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On the Stability of the Spectral Properties under Commuting Perturbations

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Abstract. In this paper, we examine the stability of several spectral properties under commuting perturbations. In particular, we show that if $T \in L(X)$ is an isoloid operator satisfying generalized Weyl's theorem and if $F \in L(X)$ is a power finite rank operator that commutes with T, then generalized Weyl's theorem holds for $T + F$. In addition, we consider the permanence of Bishop's property (β), at a point, under commuting perturbation that is an algebraic operator.

1. Introduction

Throughout this paper, $L(X)$ denotes the Banach algebra of all bounded linear operators acting on a Banach space X. For $T \in L(X)$, Let $T^*, N(T), R(T), \sigma(T), acc\sigma(T)$, $\rho(T)$ and $\sigma_a(T)$ denote the adjoint, the null space, the range, the spectrum, the accumulation points of $\sigma(T)$, the resolvent set and the approximate point spectrum of T respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} R(T)$. If the range $R(T)$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called *upper semi Fredholm* (resp. *lower semi Fredholm*). If $T \in L(X)$ is both upper and lower semi Fredholm, then T is called *Fredholm*. If $T \in L(X)$ is either upper or lower semi Fredholm, then T is called *semi Fredholm*, and its index is defined by ind $(T) = \alpha(T) - \beta(T)$. An operator $T \in L(X)$ is called a *Weyl* (resp. *upper semi Weyl*, *lower semi Weyl*) operator if it is a Fredholm operator of index 0 (resp. T is an upper semi Fredholm operator and ind $(T) \leq 0$, T is a lower semi Fredholm operator and ind $(T) \geq 0$). Recall that the *descent* and the *ascent* of $T \in L(X)$ are $q(T) = inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$ and $p(T) = inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$. An operator $T \in L(X)$ is called a *Browder* (resp. *upper semi Browder*, *lower semi Browder*) operator if T is a Fredholm operator and $p(T) = q(T) < \infty$ (resp. T is an upper semi Fredholm operator and $p(T) < \infty$, T is a lower semi Fredholm operator and $q(T) < \infty$). Let us define the Weyl spectrum, upper semi Weyl spectrum, the Browder spectrum and upper semi Browder spectrum as following respectively:

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a Weyl operator}\},$$

$$\sigma_{USW}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not an upper semi Weyl operator}\},$$

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$$\sigma_B(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a Browder operator}\},$$

$$\sigma_{USB}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not an upper semi Browder operator}\}.$$

For each integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as the map from $R(T^n)$ to $R(T^n)$ (in particular $T_{[0]} = T$). If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_{[n]}$ is upper semi Fredholm (resp. Fredholm, lower semi Fredholm), then T is called *upper semi B-Fredholm* (resp. *B-Fredholm*, *lower semi B-Fredholm*); see [12]. In this case $T_{[m]}$ is a semi Fredholm operator and $\text{ind}(T_{[m]}) = \text{ind}(T_{[n]})$ for all $m \geq n$. This enables us to define the index of a semi B-Fredholm as $\text{ind}(T) = \text{ind}(T_{[n]})$. An operator $T \in L(X)$ is called *upper semi B-Weyl* (resp. *B-Weyl*, *lower semi B-Weyl*) if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_{[n]}$ is upper semi Weyl (resp. Weyl, lower semi Weyl). Analogously, a bounded operator $T \in L(X)$ is called *upper semi B-Browder* (resp. *B-Browder*, *lower semi B-Browder*) if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_{[n]}$ is upper semi Browder (resp. Browder, lower semi Browder). Let us define the B-Weyl spectrum, upper semi B-Weyl spectrum, the B-Browder spectrum and upper semi B-Browder spectrum as following respectively:

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a B-Weyl operator}\},$$

$$\sigma_{USBW}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not an upper semi B-Weyl operator}\},$$

$$\sigma_{BB}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a B-Browder operator}\},$$

$$\sigma_{USBB}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not an upper semi B-Browder operator}\}.$$

