$-hyperideal (or fuzzy two-sided $\Gamma$-hyperideal) if it is both a fuzzy left and right $\Gamma$-hyperideal of $H$.*

Several examples of fuzzy $\Gamma$-hyperideals can be found in [15].

**Definition 6.5.** *Let $X$ be a set. Let $A$ and $B$ be two fuzzy subsets on $X$. Then:*
*(1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.*
*(2) $A = B$ if and only if if $A \subseteq B$ and $B \subseteq A$.*
*(3) $D = A \cap B$ if and only if $\mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$.*
*(4) $C = A \cup B$ if and only if $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$.*

The complement of $A$, denoted by $A^c$, is defined by $A^c(x) = 1 - \mu_A(x)$, for all $x \in X$. Notice that when the range of membership functions is restricted to the set $\{0, 1\}$, these functions perform precisely as the corresponding operators for crisp subsets. For the sake of simplicity, we shall denote every fuzzy subset by its membership function.

**Definition 6.6.** *Let $X$ be a non-empty set. For any $A \subseteq X$ and $r \in (0, 1]$, the fuzzy subset $r_A$ of $X$ is defined by*

$$r_A(x) = \begin{cases} r & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

*for all $x \in X$. In particular, when $r = 1$, $r_A$ is said to be the characteristic function of $A$, denoted by $\chi_A$, when $A = \{x\}$, $r_A$ is said to be a fuzzy point with support $x$ and value $r$ and is denoted by $x_r$.*

Let $I$ be a non-empty subset of a $\Gamma$-semihypergroup $H$ and $\chi_I$ be the characteristic function of $I$. Then $I$ is a left $\Gamma$-hyperideal (right $\Gamma$-hyperideal, $\Gamma$-hyperideal) of $H$ if and only if $\chi_I$ is a fuzzy left $\Gamma$-hyperideal (respectively, fuzzy right $\Gamma$-hyperideal, fuzzy $\Gamma$-hyperideal) of $H$.

Let $H$ be a $\Gamma$-semihypergroup and $\theta, \sigma$ be two fuzzy subsets of $H$. Then the product $\theta\Gamma\sigma$ is defined by

$$(\theta\Gamma\sigma)(x) = \begin{cases} \sup\limits_{z \in x\gamma y} \{\min\{\theta(x), \sigma(x)\}\} & \text{for } x, y \in H, \gamma \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 6.7.** *Let $H$ be a $\Gamma$-semihypergroup. A fuzzy sub $\Gamma$-semihypergroup $\mu$ in $H$ is called a fuzzy bi-$\Gamma$-hyperideal of $H$ if for all $x, y, z \in H$, $\alpha, \beta \in \Gamma$,*

$$\inf_{t \in x\alpha y\beta z} \{\mu(t)\} \geq \min\{\mu(x), \mu(z)\}.$$

Every fuzzy left (right) $\Gamma$-hyperideal is fuzzy bi-$\Gamma$-hyperideal but the converse is not true.

**Lemma 6.8.** *Let $H$ be a $\Gamma$-semihypergroup. If $B$ is a bi-$\Gamma$-hyperideal of $H$, then for any $0 < t < 1$, there exists a fuzzy bi-$\Gamma$-hyperideal $\mu$ of $H$ such that $\mu_t = B$.*

*Proof.* Let $\mu : H \to [0, 1]$ be defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

where $t$ is a fixed number in $(0, 1)$. Then, clearly $\mu_t = B$.

Now suppose that $B$ is a bi-$\Gamma$-hyperideal of $H$. For all $x, y \in B$ and $\gamma \in \Gamma$ such that $x\gamma y \subseteq B$, we have

$$\inf_{s \in x\gamma y} \mu(s) \geq t = \min\{\mu(x), \mu(y)\}.$$

Also, for all $x, y, z \in B$ and $\alpha, \beta \in \Gamma$ such that $x\alpha y\beta z \subseteq B$, we have

$$\inf_{s \in x\alpha y\beta z} \mu(s) \geq t = \min\{\mu(x), \mu(z)\}.$$

Thus $\mu$ is a fuzzy bi-$\Gamma$-hyperideal. $\square$

**Lemma 6.9.** *Let $H$ be a $\Gamma$-semihypergroup and $B$ be a non-empty subset of $H$. Then $B$ is a bi-$\Gamma$-hyperideal of $H$ if and only if $\chi_B$ is a fuzzy bi-$\Gamma$-hyperideal of $H$.*

*Proof.* Let $x, y \in B$ and $\gamma \in \Gamma$. From hypothesis, $x\gamma y \subseteq B$.