We say that $\Pi(T)$ and $\Pi_a(T)$ denote the set of all poles and the set of all left poles respectively, while $\Pi^0(T)$ and $\Pi_a^0(T)$ denote the set of all poles of finite rank and the set of all left poles of finite rank respectively. Moreover,

$$\Pi(T) = \sigma(T) \backslash \sigma_{BB}(T), \quad \Pi_a(T) = \sigma_a(T) \backslash \sigma_{USBB}(T),$$

$$\Pi^0(T) = \sigma(T) \backslash \sigma_B(T), \quad \Pi_a^0(T) = \sigma_a(T) \backslash \sigma_{USB}(T).$$

Let $E(T)$ and $E_a(T)$ denote the set of all isolated eigenvalues of T and the set of all eigenvalues of T that are isolated in $\sigma_a(T)$ respectively. That is,

$$E(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(\lambda I - T)\},$$

$$E_a(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(\lambda I - T)\}.$$

Moreover, we set

$$E^0(T) = \{\lambda \in E(T) : \alpha(\lambda I - T) < \infty\},$$

$$E_a^0(T) = \{\lambda \in E_a(T) : \alpha(\lambda I - T) < \infty\}.$$

Definition 1.1. ([13]) *Let $T \in L(X)$. We say that*
(1) *Weyl's theorem holds for T if $\sigma(T) \backslash \sigma_W(T) = E^0(T)$.*
(2) *Generalized Weyl's theorem holds for T if $\sigma(T) \backslash \sigma_{BW}(T) = E(T)$.*
(3) *Property (w) holds for T if $\sigma_a(T) \backslash \sigma_{USW}(T) = E^0(T)$.*
(4) *Property (gw) holds for T if $\sigma_a(T) \backslash \sigma_{USBW}(T) = E(T)$.*

It is well known that following implications hold [13]:

$$\begin{array}{ccc} \text{property } (gw) & \Rightarrow & \text{generalized Weyl's theorem} \\ \Downarrow & & \Downarrow \\ \text{property } (w) & \Rightarrow & \text{Weyl's theorem} \end{array}$$

Let $D(\lambda, r)$ be the open disc centred at $\lambda \in \mathbb{C}$ with radius $r > 0$, and let $O(U, X)$ denote the Fréchet algebra of all X-valued analytic functions on the open subset of $U \subset \mathbb{C}$ endowed with uniform convergence on compact subsets of U. An operator $T \in L(X)$ is said to satisfy *Bishop's property* (β) at $\lambda_0 \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subset D(\lambda_0, r)$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{O}(U, X)$, $lim_{n\to\infty}(T - \lambda I)f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$ implies $lim_{n\to\infty} f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$. We denote by $\sigma_\beta(T)$ the set where T fails to have Bishop's property (β) and we say that T satisfies *Bishop's property* (β) if $\sigma_\beta(T) = 0$. Dually, an operator $T \in L(X)$ is said to have the *decomposition property* (δ) if T^* satisfies Bishop's property (β). It is well known that T is *decomposable* if and only if T satisfies both (β) and (δ). We refer the readers to the chapter 1 and chapter 2 of [18] for more details and further definitions.

In [17], W. Y. Lee and S. H. Lee have proved that, for an isoloid operator T, Weyl's theorem is transmitted from T to $T+F$ when F is a finite rank operator commuting with T. Recently, M. Oudghiri have generalized this result replacing "finite rank" with "power finite rank" in [23]. It is known that, for an isoloid operator T, generalized weyl's theorem is also transmitted from T to $T+F$ when F is a finite rank operator commuting with T; see [10]. However, whether this result remains valid or not when we replace "finite rank" with "power finite rank" is still a question. In the present paper, we give an affirmative answer to this question. In section 2, we firstly get a useful equality $acc\sigma(T) = acc\sigma(T + F)$, where F commutes with T and the F^n is finite rank for some $n \in \mathbb{N}$. By this equality, we generalize several results in [9, 10, 14]. In particular, we show that if $T \in L(X)$ is an isoloid operator satisfying generalized Weyl's theorem and if $F \in L(X)$ commuting with T satisfies that F^n is finite rank for some $n \in \mathbb{N}$, then generalized Weyl's theorem holds for $T + F$. In section 3, we consider the permanence of Bishop's property (β), at a point, under commuting perturbation that is an algebraic operator. As its applications, we consider the permanence of decomposition property (δ) and decomposable property.