(i) If $x, y \in B$, then $\chi_B(x) = 1$ and $\chi_B(y) = 1$. In this case for every $t \in x\gamma y$

$$\chi_B(t) = 1 \geq \min\{\chi_B(x), \chi_B(y)\}.$$

(ii) If $x \in B$, $y \notin B$, then $\chi_B(x) = 1$ and $\chi_B(y) = 0$. Thus for every $t \in x\gamma y$

$$\chi_B(t) \geq 0 \geq \min\{\chi_B(x), \chi_B(y)\}.$$

(iii) If $x \notin B$, $y \in B$, then $\chi_B(x) = 0, \chi_B(y) = 1$. Thus for every $t \in x\gamma y$

$$\chi_B(t) \geq 0 \geq \min\{\chi_B(x), \chi_B(y)\}.$$

(iv) If $x \notin B$, $y \notin B$, then $\chi_B(x) = \chi_B(y) = 0$. Thus for every $t \in x\gamma y$

$$\chi_B(t) \geq 0 \geq \min\{\chi_B(x), \chi_B(y)\}.$$

Thus $\chi_B$ is a fuzzy sub $\Gamma$-semihypergroup.

Let $x, y, z \in B$ and $\alpha, \beta \in \Gamma$. From hypothesis, $x\alpha y\beta z \subseteq B$.

(i) If $x, z \in B$, then $\chi_B(x) = 1$ and $\chi_B(z) = 1$. In this case for every $t \in x\alpha y\beta z$

$$\chi_B(t) = 1 \geq \min\{\chi_B(x), \chi_B(z)\}.$$

(ii) If $x \in B$, $z \notin B$, then $\chi_B(x) = 1$ and $\chi_B(z) = 0$. Thus for every $t \in x\alpha y\beta z$

$$\chi_B(t) \geq 0 \geq \min\{\chi_B(x), \chi_B(z)\}.$$

(iii) If $x \notin B$, $y \in B$, then $\chi_B(x) = 0, \chi_B(z) = 1$. Thus for every $t \in x\alpha y\beta z$

$$\chi_B(t) \geq 0 \geq \min\{\chi_B(x), \chi_B(z)\}.$$

(iv) If $x \notin B$, $y \notin B$, then $\chi_B(x) = \chi_B(z) = 0$. Thus for every $t \in x\alpha y\beta z$

$$\chi_B(t) \geq 0 \geq \min\{\chi_B(x), \chi_B(z)\}.$$

Thus $\chi_B$ is a fuzzy bi-$\Gamma$-hyperideal.

Conversely, let us suppose that $\chi_B$ is a fuzzy bi-$\Gamma$-hyperideal of $H$. Then by Lemma 6.8, $\chi_B$ is two-valued. Hence $B$ is a bi-$\Gamma$-hyperideal of $H$. This completes the proof. □

**Lemma 6.10.** *Let $H$ be a $\Gamma$-semihypergroup. If $A$ and $B$ are subsets of $H$ then $\chi_A\Gamma\chi_B = \chi_{A\Gamma B}$.*

**Definition 6.11.** *A non-constant fuzzy bi-$\Gamma$-hyperideal $\mu$ of a $\Gamma$-semihypergroup $H$ is called a fuzzy prime bi-$\Gamma$-hyperideal of $H$ if for any two fuzzy bi-$\Gamma$-hyperideals $\theta, \sigma$ of $H$,*

$$\theta\Gamma\sigma \subseteq \mu \text{ implies } \theta \subseteq \mu \text{ or } \sigma \subseteq \mu.$$

**Definition 6.12.** *A non-constant fuzzy bi-$\Gamma$-hyperideal $\mu$ of a $\Gamma$-semihypergroup $H$ is called a fuzzy semiprime bi-$\Gamma$-hyperideal of $H$ if for any fuzzy bi-$\Gamma$-hyperideals $\theta$ of $H$,*

$$\theta\Gamma\theta \subseteq \mu \text{ implies } \theta \subseteq \mu.$$

**Definition 6.13.** *A non-constant fuzzy bi-$\Gamma$-hyperideal $\mu$ of a $\Gamma$-semihypergroup $H$ is called a fuzzy strongly prime bi-$\Gamma$-hyperideal of $H$ if for any two fuzzy bi-$\Gamma$-hyperideals $\theta, \sigma$ of $H$,*

$$\theta\Gamma\sigma \cap \sigma\Gamma\theta \subseteq \mu \text{ implies } \theta \subseteq \mu \text{ or } \sigma \subseteq \mu.$$

Obviously, every fuzzy strongly prime bi-$\Gamma$-hyperideal is a fuzzy prime bi-$\Gamma$-hyperideal and every fuzzy prime bi-$\Gamma$-hyperideal is semiprime but the converse is not true in general.