2. Weyl Type Theorems under Commuting Power Finite Rank Perturbations

In this section we study the stability of Weyl type theorems under commuting perturbation that is a power finite rank operator. An operator F is calld *power finite rank* if F^n is finite rank for some $n \in \mathbb{N}$. Let us begin with a useful Lemma.

Lemma 2.1. *Let $T \in L(X)$. If $F \in L(X)$ commuting with T satisfies that F^n is finite rank for some $n \in \mathbb{N}$, then*

$$acc\sigma(T) = acc\sigma(T + F).$$

Proof. Suppose that $\lambda \notin acc\sigma(T)$. There exists a number $\varepsilon > 0$, such that for all $\lambda \in \mathbb{C}$, if $0 < |\lambda - \mu| < \varepsilon$, then $\mu I - T$ is invertible. Therefore, there exists a bounded operator T_1 such that $T_1(\mu I - T) = (\mu I - T)T_1 = I$. Assume that $\mu \in \sigma(T + F)$, then μ is an eigenvalue of $T + F$. Indeed, since $\mu I - T$ is invertible and F is a Riesz operator commuting with $\mu I - T$, then [23, Lemma 2.2] implies that $uI - (T + F)$ is a Weyl operator. If μ is not an eigenvalue of $T + F$ - *i.e.*, $\alpha[uI - (T + F)] = 0$, then $\alpha[uI - (T + F)] = \beta[uI - (T + F)] = 0$, which leads to a contradiction. Now, we assume that $x \neq 0$ is an arbitrary eigenvector. Therefore $[uI - (T + F)]x = 0$. Thus,

$$\begin{aligned} &T_1[uI - (T + F)]x = 0 \\ &\Longrightarrow x - T_1Fx = 0 \\ &\Longrightarrow x = T_1Fx. \end{aligned}$$

As $T_1F = FT_1$, it follows $x = T_1^nF^nx$. Consequently, x is contained in a certain finite dimensional space which is isometric to $R(F^n)$. It is known that eigenvectors corresponding to distinct eigenvalue of $T + F$ are linearly independent. But we get that all such eigenvectors are contained in the finite dimensional space. It follows that $\sigma(T + F)$ may contain only finitely many points μ such that $0 < |\lambda - \mu| < \varepsilon$. This implies $\lambda \notin acc\sigma(T + F)$. The opposite implication is analogous. □

As a corollary, we generalize a result that belongs to Xiaohong Cao and Aifang Liu [14, lemma 2.1].

Corollary 2.2. *Let $T \in L(X)$. If $F \in L(X)$ commuting with T satisfies that F^n is finite rank for some $n \in \mathbb{N}$, then*

$$iso\sigma(T + F) \subset iso\sigma(T) \cup \rho(T).$$

Proof. It follows immediately from the lemmas above. □

According to [8, Theorem 2.9], we have that: T satisfies generalized Weyl's theorem if and only if T satisfies Weyl's Theorem and $E(T) = \Pi(T)$. In the next theorem, we generalize one of the main results of [10] (Theorem 3.4 of [10]), where the finite rank perturbation is extended to the power finite rank perturbation. Recall that $T \in L(X)$ is said to be *isoloid* if every isolated points of $\sigma(T)$ is an eigenvalue.

Theorem 2.3. *Suppose that $T \in L(X)$ is an isoloid operator and that $F \in L(X)$ commuting with T satisfies that F^n is finite rank for some $n \in \mathbb{N}$. If T satisfies generalized Weyl's theorem, then $T + F$ satisfies generalized Weyl's theorem.*

Proof. Assume that T satisfies generalized Weyl's theorem. This implies that T satisfies Weyl's Theorem and $E(T) = \Pi(T)$. Since, by [23, Theorem 2.4], $T+F$ satisfies Weyl's Theorem, then in order to prove $T+F$ satisfies generalized Weyl's Theorem, we need to prove $E(T + F) = \Pi(T + F)$. As the inclusion $\Pi(T + F) \subseteq E(T + F)$ is always true, we need only to prove the inclusion $E(T + F) \subseteq \Pi(T + F)$.