**Theorem 6.14.** *Let $H$ be a $\Gamma$-semihypergroup. A bi-$\Gamma$-hyperideal $B$ of $H$ is prime if and only if the characteristic function $\chi_B$ of $B$ is a fuzzy prime bi-$\Gamma$-hyperideal of $H$.*

*Proof.* Let us suppose that $B$ is a prime bi-$\Gamma$-hyperideal of $H$. Then by Lemma 6.9, $\chi_B$ is a fuzzy bi-$\Gamma$-hyperideal of $H$. Let $\theta, \sigma$ be any two fuzzy bi-$\Gamma$-hyperideals of $H$ such that $\theta\Gamma\sigma \subseteq \chi_B$ but $\theta \nsubseteq \chi_B$ and $\sigma \nsubseteq \chi_B$. Then there exist $x, y \in H$ such that $\theta(x) \neq 0$ and $\sigma(y) \neq 0$. But $\chi_B(x) = 0$ and $\chi_B(y) = 0$. Hence $x \notin B$ and $y \notin B$. Since $B$ is a prime bi-$\Gamma$-hyperideal of $H$, therefore we have $B(x)\Gamma B(y) \nsubseteq B$. Since $\theta(x) \neq 0$ and $\sigma(y) \neq 0$, therefore $\min\{\theta(x), \sigma(y)\} \neq 0$. Since $B(x)\Gamma B(y) \nsubseteq B$, therefore there exists $a \in H$ such that $a \in B(x)\Gamma B(y)$ but $a \notin B$. Thus we have $\chi_B(a) = 0$, and hence $(\theta\Gamma\sigma)(a) = 0$. Since $a \in B(x)\Gamma B(y)$ we have $a \in x_1\gamma y_1$ for some $x_1 \in B(x), y_1 \in B(y), \gamma \in \Gamma$. Thus

$$(\theta\Gamma\sigma)(a) = \sup_{a\in x_1\gamma y_1} \min\{\theta(x_1), \sigma(y_1)\} \geq \min\{\theta(x_1), \sigma(y_1)\}.$$

Since, $x_1 \in B(x) = \{x\} \cup x\Gamma x \cup x\Gamma H\Gamma x$, we have $x_1 = x$ or $x_1 \in x\Gamma x$ or $x_1 \in x\Gamma y\Gamma x$ for some $y \in H$. If $x_1 = x$, then $\theta(x_1) = \theta(x)$. If $x_1 \in x\Gamma x$, then

$$\theta(x_1) \geq \inf_{t\in x\Gamma x} \theta(t) \geq \min\{\theta(x), \theta(x)\} = \theta(x).$$

If $x_1 \in x\Gamma y\Gamma x$, then

$$\theta(x_1) \geq \inf_{t\in x\Gamma y\Gamma x} \theta(t) \geq \min\{\theta(x), \theta(x)\} = \theta(x).$$

Also, since $y_1 \in B(y) = \{y\} \cup y\Gamma y \cup y\Gamma H\Gamma y$, we have $y_1 = y$ or $y_1 \in y\Gamma y$ or $y_1 \in y\Gamma x\Gamma y$ for some $x \in H$. If $y_1 = y$, then $\sigma(y_1) = \sigma(y)$. If $y_1 \in y\Gamma y$, then

$$\sigma(y_1) \geq \inf_{t\in y\Gamma y} \sigma(t) \geq \min\{\sigma(y), \sigma(y)\} = \sigma(y).$$

If $y_1 \in y\Gamma x\Gamma y$, then

$$\sigma(y_1) \geq \inf_{t\in y\Gamma x\Gamma y} \sigma(t) \geq \min\{\sigma(y), \sigma(y)\} = \sigma(y).$$

Thus

$$(\theta\Gamma\sigma)(a) \geq \min\{\theta(x_1), \sigma(y_1)\} \geq \min\{\theta(x), \sigma(y)\} \neq 0 \Rightarrow (\theta\Gamma\sigma)(a) > 0,$$

which is a contradiction to the fact that $(\theta\Gamma\sigma)(a) = 0$. Thus for any fuzzy bi-$\Gamma$-hyperideal $\theta, \sigma$ of $H$, $\theta\Gamma\sigma \subseteq \chi_B \Rightarrow \theta \subseteq \chi_B$ or $\sigma \subseteq \chi_B$.