Let $\lambda \in E(T + F)$ be arbitrarily given, then

$$\lambda \in iso\sigma(T + F) \text{ and } \alpha[\lambda I - (T + F)] > 0.$$

So $\lambda \in iso\sigma(T) \cup \rho(T)$ by Corollary 2.2. If $\lambda \in iso\sigma(T)$, since T is isoloid, we obtain that $\alpha(\lambda I - T) > 0$. Thus, $\lambda \in E(T)$ and generalized Weyl's theorem for T ensures that $\lambda \in \Pi(T)$ - *i.e.*, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. Hence the [16, Theorem 2.2] implies $\lambda \in \Pi(T+F)$. If $\lambda \in \rho(T)$, then $\lambda I - T$ is invertible - *i.e.*, $p(\lambda I - T) = q(\lambda I - T) = 0$. As $\lambda \in iso\sigma(T + F) \subseteq \sigma(T + F)$, it then follows that $\lambda \in \Pi(T + F)$ by [16, Theorem 2.2]. Consequently, we get $E(T + F) \subseteq \Pi(T + F)$, as desired. □

Note that there exists a operator $T \in L(X)$ and a power finite rank operator $F \in L(X)$ commuting with T such that generalized Weyls theorem holds for T but it does not hold for $T + F$; see [10, Example 2]. Recall that an operaotr T is said to satisfy *property* (R) if the equality $\Pi_a^0(T) = E^0(T)$ holds for T, while T is said to satisfy *property* (gR) if the equality $\Pi_a(T) = E(T)$ holds. These two properties play an important role in studying Wely type theorems; see [4] and [5] for more details.

Theorem 2.4. *Suppose that $T \in L(X)$ is an isoloid operator and that $F \in L(X)$ commuting with T satisfies that F^n is finite rank for some $n \in \mathbb{N}$, then*

(1) *If T obeys property (gR) and $\Pi_a(T + F) \subset \sigma_a(T)$, then $T + F$ obeys property (gR).*

(2) *If T obeys property (R) and $\Pi_a^0(T + F) \subset \sigma_a(T)$, then $T + F$ obeys property (R).*

Proof. (1) Assume that T obeys property (gR) - *i.e.*, $\Pi_a(T) = E(T)$. Let $\lambda \in \Pi_a(T + F)$. From [11, Theorem 3.2] and the assumption, we have

$$\Pi_a(T + F) = \sigma_a(T + F)\backslash\sigma_{USBB}(T + F) \subseteq \sigma_a(T)\backslash\sigma_{USBB}(T) = \Pi_a(T).$$

Hence, $\lambda \in \Pi_a(T)$ and [5, Theorem 2.5] implies $\lambda \in \Pi(T)$ - *i.e.*, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. Thus $p[\lambda I - (T + F)] = q[\lambda I - (T + F)] < \infty$ by [16, Theorem 2.2]. As $\lambda \in \sigma(T + F)$, it follows that λ is a pole of the resolvent of $T + F$. So we conclude that $\lambda \in iso\sigma(T + F)$. By [3, Lemma 3.5], we have $\alpha[\lambda I - (T + F)] > 0$ which ensures that $\lambda \in E(T + F)$.

Conversely, let $\lambda \in E(T + F)$. We claim that $\lambda \in \Pi(T + F)$. Since $\lambda \in iso\sigma(T + F)$, then $\lambda \in iso\sigma(T) \cup \rho(T)$ by Corollary 2.2. If $\lambda \in iso\sigma(T)$, since T is isoloid, we have $\alpha(\lambda I - T) > 0$. So $\lambda \in E(T)$. As T satisfies property (gR), the [5, Theorem 2.5] implies that $\lambda \in \Pi(T)$ - *i.e.*, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. Hence $\lambda \in \Pi(T + F)$ by [16, Theorem 2.2]. If $\lambda \in \rho(T)$, then $\lambda I - T$ is invertible - i.e., $p(\lambda I - T) = q(\lambda I - T) = 0$. As $\lambda \in iso\sigma(T + F) \subseteq \sigma(T + F)$, it then follows that $\lambda \in \Pi(T + F)$ by [16, Theorem 2.2], as desired. Now the basic inclusion relation between $\Pi(T + F)$ and $\Pi_a(T + F)$ ensures that $T + F$ satisfies property (gR).

(2) Analogous to the proof of (1). □

Recall that an operator T is said to satisfy *generalized Browder's theorem* if $\sigma_{BW}(T) = \sigma_{BB}(T)$, while T is said to satisfy *generalized a-Browder's theorem* if $\sigma_{USBW}(T) = \sigma_{USBB}(T)$. It is known that a-Browder's theorem and generalized a-Browder's theorem are equivalent and generalized a-Browder's theorem is stable under commuting Riesz perturbations. As a corollary, we generalize the results that belong to M. Berkani and M. Amouch in [9] (Theorem 2.8 and Theorem 2.9 of [9]).

Corollary 2.5. *Suppose that $T \in L(X)$ is an isoloid operator and that $F \in L(X)$ commuting with T satisfies that F^n is finite rank for some $n \in \mathbb{N}$, then*

(1) *If T obeys property (gw) and $\Pi_a(T+F) \subset \sigma_a(T)$, then $T+F$ obeys property (gw).*

(2) *If T obeys property (w) and $\Pi_a^0(T+F) \subset \sigma_a(T)$, then $T+F$ obeys property (w).*

Proof. (1) Assume that T satisfies property (gw). Then T satisfies generalized a-Browder's theorem and property (gR) by [7, Theorem 2.6]. Hence, $T+F$ satisfies generalized a-Browder's theorem and property (gR) by Theorem 2.4. This implies $T+F$ satisfies property (gw).

(2) Analogous to the proof of (1). □

3. Bishop's Property (β) under Commuting Algebraic Perturbations

The Bishop's property (β) plays an important role in studying local spectral theory. We refer the readers to [18] for more details. For this reason, it is of interest to study the preservation of Bishop's property (β) under certain perturbations. Firstly, we collect two preliminary results of Bishop's property (β) at a point.

Lemma 3.1. *Let $T_i \in L(X_i)$, $i = 1,2$, if T_1, T_2 have Bishop's property (β) at λ_0, then $T_1 \oplus T_2$ has Bishop' s property (β) at λ_0.*

Proof. Suppose that T_1 and T_2 have the Bishop' s property (β) at λ_0 - *i.e.*, there exists $r_i > 0$, $i = 1,2$, such that for every open subset $U_i \subset D(\lambda_0, r_i)$ and for any sequence $\{f_n^i\}_{n=1}^{\infty} \subset \mathcal{O}(U_i, X_i)$, $lim_{n\to\infty}(T_i - \lambda)f_n^i(\lambda) = 0$ in $\mathcal{O}(U_i, X_i)$ implies $lim_{n\to\infty} f_n^i(\lambda) = 0$ in $\mathcal{O}(U_i, X_i)$. For an arbitrary open subset $U \subset D(\lambda_0, min\{r_1, r_2\})$, let us consider an analytic sequence $f_n = f_n^1 \oplus f_n^2 \subset \mathcal{O}(U, X_1 \oplus X_2)$, where $f_n^i \subset \mathcal{O}(U, X_i)$. The condition $lim_{n\to\infty}(\lambda I - T_1 \oplus T_2)f_n(\lambda) = 0$ in $\mathcal{O}(U, X_1 \oplus X_2)$ implies that $lim_{n\to\infty}(\lambda I - T_i)f_n^i(\lambda) = 0$ in $\mathcal{O}(U, X_i)$, $i = 1,2$. Since T_1 and T_2 have Bishop's property (β) at λ_0, then $lim_{n\to\infty} f_n^i(\lambda) = 0$ in $\mathcal{O}(U, X_i)$. Thus, $lim_{n\to\infty} f_n(\lambda) = 0$ in $\mathcal{O}(U, X_i \oplus X_2)$. □

Remark 3.2. Observe that if T has Bishop' s property (β) at λ_0, then $T - \lambda_0$ has the Bishop' s property (β) at 0. Indeed, suppose T has the Bishop' s property (β) at λ_0 – *i.e.*, there exists $r > 0$ such that for every open subset $U \subset D(\lambda_0, r)$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{O}(U, X)$, $lim_{n\to\infty}(T - \lambda I)f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$ implies $lim_{n\to\infty} f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$. Let $G \subset D(0, r)$ be open and $\{g_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{O}(G, X)$ such that $lim_{n\to\infty}(T - \lambda_0 I - \lambda I)g_n(\lambda) = 0$ in $\mathcal{O}(G, X)$. Define the sequence $\{f_n\}_{n=1}^{\infty}$ be $f_n(\lambda) = g_n(\lambda - \lambda_0)$. Then $lim_{n\to\infty}(T - (\lambda_0 + \lambda)I)f_n(\lambda_0 + \lambda) = 0$ in $\mathcal{O}(G, X)$. Since T has the Bishop' s property (β) at λ_0, then $lim_{n\to\infty} f_n(\lambda + \lambda_0) = lim_{n\to\infty} g_n(\lambda) = 0$ in $\mathcal{O}(G, X)$. Thus $T - \lambda_0$ has the Bishop' s property (β) at 0.