Conversely, let us assume that $\chi_B$ is a fuzzy prime bi-$\Gamma$-hyperideal of $H$. Let $B_1, B_2$ be any bi-$\Gamma$-hyperideal of $H$ such that $B_1\Gamma B_2 \subseteq B$. Then by Lemma 6.9, $\chi_{B_1}$ and $\chi_{B_2}$ are fuzzy bi-$\Gamma$-hyperideals of $H$. By hypothesis, $\chi_{B_1\Gamma B_2} \subseteq \chi_B$. By Lemma 6.10, $\chi_{B_1\Gamma B_2} = \chi_{B_1}\Gamma\chi_{B_2}$. Thus, $\chi_{B_1}\Gamma\chi_{B_2} \subseteq \chi_B$. Since $\chi_B$ is prime, we have $\chi_{B_1} \subseteq \chi_B$ or $\chi_{B_1} \subseteq \chi_B$. By Lemma 6.10, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. □

Similarly, we can prove the following:

**Theorem 6.15.** *Let $H$ be a $\Gamma$-semihypergroup. A bi-$\Gamma$-hyperideal $B$ of $H$ is semiprime if and only if the characteristic function $\chi_B$ of $B$ is a fuzzy semiprime bi-$\Gamma$-hyperideal of $H$.*

**Example 6.16.** *Let $S = (0, 1)$, $\Gamma = \{\gamma_n \,|n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ we define hyperoperation $\gamma_n$ on $S$ as follows*

$$x\gamma_n y = \left\{\frac{xy}{2^k} \,|0 \leq k \leq n\right\}, \forall x, y \in S.$$

*Then, $x\gamma_n y \subset S$ and for every $m; n \in \mathbb{N}$ and $x; y; z \in S$*

$$(x\gamma_n y)\gamma_m z = \left\{\frac{xyz}{2^k} \,|0 \leq k \leq n + m\right\} = x\gamma_n\,(y\gamma_m z)$$

*So $S$ is a $\Gamma-$semihypergroup. Now let $S_i = \left(0, \frac{1}{2^i}\right)$, where $i \in \mathbb{N}$. It easily to see that $S_1$ is a prime bi-$\Gamma$-hyperideal of $S$. It is clear that its characteristic function is fuzzy prime bi-$\Gamma$-hyperideal of $S$.*

**Example 6.17.** *Let $S = N$ with natural order and $\Gamma$ be a non-empty subset of $S$. We define $x\gamma y = \{z \in S : z \geq \max\{x, \gamma, y\}\}$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then $S$ is a $\Gamma$-semihypergroup. Then $I_i = \{i, i+1, ...\}$ are bi-$\Gamma$-hyperideals of $S$ for all $i \in S$ and $I_2$ is a prime bi-$\Gamma$-hyperideal of $S$. It is clear that its characteristic function is a fuzzy prime bi-$\Gamma$-hyperideals of $S$. We define now the fuzzy subset $\mu_n$ of $S$ as follows:*

$$\mu_n(x) = \begin{cases} 0 & \text{if } x < y \\ 1 - \frac{1}{n+i} & \text{if } x = n + i, i = 1, 2, ... \end{cases}$$

*It can be easily verified that for every $n$, $\mu_n$ is a prime fuzzy bi-$\Gamma$-hyperideal of $S$.*

**Theorem 6.18.** *Let $H$ be a $\Gamma$-semihypergroup. A bi-$\Gamma$-hyperideal $B$ of $H$ is strongly prime if and only if the characteristic function $\chi_B$ of $B$ is a fuzzy strongly prime bi-$\Gamma$-hyperideal of $H$.*

*Proof.* Let $B$ be a strongly prime bi-$\Gamma$-hyperideal of $H$. Then by Lemma 6.9, $\chi_B$ is a fuzzy bi-$\Gamma$-hyperideal of $H$. Let $\theta, \sigma$ be any fuzzy bi-$\Gamma$-hyperideals of $H$ such that $\theta\Gamma\sigma \cap \sigma\Gamma\theta \subseteq \chi_B$, but $\theta \nsubseteq \chi_B$ and $\sigma \nsubseteq \chi_B$. Then there exist $x, y \in H$ such that $\theta(x) \neq 0$ and $\sigma(y) \neq 0$, but $\chi_B(x) = 0$ and $\chi_B(y) = 0$. Hence $x \notin B$ and $y \notin B$. Since $B$ is a strongly prime bi-$\Gamma$-hyperideal of $H$, we have $B(x)\Gamma B(y) \cap B(y)\Gamma B(x) \nsubseteq B$. Since

$$\theta(x) \neq 0 \text{ and } \sigma(y) \neq 0.$$

We have

$$\min\{\theta(x), \sigma(y)\} \neq 0.$$

Since $B(x)\Gamma B(y) \cap B(y)\Gamma B(x) \nsubseteq B$, so there exists $a \in H$ such that $a \in B(x)\Gamma B(y)$ and $a \in B(y)\Gamma B(x)$, but $a \notin B$. Thus we have $\chi_B(a) = 0$, and hence

$$(\theta\Gamma\sigma)(a) \wedge (\sigma\Gamma\theta)(a) = 0.$$

Since $a \in B(x)\Gamma B(y)$ and $a \in B(y)\Gamma B(x)$, we have $a \in x_1\Gamma y_1$ and $a \in y_2\Gamma x_2$ for some $x_1, x_2 \in B(x)$ and $y_1, y_2 \in B(y)$. Thus

$$(\theta\Gamma\sigma)(a) = \sup_{a \in x_1\Gamma y_1} \min\{\theta(x_1), \sigma(y_1)\} \geq \min\{\theta(x_1), \sigma(y_1)\}.$$

Since, $x_1 \in B(x) = \{x\} \cup x\Gamma x \cup x\Gamma H\Gamma x$, we have $x_1 = x$ or $x_1 \in x\Gamma x$ or $x_1 \in x\Gamma y\Gamma x$ for some $y \in H$. If $x_1 = x$, then $\theta(x_1) = \theta(x)$. If $x_1 \in x\Gamma x$, then

$$\theta(x_1) \geq \inf_{t \in x\Gamma x} \theta(t) \geq \min\{\theta(x), \theta(x)\} = \theta(x).$$

If $x_1 \in x\Gamma y\Gamma x$, then

$$\theta(x_1) \geq \inf_{t \in x\Gamma y\Gamma x} \theta(t) \geq \min\{\theta(x), \theta(x)\} = \theta(x).$$

Also since $y_1 \in B(y) = \{y\} \cup y\Gamma y \cup y\Gamma H\Gamma y$, we have $y_1 = y$ or $y_1 \in y\Gamma y$ or $y_1 \in y\Gamma x\Gamma y$ for some $x \in H$. If $y_1 = y$, then $\sigma(y_1) = \sigma(y)$. If $y_1 \in y\Gamma y$, then

$$\sigma(y_1) \geq \inf_{t \in y\Gamma y} \sigma(t) \geq \min\{\sigma(y), \sigma(y)\} = \sigma(y).$$

If $y_1 \in y\Gamma x\Gamma y$, then

$$\sigma(y_1) \geq \inf_{t \in y\Gamma x\Gamma y} \sigma(t) \geq \min\{\sigma(y), \sigma(y)\} = \sigma(y).$$

Thus

$$(\theta\Gamma\sigma)(a) \geq \min\{\theta(x_1), \sigma(y_1)\} \geq \min\{\theta(x), \sigma(y)\} \neq 0 \Rightarrow (\theta\Gamma\sigma)(a) > 0.$$

Similarly, we can show that $(\sigma\Gamma\theta)(a) > 0$. Thus, $(\theta\Gamma\sigma)(a) \wedge (\sigma\Gamma\theta)(a) > 0$ which is a contradiction to the fact that $(\theta\Gamma\sigma)(a) \wedge (\sigma\Gamma\theta)(a) = 0$. Thus, for any fuzzy bi-Γ-hyperideal $\theta, \sigma$ of $H$, $\theta\Gamma\sigma \cap \sigma\Gamma\theta \subseteq \chi_B \Rightarrow \theta \subseteq \chi_B$ or $\sigma \subseteq \chi_B$.

Conversely, let us assume that $\chi_B$ is a fuzzy strongly prime bi-Γ-hyperideal of $H$. Let $B_1, B_2$ be bi-Γ-hyperideals of $H$ such that $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$, then by Lemma 6.9, $\chi_{B_1}$ and $\chi_{B_2}$ are fuzzy bi-Γ-hyperideals of $H$. By hypothesis, $\chi_{B_1\Gamma B_2} \cap \chi_{B_2\Gamma B_1} \subseteq \chi_B$. By Lemma 6.10, we have $\chi_{B_1\Gamma B_2} = \chi_{B_1}\Gamma\chi_{B_2}$. Thus, $\chi_{B_1}\Gamma\chi_{B_2} \cap \chi_{B_2}\Gamma\chi_{B_1} \subseteq \chi_B$. Since $\chi_B$ is strongly prime, we have $\chi_{B_1} \subseteq \chi_B$ or $\chi_{B_2} \subseteq \chi_B$. By Lemma 6.10, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. □

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