Recall from [18], the commutator of two operators $S, T \in L(X)$ is the operator $C(S, T)$ on $L(X)$ defined by $C(S, T)(A) := SA - AT$. By induction it is easy to show the binomial identity

$$C(S,T)^k(A) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i S^{k-i} A T^i.$$

Obviously, $C(S - \lambda I, T - \lambda I)^k(A) = C(S, T)^k(A)$ for all $\lambda \in \mathbb{C}$. If there exists an integer $k \in \mathbb{N}$ for which $C(S, T)^k(I) = C(T, S)^k(I) = 0$, then the operators S and T are said to be *nilpotent equivalent*. For $T, S \in L(X)$ with $ST = TS$, it is easily seen that $C(S, T)^k(I) = (S - T)^k$ for all $k \in \mathbb{N}$. Thus, S and T are nilpotent equivalent precisely when $S - T$ is nilpotent.

Theorem 3.3. *Let $T, S \in L(X)$ be nilpotent equivalent. Then T has Bishop' s property (β) at λ_0 if and only if S has Bishop's property (β) at λ_0. In particular, if T has Bishop's property (β) at λ_0 and N is nilpotent that commutes with T, then $T + N$ also has Bishop's property (β) at λ_0.*

Proof. By symmetry, it suffices to show that Bishop's property (β) at λ_0 is transferred from S to T. Suppose that S has the Bishop' s property (β) at λ_0 – *i.e.*, there exists $r > 0$ such that for every open subset $U \subset D(\lambda_0, r)$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{O}(U, X)$, $lim_{n\to\infty}(S - \lambda I)f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$ implies $lim_{n\to\infty} f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$. Let $lim_{n\to\infty}(T - \lambda I)f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$. Thus, we have

$$\begin{aligned} & C(S,T)^k(I)f_n(\lambda) - (S-\lambda I)^k f_n(\lambda) \\ &= C(S-\lambda I, T-\lambda I)^k(I)f_n(\lambda) - (S-\lambda I)^k f_n(\lambda) \\ &= \sum_{i=1}^{k}(-1)^i(S-\lambda I)^{k-i}(T-\lambda I)^i f_n(\lambda) \\ &= [\sum_{i=1}^{k}(-1)^i(S-\lambda I)^{k-i}(T-\lambda I)^{i-1}](T-\lambda I)f_n(\lambda) \\ &\longrightarrow 0 \quad \text{in } \mathcal{O}(U,X), \quad \text{as } n \to \infty. \end{aligned}$$

Since $C(S, T)^k(I) = 0$, we get that $lim_{n\to\infty}(S - \lambda I)^k f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$. As S has Bishop's property (β) at λ_0, it follows that $lim_{n\to\infty}(S - \lambda I)^{k-1} f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$. By induction we have that $lim_{n\to\infty} f_n(\lambda) = 0$ in $\mathcal{O}(U, X)$. Therefore, T has Bishop's property (β) at λ_0. □

Recall that an operator $K \in L(X)$ is said to be *algebraic* if there exists a non-trivial complex polynomial h such that $h(K) = 0$. Trivially, every nilpotent operator is algebraic and it is know that every finite-dimensional operator is algebraic. It is also known that every algebraic operator has a finite spectrum. The proof of the following theorem is strongly inspired by that of [6, theorem 2.3].

Theorem 3.4. *Let $T \in L(X)$, suppose that $K \in L(X)$ is an algebraic operator which commutes with T, and let h be a non-zero polynomial for which $h(K) = 0$. If T has Bishop's property (β) at each of the zeros of h, then $T - K$ has Bishop's property (β) at 0. In partircular, if T has Bishop's property (β), then $T + K$ has Bishop's property (β).*

Proof. An algebraic operator has a finite spectrum. Indeed, by spectral mapping theorem $h(\sigma(K)) = \sigma(h(K)) = \{0\}$, so that $\sigma(K)$ is finite. Let $\sigma(K) = \{\lambda_1, ..., \lambda_n\}$ and denote P_i as the spectral projection associated with K and the spectral set $\{\lambda_i\}$. Set $Y_i := R(P_i)$. Therefore, from the classical spectral decomposition, $Y_1, ..., Y_n$ are closed linear subspaces of X each of which is invariant under both K and T, and $X = Y_1 \oplus ... \oplus Y_n$. Moreover, for arbitrary $i = 1, ..., n$, $K_i := K|Y_i$ and $T_i := T|Y_i$ commute and we have $\sigma(K_i) = \{\lambda_i\}$. We claim that $N_i := \lambda_i I - K_i$ is nilpotent for every $i = 1, ..., n$. Since

$$h(\{\lambda_i\}) = h(\sigma(K_i)) = \sigma(h(K_i)) = \{0\},$$

it then follows that $h(\lambda_i) = 0$. Write

$$h(\mu) = (\lambda_i - \mu)^v q(\mu) \;\text{ with }\; q(\lambda_i) \neq 0.$$

Then

$$(\lambda_i I - K_i)^v q(K_i) = h(K_i) = 0$$

with $q(K_i)$ invertible. Therefore, $(\lambda_i I - K_i)^v = 0$ and hence, $N_i := \lambda_i I - K_i$ is nilpotent for every $i = 1, ..., n$, as desired.

Now observe that

$$T_i - K_i = (T_i - \lambda_i I) - (K_i - \lambda_i I) = T_i - \lambda_i I - N_i.$$

Since T has Bishop's property (β) at λ_i, we get that $T - \lambda_i I$ has Bishop's property (β) at 0 by Remark 3.2. Since Bishop's property (β) is inherited by restrictions to closed invariant subspaces, we conclude that $T_i - \lambda_i I$ has Bishop's property (β) at 0, and hence, by Theorem 3.3, $T_i - K_i = T_i - \lambda_i I - N_i$ also has Bishop's property (β) at 0 for all $i = 1, ..., n$. From Theorem 3.1, it then follows that

$$T - K = (T_1 - K_1) \oplus ... \oplus (T_n - K_n)$$

has Bishop's property (β) at 0, as desired. With an application of the main result, we can establish the final claim. □

In the sequel we give some applications of the Theorem 3.4.

Corollary 3.5. *If $T \in L(X)$ has decomposition property (δ) and $K \in L(X)$ is an algebraic operator which commutes with T, then $T + K$ has decomposition property (δ).*

Proof. Assume that T has decomposition property (δ), then T^* has Bishop's property (β). Obviously, K^* is algebraic and commutes with T^*. Hence, $(T + K)^* = T^* + K^*$ has Bishop's property (β). This implies $T + K$ has decomposition property (δ). □

Since T is decomposable if and only if T satisfies both (β) and (δ), then we get the following consequence immediately from the Theorem 3.4 and Corollary 3.5.

Corollary 3.6. *If $T \in L(X)$ is decomposable and $K \in L(X)$ is an algebraic operator which commutes with T, then $T + K$ is decomposable.*

Now we give some classes of operators which have attracted the attention of several authors in connection with Bishop's property (β); see also [15, 19–21] for more details. Let H be a Hilbert space and $T \in L(H)$:

(1) An operator T is called *k-quasi-∗-paranormal* if $\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\|$.
(2) An operator T is called *k-quasi-paranormal* if $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^kx\|$.
(3) An operator T is called *k-quasi-∗-class A* if $T^{*k}(|T^2| - |T^*|^2)T^k \geq 0$.
(4) An operator T is called *k-quasi-class A* if $T^{*k}(|T^2| - |T|^2)T^k \geq 0$.

It is not hard to check that T is a k-quasi-class A operator implies that T is a k-quasi-paranormal operator. Indeed, suppose that T is k-quasi-class A - i.e., $T^{*k}|T|^2T^k \leq T^{*k}|T^2|T^k$. Let $x \in H$, then $\|T^{k+1}x\|^2 = \langle T^{k+1}x, T^{k+1}x\rangle = \langle T^{*k}|T|^2T^kx, x\rangle \leq \langle T^{*k}|T^2|T^kx, x\rangle \leq \||T^2|T^kx\|\|Tx\| = \|T^{k+2}x\|\|T^kx\|$. Hence, T is k-quasi-paranormal.

In [22], Mecheri has proved that if T is k-quasi-class A, then T has Bishop's property (β). However, for the k-quasi-∗-class A, k-quasi-∗-paranormal or k-quasi-∗-paranormal operator T, whether T has Bishop's property (β) or not is still an open question; see [21]. Therefore, from our Theorem 3.4, we obtain the following result immediately.

Corollary 3.7. *Let T be an k-quasi-class A operator, if K is an algebraic operator that commutes with T, then $T + K$ has the Bishop's property (β).*

References

[1] P. Aiena, Fredholm and local spectral Theory, with application to Multipliers, Kluwer Academic Publishers, Dordrecht 2004.
[2] P. Aiena, E. Aponte, Polaroid type operators under perturbations, Studia Math. 214 (2013) 121–136.
[3] P. Aiena, E. Aponte, E. Bazan, Weyl type theorems for left and right polaroid operators, Integr. equ. oper. theory 66 (2010) 1–20.
[4] P. Aiena, J. R. Guillen, P. Pena, Property (R) for bounded linear operators, Mediterr. J. Math. 8 (2011) 491–508.
[5] P. Aiena, J. R. Guillen, P. Pena, Property (gR) and perturbations, Acta Sci. Math. (Szeged) 78 (2012) 569–588.
[6] P. Aiena, M. M. Neumann, On the stability of the localized single-valued extension property under commuting perturbations, Proc. Amer. Math. Soc. 141 (2013) 2039–2050.
[7] M. Amouch, M. Berkani, On the property (gw), Mediterr. J. Math. 5 (2008) 371–378.
[8] M. Berkani, On the equivalence of weyl theorem and generalized weyl theorem, Acta Mathematica Sinica, English Series 23 (2007) 103–110.
[9] M. Berkani, M. Amouch, Preservation of property (gw) under perturbations, Acta Sci. Math. (Szeged) 74 (2008) 769–781.
[10] M. Berkani, A. Arroud, Generalized Weyl's theorem and hyponormal operators, J. Aust. Math. Soc. 76 (2004) 291–302.
[11] O. Bel Hadj Fredj, M. Burgos, M. Oudghiri, Ascent spectrum and essential ascent spectrum, Studia Math. 187 (2008) 59–73.
[12] M. Berkani, M. Sarih, On semi B-Fredholm operators, Glasgow Math. J. 43 (2001) 457-465.
[13] M. Berkani, H. Zariouh, Extended Weyl type theorems, Math. Bohemica 134 (2009) 369–378.
[14] X. H. Cao, A. F. Liu, Generalized Kato type operators and property (w) under perturbations, Linear Algebra Appl. 436 (2012) 2231–2239.
[15] F. G. Gao, X. C. Fang, On k-quasiclass A operators, J. Inequal. Appl. 2009 (2009), 1–10.
[16] M. A. Kaashoek, D. C. Lay, Ascent, descent, and commuting perturbations, Trans. Amer. Math. Soc. 169 (1972) 35–47.

[17] S. H. Lee, W. Y. Lee, On Weyls theorem II, Math. Japon. 43 (1996) 549–553.
[18] K. B. Laursen, M. M. Neumann, Introduction to local spectral theory, Clarendon Press, Oxford, 2000.
[19] S. Mecheri, Isolated points of spectrum of k-quasi-$*$-class A operators, Studia Math. 208 (2012) 87–96.
[20] S. Mecheri, Bishop property (β) and Riesz idempotent for k-quasi-paranormal operators, Banach J. Math. Anal. 6 (2012) 147–154.
[21] S. Mecheri, On a new class of operators and Weyl type theorems, Filomat 27 (2013) 629–636.
[22] S. Mecheri, Bishop's property (β), hypercyclicity and hyperinvariant subspaces, Operators and Matrices 2 (2014) 555–562.
[23] M. Oudghiri, Weyl 's theorem and perturbations, Integr. equ. oper. theory 53 (2005) 535–545.